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Fourier expansion of periodic U-Bernoulli, U-Euler and U-Genocchi functions and their relation with the Riemann zeta function

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Abstract. Fourier series and generating function theory is an active branch of modern analysis that has gained importance due to its applications in methods of analysis for mathematical solutions to boundary value problems, engineering, and signal processing in communications. On the other hand, the Riemann zeta function and its generalizations are useful in the investigation of analytic number theory and allied disciplines, especially in the role played by their special values in integral arguments (see [4, 14]).

Key Words and Phrases: New U-Bernoulli polynomials, New U-Euler polynomials, New U-Genocchi polynomials, Fourier expansion, Integral representation, Riemann zeta function.

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1. Introduction

Fourier series and generating function theory is an active branch of modern analysis that has gained importance due to its applications in methods of analysis for mathematical solutions to boundary value problems, engineering, and signal processing in communications. On the other hand, the Riemann zeta function and its generalizations are useful in the investigation of analytic number theory and allied disciplines, especially in the role played by their special values in integral arguments (see [4, 14]).

It is well known that the classical Bernoulli, Euler, and Genocchi polynomials, with $x \in$, are respectively defined by the following generating functions (see [7, 16]):

$$\frac{ze^{zx}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}, \quad (|z| < 2\pi), \tag{1}$$

$$\frac{2e^{zx}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!}, \quad (|z| < \pi), \tag{2}$$

$$\frac{2ze^{zx}}{e^z+1}=\sum_{n=0}^{\infty}G_n(x)\frac{z^n}{n!},\quad (|z|<\pi). \tag{3}$$

For $x=0$, the numbers $B_n(0)=B_n$, $E_n(0)=E_n$ and $G_n(0)=G_n$ appear.

If $n>2$, the Fourier series for periodic Bernoulli functions of period 1 for even index is given by (see [10])

$$\tilde{B}_{2n}(x)=(-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\sum_{m=1}^{\infty}\frac{\cos(2m\pi x)}{m^{2n}};\quad x\in,\quad (n\in), \tag{4}$$

and for odd index

$$\tilde{B}_{2n+1}(x)=(-1)^{n+1}\frac{2(2n+1)!}{(2\pi)^{2n+1}}\sum_{m=1}^{\infty}\frac{\sin(2m\pi x)}{m^{2n+1}};\quad x\in,\quad (n\in). \tag{5}$$

By (5) and integration of the series for $\frac{\tilde{B}_{2n+1}(x)}{x}$ leads to the series representation of the Riemann zeta function $\zeta(2n+1)$, $n\in$ (cf. [10]):

$$\zeta(2n+1)=(-1)^{n+1}\frac{(2\pi)^{2n+1}}{2(2n+1)!}\int_0^1 B_{2n+1}(x)\cot\left(\frac{\pi x}{2}\right)\,dx. \tag{6}$$

Given that, $\tilde{E}_n(x+2)=-\tilde{E}_n(x+1)=\tilde{E}_n(x)$, $x\in$, if $n\in$, the Fourier series for the periodic Euler functions of period 2 is given by (see [11]):

$$\tilde{E}_{2n}(x)=4(-1)^n(2n)!\sum_{m=0}^{\infty}\frac{\sin((2m+1)\pi x)}{[(2m+1)\pi]^{2n+1}};\quad (x\in). \tag{7}$$

Using (7) and integration of the series for $\frac{\tilde{E}_{2n}(x)}{x}$ leads to the integral representation of $\zeta(2n+1)$, $n\in$ (cf. [11]).

Given that, $\tilde{G}_n(x+2)=-\tilde{G}_n(x+1)=\tilde{G}_n(x)$, for $x\in$, the Fourier series for the periodic Genocchi functions of period 2, is given by the relation (see [12]):

$$\tilde{G}_{2n+1}(x)=4(-1)^n(2n+1)!\sum_{m=0}^{\infty}\frac{\sin((2m+1)\pi x)}{[(2m+1)\pi]^{2n+1}};\quad (x\in). \tag{8}$$

By using (8) and integration of the series for $\frac{\tilde{G}_{2n+1}(x)}{x}$ leads to the integral representation of $\zeta(2n+1)$, $n\in$ (cf. [12]).

In the present work, considering [9], we investigate Fourier expansions for the new periodic U-Bernoulli, U-Euler, and U-Genocchi functions for finding new relationships with the Riemann zeta function, respectively. Moreover, we study special bounds for the respective numbers. In the next section, some formulas for the U-Bernoulli, U-Euler, and U-Genocchi polynomials will be derived, and we will recall some known results necessary for the present investigation. In Section 3, we give the Fourier series of the functions f_n. Also, we give a connection with U-Bernoulli numbers and the Riemann zeta function, and we find a bound for these numbers. We do the same job in sections 4 and 5 for the two families, $g_n(x)$ and $h_n(x)$.

