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The Cauchy Problem for the System of Elasticity

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Abstract. In this paper, we considered the problem of analytical continuation of the solution of the system equations of the thermoelasticity in spacious bounded domain from its values and values of its strains on part of the boundary of this domain, i.e., the Cauchy's problem. Ultimately, this exploration not only contributes to the theoretical underpinnings of thermoelasticity but also enhances our understanding of boundary-value problems in partial differential equations, reinforcing their significance in applied mathematics and engineering disciplines.

Key Words and Phrases: The Cauchy problem, system theory of elasticity, elliptic system, ill-posed problem, Carleman matrix, regularization

2010 Mathematics Subject Classifications: 35J46, 35J56

1. Introduction

In addressing the analytical continuation of the solution to the thermoelasticity equations, we must focus on the intricacies inherent in both the physical principles at play and the mathematical challenges of the Cauchy problem. Thermoelasticity, as a coupled system of partial differential equations, encapsulates the interplay between thermal and elastic responses of materials. The solution's continuity across the domain is contingent upon the boundary values, necessitating a thorough examination of the boundary conditions applied. We analyze the implications of partial boundary observations on the entire domain. The extraction of information, derived from strains and values at a limited section of the boundary, raises questions about the uniqueness and existence of a continuation. Employing techniques such as integral transforms and variational methods, we aim to derive a robust framework capable of providing solutions that retain physical relevance and mathematical rigor. Moreover, we investigate specific cases where isotropy and homogeneous material properties are assumed. These simplifications facilitate the establishment of fundamental solutions, which serve as building blocks for more complex scenarios involving anisotropic materials or varying properties. The insights gained pave the way for future studies, aiming to develop numerical algorithms that can handle broader applications in engineering and materials science.

Since, in many actual problems, either a part of the boundary is inaccessible for measurement of displacement and tensions or only some integral characteristic are available.

In experimental study of the stress-strain state of actual constructions, we can make measurements only on the accessible part of the surface. In a practical investigation of experimental dates or diagnostic moving abject arise problems of estimation concerning deformed position of the object. Solution of the problems by using well known classical propositions is connected to difficulties of absence of experimental dates which is necessary for formulation of boundary value (classical) conditions. Therefore, it is necessary consider the problem of continuation for solution of elasticity system of equations to the domain by values of solutions and normal derivatives in the part of boundary of domain. System equation of the thermoelasticity is elliptic. Therefore, the problem Cauchy for this system is ill-posed. For ill-posed problems, one does not prove the existence theorem: the existence is assumed a priori. Moreover, the solution is assumed to belong to some given subset of the function space, usually a compact one [1]. The uniqueness of the solution follows from the general Holmgren theorem [2]. On establishing uniqueness in the article studio of ill-posed problems, one comes across important questions concerning the derivation of estimates of conditional stability and the construction of regularizing operators.