2. Preliminary and basic results

Let $s = \sigma + i\rho$, be a complex number with σ, $\rho \in$. The Riemann zeta function is defined by (see [1, p. 249])

$$\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}, \qquad \sigma > 0. \tag{9}$$

The Euler formula is given by

$$e^{ix} = \cos(x) + i \sin(x). \tag{10}$$

We can see from (10) that (cf. [5])

$$i \tan x = 1 - \frac{2}{e^{2ix} + 1}. \tag{11}$$

Moreover, let us notice that:

$$i \tan x = \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} = 1 - \frac{2}{e^{2ix} - 1} + \frac{4}{e^{4ix} - 1}. \tag{12}$$

From (12), it follows that

$$\tan(-x) = -i + \frac{2i}{e^{-2ix} - 1} - \frac{4i}{e^{-4ix} - 1},$$

therefore

$$x \tan(-x) = -ix + \frac{2ix}{e^{-2ix} - 1} - \frac{4ix}{e^{-4ix} - 1}.$$

It is known that the integer part function $\lfloor x \rfloor$ is defined by

$$\lfloor x \rfloor := \max\{k \in : k \leq x\}.$$

The new family of U-Bernoulli polynomials $M_n(x)$ of degree n in variable x is defined by the following generating function (see [9])

$$F_M(x,z) = \left(\frac{z}{e^{-z}-1}\right) e^{-xz} = \sum_{n=0}^{\infty} M_n(x) \frac{z^n}{n!}, \quad (|z| < 2\pi). \tag{13}$$

Some U-Bernoulli polynomials are:

$M_0(x) = -1,$ $\quad$ $M_1(x) = x - \frac{1}{2},$
$M_1(x) = x - \frac{1}{2},$ $\quad$ $M_4(x) = -x^4 + 2x^3 - x^2 + \frac{1}{30},$
$M_2(x) = -x^2 + x - \frac{1}{6},$ $\quad$ $M_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x.$

$$M_0(x) = -1, \qquad M_1(x) = x - \frac{1}{2}, \qquad M_2(x) = -x^2 + x - \frac{1}{6}.$$

If $x = 0$ in (13), $M_n = M_n(0)$ will be called the U-Bernoulli numbers, hence, we have

$$\frac{z}{e^{-z}-1} = \sum_{n=0}^{\infty} M_n \frac{z^n}{n!}, \quad (|z| < 2\pi). \tag{14}$$

Some of these numbers are:

$$M_0 = -1; \quad M_1 = -\frac{1}{2}; \quad M_2 = -\frac{1}{6}; \quad M_3 = 0.$$

We can easily prove that for $n \in$ odd,

$$M_n = 0, \qquad \forall\, n \geq 3. \tag{15}$$

The new family of U-Euler polynomials $A_n(x)$ of degree n in variable x is defined by the following generating function (see [9])

$$E_A(x;z) = \left(\frac{2}{e^{-\frac{z}{2}}+1}\right) e^{-\frac{xz}{2}} = \sum_{n=0}^{\infty} A_n(x) \frac{z^n}{n!}, \quad (|z| < 2\pi). \tag{16}$$

Some U-Euler polynomials are:

$A_0(x) = 1,$ $\quad$ $A_3(x) = -\frac{1}{8}x^3 + \frac{3}{16}x^2 - \frac{1}{32},$
$A_1(x) = \frac{1}{4} - \frac{x}{2},$ $\quad$ $A_4(x) = \frac{1}{16}x^4 - \frac{1}{8}x^3 + \frac{1}{16}x,$
$A_2(x) = \frac{1}{4}x^2 - \frac{1}{4}x,$ $\quad$ $A_5(x) = -\frac{1}{32}x^5 + \frac{5}{64}x^4 - \frac{5}{64}x^2 + \frac{1}{64}.$

$$A_2(x) = \frac{1}{4}x^2 - \frac{1}{4}x, \qquad A_3(x) = -\frac{1}{8}x^3 + \frac{3}{16}x^2 - \frac{1}{32}, \qquad A_4(x) = \frac{1}{16}x^4 - \frac{1}{8}x^3 + \frac{1}{16}x.$$

When $x = 0$ in (16), $A_n = A_n(0)$ we will call them the U-Euler numbers. We have therefore

$$\frac{2}{e^{-\frac{z}{2}}+1} = \sum_{n=0}^{\infty} A_n \frac{z^n}{n!}, \quad (|z| < 2\pi). \tag{17}$$

Some of these numbers are:

$$A_0 = 1; \quad A_1 = \frac{1}{4}; \quad A_2 = 0; \quad A_3 = -\frac{1}{32}.$$

In (17), we can prove that if $n \in$ is an even number,

$$A_n = 0, \qquad \forall n \geq 2. \tag{18}$$

The new family of U-Genocchi polynomials $V_n(x)$ of degree n in variable x is defined by (see [9])

$$G_V(x,z) = \left(\frac{2z}{e^{-\frac{z}{2}}+1}\right) e^{-\frac{xz}{2}} = \sum_{n=0}^{\infty} V_n(x)\frac{z^n}{n!}, \quad (|z| < 2\pi). \tag{19}$$

Some U-Genocchi polynomials are:

$V_0(x) = 0,$ $\qquad V_3(x) = \frac{3}{4}x^2 - \frac{3}{4}x,$
$V_1(x) = 1,$ $\qquad V_4(x) = -\frac{1}{2}x^3 + \frac{3}{4}x^2 - \frac{1}{8},$
$V_2(x) = -x + \frac{1}{2},$ $\qquad V_5(x) = \frac{5}{16}x^4 - \frac{5}{8}x^3 + \frac{5}{16}x.$

$$V_2(x) = -x + \frac{1}{2}, \qquad V_3(x) = \frac{3}{4}x^2 - \frac{3}{4}x, \qquad V_4(x) = -\frac{1}{2}x^3 + \frac{3}{4}x^2 - \frac{1}{8}.$$

For $x = 0$ in (19), $V_n = V_n(0)$, we will call them the U-Genocchi numbers. So

$$\frac{2z}{e^{-\frac{z}{2}}+1} = \sum_{n=0}^{\infty} V_n \frac{z^n}{n!}, \qquad (|z| < 2\pi). \tag{20}$$

Some of these numbers are:

$$V_0 = 0; \quad V_1 = 1; \quad V_4 = -\frac{1}{8}.$$

Starting from (19), it is possible to establish, for $n \geq 1$ a relationship with the U-Euler polynomials employing

$$V_n(x) = nA_{n-1}(x). \tag{21}$$

Furthermore, by setting $x = 0$ in (21), it follows

$$V_n = nA_{n-1}, \qquad \forall\, n \geq 1. \tag{22}$$

Proposition 2.1. *The polynomials $M_n(x)$ satisfy*

$$M_n^{'}(x) = -nM_{n-1}(x), \qquad \forall n \in . \tag{23}$$

$$M_n(x+1) - M_n(x) = n(-1)^{n-1}x^{n-1}, \qquad \forall n \in . \tag{24}$$

Proof. In view of (13), we have

$$\frac{\partial}{\partial x}\left(\frac{ze^{-xz}}{e^{-z}-1}\right) = -n\sum_{n=1}^{\infty} M_{n-1}(x)\frac{z^n}{n!}.$$

Besides,

$$\frac{\partial}{\partial x}\left(\frac{ze^{-xz}}{e^{-z}-1}\right) = \sum_{n=0}^{\infty} M_n^{'}(x)\frac{z^n}{n!}.$$

So, the uniqueness of the power series leads us to

$$\frac{M_n^{'}(x)}{n!} = \frac{-nM_{n-1}(x)}{n!}.$$

Hence, assertion (23) follows.

We will now provide (24). We start from (13)

$$\frac{z}{e^{-z}-1}e^{-(x+1)z} - \frac{z}{e^{-z}-1}e^{-xz} = \frac{ze^{-xz}(e^{-z}-1)}{e^{-z}-1} = ze^{-xz}.$$

Turning to the power series, we have that

$$\sum_{n=0}^{\infty}[M_{n+1}(x+1) - M_{n+1}(x)]\frac{z^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty}(-1)^n x^n (n+1)\frac{z^{n+1}}{(n+1)!}.$$

So, assertion (24) holds. Proposition 2.1 is proved.

□

Taking $x = 0$ in (24) for $n \geq 2$, we get

$$M_n(1) \quad = \quad M_n. \tag{25}$$

Proposition 2.2. *Let $n \in$, then the polynomials $A_n(x)$ satisfy*

$$A_n^{'}(x) = -\frac{n}{2}A_{n-1}(x), \tag{26}$$

$$A_n(x+1) + A_n(x) = \frac{(-1)^n}{2^{n-1}}x^n. \tag{27}$$

Proof. By differentiating (16), with respect to x, the power series' uniqueness leads us to (26).

We will show the assertion (27). Starting from (16) we have

$$\frac{2e^{-\frac{(x+1)z}{2}}}{e^{-\frac{z}{2}}+1}+\frac{2e^{-\frac{xz}{2}}}{e^{-\frac{z}{2}}+1}=2e^{-\frac{xz}{2}}.$$

Turning to the power series we have that

$$\sum_{n=0}^{\infty}A_n(x+1)\frac{z^n}{n!}+\sum_{n=0}^{\infty}A_n(x)\frac{z^n}{n!}=2\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\frac{x^nz^n}{n!}, \tag{28}$$

from 28 we obtain (27). Proposition 2.2 is proved. □

Let's notice that, making $x=0$ in (27), we obtain:

$$\begin{aligned} A_n(1)+A_n(0) &= 0, \\ A_n(1) &= -A_n. \end{aligned} \tag{29}$$

Theorem 2.3. *For every $n\in$, the polynomials $V_n(x)$ satisfy the properties:*

$$V_n'(x)=-\frac{n}{2}V_{n-1}(x), \tag{30}$$

$$V_n(x+1)+V_n(x)=\frac{n(-1)^{n-1}}{2^{n-2}}x^{n-1}. \tag{31}$$

Proof. From (26), we have

$$A_n'(x)=-\frac{n}{2}A_{n-1}(x).$$

By using (21), we obtain

$$A_n'(x)=-\frac{1}{2}V_n(x),$$

then

$$V_n(x)=-2A_n'(x).$$

Whence it follows that

$$\begin{aligned} V_n'(x) &= -\frac{n(n-1)}{2}\frac{V_{n-1}(x)}{n-1}.[-2A_n'(x)]' \\ &= -2\left[-\frac{n}{2}A_{n-1}(x)\right]' \\ &= n\left[-\frac{n-1}{2}A_{n-2}(x)\right]'=-\frac{n(n-1)}{2}A_{n-2}(x) \\ &= -\frac{n(n-1)}{2}\frac{V_{n-1}(x)}{n-1}. \end{aligned}$$

Then, the U-Genocchi polynomials satisfy (30).

We will now prove the assertion (31). Starting from (27) and using (21), we have that

$$\begin{aligned} A_n(x+1) + A_n(x) &= \frac{(-1)^n}{2^{n-1}} x^n \\ \Leftrightarrow \frac{V_{n+1}(x+1)}{n+1} + \frac{V_{n+1}(x)}{n+1} &= \frac{(-1)^n}{2^{n-1}} x^n. \end{aligned}$$

So, (31) follows. Thus, Theorem 2.3 is completely proved.