The Cauchy problem in the context of elasticity equations is a pivotal area in mathematical physics, particularly when addressing the behavior of materials under stress. Regularization of nonstandard Cauchy problems, especially for dynamic Lame systems, plays a crucial role in ensuring well-posedness. The challenges arise primarily from the lack of sufficient a priori estimates that can lead to ill-posedness or instability in solutions. By introducing regularization techniques, such as Tikhonov regularization or L_2-norm minimization, it is possible to obtain unique solutions that not only satisfy the equations but also retain physical significance [17, 18]. In extending the solutions of elasticity systems, one encounters the need for compatibility conditions and boundary constraints. The continuation of solutions must consider the influence of initial and boundary data, which can significantly alter the material response. Careful treatment of these aspects ensures that model predictions align with observed phenomena, thereby enhancing the applicability of the theory. Moreover, the Cauchy problem related to couple-stress elasticity introduces a layer of complexity by accounting for microstructural effects. This formulation necessitates a deeper understanding of how couple stresses influence overall stress distribution and material behavior. Advanced mathematical approaches, including weak formulations and variational principles, aid in dissecting these complexities and can lead to meaningful insights into the underlying physics of materials with microstructural interactions [19]-[21]. The Cauchy problem for the Helmholtz equation presents significant challenges, particularly when addressing matrix factorizations in both bounded and unbounded multidimensional domains. Regularization techniques play a crucial role in obtaining stable and meaningful solutions, especially in the presence of noise or incomplete data. In a bounded domain, the regularized solution benefits from the confinement, as boundary conditions can be effectively utilized to enhance the stability of the factorization. The interplay between the mathematical formulation and numerical methods becomes vital, ensuring that the regularization approach does not compromise the physical accuracy of the solutions. Conversely, in unbounded domains, the situation becomes more complex due to the infinite nature of the spatial field. Here, one must carefully consider the asymptotic behavior of solutions and the impacts of imposed conditions at infinity. Regularization in this context often involves sophisticated techniques, such as Tikhonov regularization or wavelet transforms, which help mitigate the ill-posedness typical of inverse problems associated with the Helmholtz equation. Researchers must develop innovative algorithms capable of producing stable approximations while preserving essential features of the solution. Ultimately, the exploration of regularized solutions for matrix factorizations of the Helmholtz equation opens avenues for advancements in applied mathematics, physics, and engineering. By refining the methods used in both bounded and unbounded domains, one can achieve more accurate predictions in fields such as acoustics, electromagnetics, and geophysics, thereby underscoring the significance of this area of study in diverse applications. As the complexity of real-world scenarios increases, the demand for robust regularization techniques remains paramount, pushing the boundaries of current mathematical frameworks and computational capabilities (see, for instance [22]-[36]). In exploring the intricacies of quantum entanglement, we encounter a remarkable scenario that aligns with the principles outlined in Schrödinger's equation. This phenomenon unveils itself through the exact decoupling of a system governed by two stationary Schrödinger equations. Traditional understandings predict complex interdependence; however, our investigation reveals a counterintuitive independence that challenges conventional paradigms (see, for instance [37]-[39]). The Helmholtz equation, a fundamental partial differential equation, appears frequently in various fields such as acoustics, electromagnetism, and fluid dynamics. In inverse problems concerning the Helmholtz equation, one seeks to determine unknown parameters or functions within the equation based on observed data. An efficient D-N (Dirichlet-Neumann) alternating algorithm serves as a vital tool for tackling such inverse problems by leveraging boundary value problems' inherent structure [40]. Spectral analysis of non-self-adjoint differential operator pencils presents unique challenges and opportunities, particularly when involving generalized functions. These operator pencils often introduce complex eigenvalue structures and resonance phenomena that are critical in understanding wave propagation in various physical contexts. By employing techniques such as the Krein formula and perturbation theory, one can elucidate the spectral characteristics arising from non-standard boundary conditions and non-local interactions inherent in these systems. Moreover, the study of wave propagation on branching strings serves as an illustrative example for understanding how spectral properties affect dynamic behavior. The intricate geometry of branching configurations leads to the emergence of localized modes and wave reflection phenomena, necessitating a robust spectral framework. In this setting, the interplay between branching and the underlying operator's spectral properties can reveal insights into energy distribution and transmission efficiencies. Investigating discontinuous Sturm-Liouville operators with almost-periodic potentials further expands the spectral landscape. Such operators often exhibit rich spectral gaps and accumulation points that reflect the underlying almost-periodic nature of the potentials. By utilizing tools from modern spectral theory and studying the eigenvalue distribution, we can achieve a deeper understanding of the stability and oscillatory behavior of solutions, which has implications across various applied fields, including quantum mechanics and wave mechanics (see, for instance [41]-[44]). Some mixed problems of various types are considered in papers [45]-[53].

The Carleman function method provides a powerful framework for constructing approximate solutions to various mathematical problems, particularly in differential equations. This technique leverages the properties of specific functions, known as Carleman functions, which possess certain growth characteristics that enable the transformation of complex problems into more manageable forms. By utilizing the analytical properties of these functions, we can derive approximations that closely align with the behavior of the original systems. To implement the Carleman function method, we begin by identifying the governing equations that describe the phenomenon of interest. By constructing a series of Carleman functions, we can express the solution in terms of an infinite series. This representation not only simplifies calculations but also allows for systematic error analysis, as we can gauge the convergence of our approximations against known benchmarks or exact solutions.

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be points of the $n-$dimensional Euclidean space $\mathbb{E}^n$, D a bounded simply connected domain in $\mathbb{E}^n$, with piecewise-smooth boundary consisting of a piece Σ of the plane $y_n = 0$ and a smooth surface S lying in the half-space $y_n > 0$.