□

For $x = 0$ in (31), we get

$$V_n(1) = -V_n. \tag{32}$$

3. Fourier series of the periodic U-Bernoulli functions

In this section, we give the Fourier series of a periodic function involving $U-$Bernoulli polynomials and introduce the relationship of the U-Bernoulli numbers with the Riemann zeta function. Furthermore, we give a bound for U-Bernoulli numbers.

Theorem 3.1. *Let $n \in$,*

$$f_n(x) := M_n(x - \lfloor x \rfloor), \quad (x \in). \tag{33}$$

Then, the Fourier series for $f_n(x)$ is

$$f_n(x) = M_n(x - \lfloor x \rfloor) = \frac{(-1)^n n!}{(2\pi i)^n} {\sum}' \frac{e^{2\pi i k x}}{k^n}, \tag{34}$$

where $\sum'$ denotes a sum over the integers not including 0.

Proof. The function f_n is a periodic function with a period $T = 1$.

$$f_n(x+1) = M_n(x + 1 - \lfloor x+1 \rfloor) = M_n(x - \lfloor x \rfloor) = f_n(x).$$

So, $f_n(x)$ has a representation in Fourier series, which is given by (see [4]):

$$f_n(x) = \sum_{k \in} \hat{f}_n(k) e^{2\pi i n x}.$$

Firstly, note that for every $x \in [0, 1)$,

$$f_n(x) = M_n(x - \lfloor x \rfloor) = M_n(x - 0) = M_n(x).$$

Now, let's calculate the coefficients $\hat{f}_n(k)$. For $k = 0$, applying (23) we have

$$\hat{f}_n(0) = \int_0^1 f_n(x)e^{-2\pi i0x}dx = \frac{-1}{n+1}[M_{n+1}(0) - M_{n+1}(1)] = 0, \qquad \forall n \geq 1.$$

Now, for $k > 0$, $n = 1$, we obtain

$$\hat{f}_1(k) = \int_0^1 M_1(x - \lfloor x \rfloor)e^{-2\pi ikx}dx = \int_0^1 \left(x - \frac{1}{2}\right) e^{-2\pi ikx}dx.$$

Integrating by parts, it follows

$$\hat{f}_1(k) = \left[-\left(x - \frac{1}{2}\right)\left(\frac{e^{-2\pi ikx}}{2\pi ikx}\right)\right]_0^1 + \frac{1}{2\pi ik}\int_0^1 e^{-2\pi ikx}.$$

If $u = 2\pi ikx$, we see that

$$\int_0^1 e^{-2\pi ikx} = \int_0^1 [\cos(2\pi kx) - i\sin(2\pi kx)]dx = 0.$$

This implies

$$\hat{f}_1(k) = \frac{-1}{2\pi ik}. \tag{35}$$

Let's see now the case $k > 0$, $n > 1$.

$$\hat{f}_n(k) = \int_0^1 M_n(x - \lfloor x \rfloor)e^{-2\pi ikx}dx = \int_0^1 M_n(x)e^{-2\pi ikx}dx.$$

Integrating by parts, with $u = M_n(x)$ and $dv = e^{-2\pi ikx}dx$, and using (25) we get

$$\begin{aligned} \hat{f}_n(k) &= \frac{-1}{2\pi ik}\left[M_n(1) - M_n(0)\right] - \frac{n}{2\pi ik}\int_0^1 M_{n-1}(x)e^{-2\pi ikx}dx, \\ &= -\frac{n}{2\pi ik}\int_0^1 M_{n-1}(x)e^{-2\pi ikx}dx. \end{aligned}$$

Integrating by parts again, with $u = M_{n-1}(x)$ and $dv = e^{-2\pi ikx}dx$, we can apply (25), to get

$$\hat{f}_n(k) = \frac{-n}{2\pi ik}\left\{\left[-\frac{M_{n-1}(x)}{2\pi ik}e^{-2\pi ikx}\right]_0^1 - \frac{n-1}{2\pi ik}\int_0^1 M_{n-2}(x)e^{-2\pi ikx}dx\right\},$$

$$= \frac{(-1)^2 n(n-1)}{(2\pi ik)^2} \int_0^1 M_{n-2}(x) e^{-2\pi ikx} dx.$$

So, integrating by parts $(n-1)$ times we get

$$\hat{f}_n(k) = \frac{(-1)^{n-1} n!}{(2\pi ik)^{n-1}} \int_0^1 M_1(x) e^{-2\pi ikx} dx.$$

Now, it follow from (35) that

$$\hat{f}_n(k) = \frac{(-1)^n n!}{(2\pi ik)^n}.$$

This finishes the proof of Theorem 3.1. □

Theorem 3.2. *For $n \geq 1$, the U-Bernoulli numbers satisfy the following relationship with the Riemann zeta function:*

$$M_{2n} = \frac{2(2n)!(-1)^n}{(2\pi)^{2n}} \zeta(2n). \tag{36}$$

Proof. We employ the Theorem 3.1. By making $x = 0$ in (33), we have

$$\begin{aligned} f_n(0) &= \frac{(-1)^n n!}{(2\pi i)^n} \sum_{k\in}{}' \frac{e^{2\pi ik(0)}}{k^n} \\ &= M_n(0) = M_n = \frac{(-1)^n n!}{(2\pi i)^n} \sum_{k\in}{}' \frac{1}{k^n}. \end{aligned}$$

Therefore,

$$M_{2n} = \frac{(2n)!}{(2\pi i)^{2n}} \sum_{k\in}{}' \frac{1}{k^{2n}}. \tag{37}$$

Notice that, from (9)

$$\begin{aligned} \sum_{k\in}{}' \frac{1}{k^{2n}} &= \cdots + \frac{1}{(-2)^{2k}} + \frac{1}{(-1)^{2k}} + \frac{1}{(1)^{2k}} + \frac{1}{(2)^{2k}} + \cdots \\ &= 2 \sum_{k\in}{}' \frac{1}{k^{2n}} = 2\zeta(2n). \end{aligned} \tag{38}$$

Consequently, (37) becomes

$$M_{2n} = \frac{(2n)!}{(2\pi)^{2n}(i^2)^n} 2\zeta(2n) = \frac{2(2n)!}{(2\pi)^{2n}(-1)^n} \zeta(2n).$$

So, (36) follows and Theorem 3.2 is proved.

□

Theorem 3.3. *For all $n \geq 1$, the U-Bernoulli numbers M_n satisfy the inequality,*

$$\frac{(2n)!}{(2\pi)^{2n}} \leq |M_{2n}| \leq \frac{(2n)!}{(2\pi)^{2n}} \frac{\pi^2}{3}. \tag{39}$$

Proof. From (36), gives

$$|M_{2n}| = \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k\geq 1} \frac{1}{k^{2n}}. \tag{40}$$

Now, let's consider the following chain of inequalities:

$$\begin{aligned} & k^{2n} \geq k^2 \\ \Leftrightarrow \quad & \sum_{k\geq 1} \frac{1}{k^{2n}} \leq \sum_{k\geq 1} \frac{1}{k^2} \\ \Leftrightarrow \quad & \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k\geq 1} \frac{1}{k^{2n}} \leq \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k\geq 1} \frac{1}{k^2}, \end{aligned}$$

hence

$$|M_{2n}| \leq \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k\geq 1} \frac{1}{k^2}.$$

Now, by Theorem 3.2 with $n = 1$, we have

$$\sum_{k\geq 1} \frac{1}{k^2} = \frac{(2\pi)^2}{2(-1)2} M_2 = \frac{4\pi^2}{-4}\left(-\frac{1}{6}\right) = \frac{\pi^2}{6},$$

therefore

$$\frac{2(2n)!}{(2\pi)^{2n}} \sum_{k\geq 1} \frac{1}{k^2} = \frac{(2n)!}{(2\pi)^{2n}} \frac{\pi^2}{3}.$$

So that,

$$|M_{2n}| \leq \frac{(2n)!}{(2\pi)^{2n}} \frac{\pi^2}{3}, \qquad n \geq 1. \tag{41}$$

On the other hand, notice that

$$\frac{2(2n)!}{(2\pi)^{2n}} \sum_{k\geq 1} \frac{1}{k^2} \geq \frac{(2n)!}{(2\pi)^{2n}},$$

then

$$|M_{2n}| \geq \frac{(2n)!}{(2\pi)^{2n}}, \qquad k \geq 1. \tag{42}$$

Therefore, we derive (39) from (41) and (42). Theorem 3.3 is proved.

□

4. Fourier series of the periodic U-Euler functions

In this section, we give the Fourier series of a periodic function involving $U-$Euler polynomials. Furthermore, we give a bound for U-Euler numbers.

Theorem 4.1. *Let $n \in$,*

$$g_n(x) := A_n(x - \lfloor x \rfloor), \qquad (x \in). \tag{43}$$

Then, the Fourier series for $g_n(x)$ is

$$g_n(x) = \sum_{k\in} \left[2\sum_{j=1}^{n-1} \left(-\frac{1}{2}\right)^{j-1} \binom{n}{j-1} \frac{A_{n-j+1}(j-1)!}{(2k\pi i)^j} + \left(-\frac{1}{2}\right)^n \frac{n!}{(2k\pi i)^n} \right] e^{2k\pi i x}. \tag{44}$$

Proof. In analogy to (33), we find that g_n is also a periodic function with period $T = 1$, hence, it admits a Fourier series expansion given by

$$g_n(x) = \sum_{k\in} \hat{g}_n(k) e^{2\pi i k x},$$

where

$$\hat{g}_n(k) = \int_0^1 g_n(x) e^{-2\pi i k x} dx.$$

Now, let's calculate the coefficients $\hat{g}_n(k)$.

For $k = 0,\ n \in$ and using (26) and (29), we get

$$\begin{aligned} \hat{g}_n(0) &= \int_0^1 A_n(x - \lfloor x \rfloor) e^{-2\pi i 0 x} dx \\ &= -\frac{2}{n+1}[-2A_{n+1}] = \frac{(-2)^2 A_{n+1}}{n+1}, \end{aligned}$$

this is true for all $n \geq 1$. For $k > 0$ and $n = 1$, from (26) and (29), follows

$$\hat{g}_1(k) = \int_0^1 A_1(x - \lfloor x \rfloor) e^{-2\pi i k x} dx = \int_0^1 \left(\frac{1}{4} - \frac{x}{2}\right) e^{-2\pi i k x} dx.$$

Integrating by parts, with $u = \frac{1}{4} - \frac{x}{2}$, $dv = e^{-2\pi i k x} dx$, using (26) and (29) gives

$$
\begin{aligned}
\hat{g}_1(k) &= \left[-\frac{1}{2\pi ik}\left(\frac{1}{4}-\frac{x}{2}\right)e^{-2\pi ikx}\right]_0^1 - \int_0^1 -\frac{1}{2\pi ik}e^{-2\pi ikx}\left(-\frac{1}{2}\right)dx \\
&= -\frac{1}{2\pi ik}\left[\left(\frac{1}{4}-\frac{x}{2}\right)e^{-2\pi ikx}\right]_0^1 - \frac{1}{2\pi ik}\frac{1}{2}\int_0^1 e^{-2\pi ikx}dx.
\end{aligned}
$$

Now, it follows that,

$$
\hat{g}_1(k) = -\frac{1}{2\pi ik}\left[-\frac{1}{4}(1+1)\right] = -\frac{1}{2\pi ik}\left(-\frac{1}{2}\right).
$$

Therefore,

$$
\hat{g}_1(k) = -\frac{1}{2k\pi i}\left(-\frac{1}{2}\right). \tag{45}
$$

Let's consider now, $n > 1$ and $k > 0$. We have in this case,

$$
\hat{g}_n(k) = \int_0^1 A_n(x - \lfloor x \rfloor)e^{-2\pi ikx}dx = \int_0^1 A_n(x)e^{-2\pi ikx}dx,
$$

integrating by parts, with $u = A_n(x)$, $dv = e^{-2\pi ikx}dx$, and using (26), (29) we get

$$
\hat{g}_n(k) = -\frac{1}{2\pi ik}\left[A_n(1)e^{-2\pi ik} - A_n(0)\right] - \frac{n}{2}\frac{1}{2\pi ik}\int_0^1 A_{n-1}(x)e^{-2\pi ikx}dx.
$$

So that,

$$
\hat{g}_n(k) = -\frac{1}{2\pi ik}\left[-2A_n\right] - \frac{n}{2}\frac{1}{2\pi ik}\int_0^1 A_{n-1}(x)e^{-2\pi ikx}dx.
$$

Integrating by parts the last integral, with $u = A_{n-1}(x)$, $du = -\frac{(n-1)}{2}A_{n-2}(x)$ and using (26) and (29) it follows that

$$
\begin{aligned}
\hat{g}_n(k) &= -\frac{[-2A_n]}{2k\pi i} - \frac{n}{2}\frac{1}{2\pi ik}\left\{-\frac{1}{2\pi ik}\left[A_{n-1}(x)e^{-2k\pi ix}\right]_0^1 - \frac{(n-1)}{2}\frac{1}{2\pi ik}\int_0^1 A_{n-2}(x)e^{-2k\pi ix}dx\right\} \\
&= \frac{2A_n}{2k\pi i} + \left(-\frac{1}{2}\right)\frac{2nA_{n-1}}{(2\pi ik)^2} + \left(-\frac{1}{2}\right)^2\frac{n(n-1)}{(2k\pi i)^2}\int_0^1 A_{n-2}(x)e^{-2k\pi ix}dx.
\end{aligned}
$$

Integrating by parts the last integral, with $u = A_{n-2}(x)$, $du = -\frac{(n-2)}{2}A_{n-3}(x)dx$ by (26), (29) and using the notation

$\Theta = \dfrac{2A_n}{2k\pi i} + \left(-\dfrac{1}{2}\right)\dfrac{2nA_{n-1}}{(2\pi ik)^2}$, we obtain

$$\begin{aligned}
\hat{g}_n(k) &= \Theta + \left(-\frac{1}{2}\right)^2 \frac{n(n-1)}{(2k\pi i)^2}\left\{-\frac{1}{2k\pi i}[A_{n-2}(x)e^{-2k\pi i}]_0^1 - \frac{(n-2)}{2}\frac{1}{2k\pi i}\int_0^1 A_{n-3}(x)e^{-2k\pi ix}dx\right\}\\
&= \Theta + \left(-\frac{1}{2}\right)^2 \frac{n(n-1)}{(2k\pi i)^2}\frac{2A_{n-2}}{2k\pi i} + \left(-\frac{1}{2}\right)^2 \frac{n(n-1)}{(2k\pi i)^2}\frac{-(n-2)}{2}\frac{1}{2k\pi i}\int_0^1 A_{n-3}(x)e^{-2k\pi ix}dx\\
&= \Theta + \left(-\frac{1}{2}\right)^2 \frac{2n(n-1)A_{n-2}}{(2k\pi i)^3} + \left(-\frac{1}{2}\right)^3 \frac{n(n-1)(n-2)}{(2k\pi i)^3}\int_0^1 A_{n-3}(x)e^{-2k\pi ix}dx.
\end{aligned}$$

Integrating by parts the last integral $(n-1)$ times, we can say

$$\begin{aligned}
\hat{g}_n(k) &= \sum_{j=0}^{n-2}\left(-\frac{1}{2}\right)^j \frac{2A_{n-j}}{(2k\pi i)^{j+1}}\frac{n!}{(n-j)!} + \left(-\frac{1}{2}\right)^{n-1}\frac{n(n-1)\cdots(n-(n-1))}{(2k\pi i)^{n-1}}\int_0^1 A_1(x)e^{-2k\pi ix}dx\\
&= \sum_{j=1}^{n-1}\left(-\frac{1}{2}\right)^{j-1}\frac{2A_{n-j+1}(j-1)!}{(2k\pi i)^j}\frac{n!}{(j-1)!(n-j+1)!} + \left(-\frac{1}{2}\right)^n \frac{n!}{(2k\pi i)^n}.
\end{aligned}$$

Then,

$$\hat{g}_n(k) = \left(-\frac{1}{2}\right)^n \frac{n!}{(2k\pi i)^n} + \sum_{j=1}^{n-1}\left(-\frac{1}{2}\right)^{j-1}\binom{n}{j-1}\frac{2A_{n-j+1}(j-1)!}{(2k\pi i)^j}. \tag{46}$$

Therefore, from (4) and (46) follows (44). Theorem 4.1 is completely the proved. □

Proposition 4.2. *The $U-$Bernoulli numbers defined in (14) and the tangent function are related by*

$$\tan(-x) = \sum_{n=0}^{\infty}\frac{M_{2n+2}}{(2n+2)!}4^{n+1}[1-4^{n+1}](-1)^{n+1}x^{2n+1}. \tag{47}$$

Proof. By (14) and (2) with $z = 2ix$ and $z = 4ix$ and taking into account (15), we have the following

$$\begin{aligned}
x\tan(-x) &= -ix + \sum_{n=0}^{\infty}\frac{M_n}{n!}[2^n - 4^n]i^n x^n\\
&= -ix + \left(-\frac{1}{2}\right)[-2]ix + \sum_{n=2}^{\infty}\frac{M_n}{n!}[2^n - 4^n]i^n x^n
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{M_{2n}}{(2n)!}[4^n - 4^n 4^n](-1)^n x^{2n}.$$

Hence,

$$x \tan(-x) = \sum_{n=1}^{\infty} \frac{M_{2n}}{(2n)!} 4^n [1 - 4^n](-1)^n x^{2n}.$$

So (47) follows. Proposition 4.2 is demonstrated.

□

Proposition 4.3. *The U-Euler numbers defined in (17) and the tangent function are related by*

$$\tan(-x) = \sum_{n=0}^{\infty} A_{2n+1} \frac{4^{2n+1}(-1)^{n+1} x^{2n+1}}{(2n+1)!}. \tag{48}$$

Proof. Using together (16) and (11) with $z = 4ix$ and taking into account (18), it follows that

$$\begin{aligned} i \tan(-x) &= 1 - \sum_{n=0}^{\infty} A_n \frac{(4ix)^n}{n!} = 1 - \sum_{n=0}^{\infty} A_n \frac{4^n i^n x^n}{n!} \\ &= -\sum_{n=0}^{\infty} A_{n+1} \frac{4^{n+1} i^{n+1} x^{n+1}}{(n+1)!} = -i \sum_{n=0}^{\infty} A_{n+1} \frac{4^{n+1} i^n x^{n+1}}{(n+1)!} \\ &= -i \sum_{n=0}^{\infty} A_{2n+1} \frac{4^{2n+1} i^{2n} x^{2n+1}}{(2n+1)!}. \end{aligned}$$

So,

$$i \tan(-x) = i \sum_{n=0}^{\infty} A_{2n+1} \frac{4^{2n+1}(-1)^{n+1} x^{2n+1}}{(2n+1)!}. \tag{49}$$

Thus, by (49), the desire result follows. Thus, we complete the proof of the Proposition 4.3.

□

Proposition 4.4. *For $n \geq 1$, the numbers M_n and A_n are related by means of the following formula*

$$M_{2n} = \frac{n 2^{2n-1}}{(1 - 4^n)} A_{2n-1}. \tag{50}$$

Proof. We have from (47) and (48) that

$$\begin{aligned}
\frac{M_{2n+2}}{(2n+2)!}4^{n+1}[1-4^{n+1}](-1)^{n+1} &= \frac{A_{2n+1}}{(2n+1)!}4^{2n+1}(-1)^{n+1} \\
&= \frac{A_{2n+1}4^{2n+1}(-1)^{n+1}(2n+2)!}{4^{n+1}[1-4^{n+1}](-1)^{n+1}(2n+1)!} \\
&= \frac{A_{2n+1}4^{n}4^{n+1}(2n+1)!(2n+2)}{4^{n+1}[1-4^{n+1}](2n+1)!} \\
&= \frac{A_{2n+1}4^{n}(2n+2)}{[1-4^{n+1}]}.
\end{aligned}$$

Then,

$$M_{2(n+1)} = \frac{4^n(2n+2)}{[1-4^{n+1}]}A_{2n+1}.$$

□

From the previous equality, it follows that

$$\begin{aligned}
M_{2n} &= \frac{A_{2(n-1)+1}4^{n-1}(2(n-1)+2)}{[1-4^{(n-1)+1}]} \\
&= \frac{A_{2n-1}4^{n-1}(2n)}{1-4^n} \\
&= \frac{A_{2n-1}4^n n}{2(1-4^n)} = \frac{n2^{2n-1}}{1-4^n}A_{2n-1}.
\end{aligned}$$

Proposition 4.4 is demonstrated.

Proposition 4.5. *For $n \geq 1$, the U-Euler numbers defined in* (17) *are related to the Riemann zeta function as follows:*

$$\zeta(2n) = \frac{4^{n-1}(2\pi)^{2n}(-1)^n}{2(2n-1)![1-4^n]}A_{2n-1}. \tag{51}$$

Furthermore,

$$\frac{(4^n-1)(2n)!}{n2^{2n-1}(2\pi)^{2n}} \leq |A_{2n-1}| \leq \frac{2(4^n-1)(2n)!\pi^2}{3n(4\pi)^{2n}}. \tag{52}$$

Proof. The representation (51) follows from (36) and (50)

$$\frac{2(2n)!(-1)^n}{(2\pi)^{2n}}\zeta(2n) = \frac{nA_{2n-1}2^{2n-1}}{1-4^n}.$$

So, we obtain

$$\zeta(2n) = \frac{A_{2n-1}4^{n-1}(2\pi)^{2n}(-1)^n}{2(2n-1)![1-4^n]}.$$

From (50) it follows that

$$A_{2n-1} = \frac{(1-4^n)}{n2^{2n-1}}M_{2n}.$$

This implies

$$|A_{2n-1}| = \left|\frac{(1-4^n)}{n2^{2n-1}}M_{2n}\right| = \frac{(4^n-1)}{n2^{2n-1}}|M_{2n}|. \tag{53}$$

Now, from (41) and (53) we have

$$|A_{2n-1}| \le \frac{2(4^n-1)(2n)!\pi^2}{3n(4\pi)^{2n}}. \tag{54}$$

On the other hand, from (42) and (53), we get

$$|A_{2n-1}| \quad \ge \quad \frac{4^n-1}{n2^{2n-1}}\frac{(2n)!}{(2\pi)^{2n}}. \tag{55}$$

Then, from (54) and (55), (52) follows. Proposition 4.5 is completely proved.

$$\frac{(4^n-1)(2n)!}{n2^{2n-1}(2\pi)^{2n}} \le |A_{2n-1}| \le \frac{2(4^n-1)(2n)!\pi^2}{3n(4\pi)^{2n}}.$$

□

5. Fourier series of the periodic U-Genocchi functions

In this section, we give the Fourier series of a periodic function involving U-Genocchi polynomials. Furthermore, we give a bound for U-Genocchi numbers.

Theorem 5.1. *Let $n \ge 1$, $x \in$ and let's consider the function*

$$h_n(x) := V_n(x - \lfloor x \rfloor), \tag{56}$$

where $\lfloor x \rfloor$ is the integer part function. Then, the Fourier series for $h_n(x)$ is given by

$$h_n(k) = n\sum_{k\in}\left[2\sum_{j=1}^{n-2}\left(-\frac{1}{2}\right)^{j-1}\binom{n-1}{j-1}\frac{V_{n-j+2}}{n-j+2}\frac{(j-1)!}{(2k\pi i)^j} + \left(-\frac{1}{2}\right)^{n-1}\frac{(n-1)!}{(2k\pi i)^n}\right]e^{2k\pi i x}. \tag{57}$$

Proof. Analogously to (33), the function h_n is also a periodic function with period $T = 1$, hence, it admits a Fourier series expansion given by

$$h_n(x) = \sum_{k\in} \hat{h}_n(k)e^{2\pi ikx},$$

where

$$\hat{h}_n(k) = \int_0^1 h_n(x)e^{-2\pi ikx}dx.$$

Using the relationship between U-Euler and U-Genocchi polynomials given by (21), we can get a Fourier expansion for the U-Genocchi polynomial as follows: Firstly, let's see that

$$h_n(x) = V_n(x - \lfloor x \rfloor) = nA_{n-1}(x - \lfloor x \rfloor) = ng_{n-1}(x).$$

Hence, applying (21) to the Fourier series for $g_n(x)$ given by (44) yields (57). This completes the proof. □

On the other hand, the relationship between U-Euler and U-Genocchi numbers given by (22) allows us to find directly, the relationship between the U-Genocchi numbers and the Riemann zeta function. We can now state the following result.

Proposition 5.2. *For $n \geq 1$, numbers V_n are related to the Riemann zeta function as follows*

$$\zeta(2n) = \frac{4^{n-2}(2\pi)^{2n}(-1)^n}{n(2n-1)![1-4^n]}V_{2n}. \tag{58}$$

Furthermore,

$$\frac{4(4^n-1)(2n)!}{2^{2n}(4\pi)^{2n}} \leq |V_{2n}| \leq \frac{4(4^n-1)(2n)!\pi^2}{3(4\pi)^{2n}}. \tag{59}$$

Proof. From (21) and (51), (58) follows.

$$\begin{aligned} \zeta(2n) &= \frac{4^{n-1}(2\pi)^{2n}(-1)^n}{2(2n-1)![1-4^n]}A_{2n-1} = \frac{4^{n-1}(2\pi)^{2n}(-1)^n}{2(2n-1)![1-4^n]}\frac{V_{2n}}{2n} \\ &= \frac{4^{n-1}(2\pi)^{2n}(-1)^n}{4n(2n-1)![1-4^n]}V_{2n} = \frac{4^{n-2}(2\pi)^{2n}(-1)^n}{n(2n-1)![1-4^n]}V_{2n}. \end{aligned}$$

Using (52), we get a bound for the U-Genocchi numbers as follows:

$$\frac{(4^n-1)(2n)!}{n2^{2n-1}(2\pi)^{2n}} \leq |A_{2n-1}| \leq \frac{2(4^n-1)(2n)!\pi^2}{3n(4\pi)^{2n}}$$

$$\begin{aligned}\Leftrightarrow\ \frac{(4^n-1)(2n)!}{n2^{2n-1}(2\pi)^{2n}} &\leq \left|\frac{V_{2n}}{2n}\right| \leq \frac{2(4^n-1)(2n)!\pi^2}{3n(4\pi)^{2n}}\\ \Leftrightarrow\ \frac{2(4^n-1)(2n)!}{2^{2n}2^{-1}(2\pi)^{2n}} &\leq |V_{2n}| \leq \frac{4(4^n-1)(2n)!\pi^2}{3(4\pi)^{2n}}\\ \Leftrightarrow\ \frac{4(4^n-1)(2n)!}{2^{2n}(4\pi)^{2n}} &\leq |V_{2n}| \leq \frac{4(4^n-1)(2n)!\pi^2}{3(4\pi)^{2n}}.\end{aligned}$$

Proposition 5.2 is completely proved.

□

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