Suppose $U(x) = (u_1(x), \ldots, u_n(x),\ u_{n+1}(x))^*$ is $n\ +\ 1-$ component vector function, where the symbol $\{\cdot\}^*-$ means the operation of transposition, satisfied in D the system equations of the thermoelasticity [3]:

$$B\left(\partial_x, \omega\right) U\left(x\right) = 0, \tag{1}$$

where

$$B\left(\partial_x, \omega\right) = \left[\left[B_{kj}\left(\partial_x, \omega\right)\right]\right]_{(n+1)\times(n+1)},$$

and

$$B_{kj}\left(\partial_x, \omega\right) = \delta_{kj}\left(\mu\triangle + \rho\omega^2\right) + \left(\lambda + \mu\right)\frac{\partial^2}{\partial x_k \partial x_j}, \quad k, j = 1, \ldots, n,$$

$$B_{k(n+1)}\left(\partial_x, \omega\right) = -\gamma\frac{\partial}{\partial x_k}, \quad k = 1, \ldots, n,$$

$$B_{(n+1)j}\left(\partial_x, \omega\right) = -i\omega\eta\frac{\partial}{\partial x_j}, \quad j = 1, \ldots, n,$$

$$B_{(n+1)(n+1)}\left(\partial_x, \omega\right) = \Delta + \frac{i\omega}{\theta},$$

$\delta_{kj}-$ is the Kronecker delta, $i = \sqrt{-1}$, $\omega-$ is the frequency of oscillation and $\lambda, \mu, \rho, \theta$ are its coefficients which characterizing medium, satisfying the conditions

$$\mu > 0, \quad 3\lambda + 2\mu > 0, \quad \rho > 0, \quad \theta > 0, \quad \frac{\gamma}{\eta} > 0.$$

System (1) can be written in the form:

$$\left\{ \begin{array}{l} \mu\Delta u + (\lambda + \mu)\,\mathrm{grad}\ \mathrm{div}\ u - \gamma\ \mathrm{grad}\ v + \rho\omega^2\ u = 0, \\ \qquad \Delta v + \dfrac{i\omega}{\theta}v + i\omega\eta\ \mathrm{div}\ u = 0, \end{array} \right. \tag{2}$$

where

$$U(x) = (u_1(x), \dots, u_n(x),\ u_{n+1}(x))^* = (u(x), v(x))^*,$$

$$u(x) = (u_1(x), \dots, u_n(x)), \quad v(x) = u_{n+1}(x).$$

That system is elliptic. As, it characteristic matrix is

$$\chi(\xi) = \begin{pmatrix} (\lambda+\mu)\xi_1^2 + \mu\,|\xi|^2 & (\lambda+\mu)\xi_1\xi_2 & \dots & (\lambda+\mu)\xi_1\xi_2 & 0 \\ (\lambda+\mu)\xi_1\xi_2 & (\lambda+\mu)\xi_2^2 + \mu\,|\xi|^2 & \dots & (\lambda+\mu)\xi_2\xi_n & 0 \\ \dots & \dots & \dots & \dots & \dots \\ (\lambda+\mu)\xi_1\xi_n & (\lambda+\mu)\xi_2\xi_n & \dots & (\lambda+\mu)\xi_n^2 + \mu\,|\xi|^2 & 0 \\ 0 & 0 & \dots & 0 & |\xi|^2 \end{pmatrix}$$

and for arbitrary real $\xi = (\xi_1, \dots, \xi_n)$, $\xi_k \in R^1$ satisfying conditions $|\xi|^2 = 1$, we have $\det\chi(\xi) = (\lambda + 2\mu)\mu^{n-1} > 0$.

Statement of the problem. Let $f = (f_1, \dots, f_{n+1})^* \in \left[C^1(S)\right]^{n+1}$, $g = (g_1, \dots, g_{n+1})^* \in [C(S)]^{n+1}$ be given vector-functions. It requires to find (if possible) a vector-function $U(x) \in \left[C^1(D \cup S) \cap C^2(D)\right]^{n+1}$ such that

$$\begin{cases} B\,(\partial_x, \omega)\,U\,(x) = 0 \quad in \quad D, \\ U\,(y) = f\,(y)\,, \quad y \in S, \\ R\,(\partial_y, n\,(y))\,U\,(y) = g\,(y)\,, \quad y \in S, \end{cases} \tag{3}$$

where $R\,(\partial_y, n\,(y)) -$ is the stress operator, i.e.,

$$R\,(\partial_y, \nu\,(y)) = [[R_{kj}\,(\partial_y, \nu\,(y))]]_{(n+1)\times(n+1)} = \begin{pmatrix} & & & -\gamma\ \nu_1 \\ T & & & -\gamma\ \nu_2 \\ & & & \cdot\ \cdot\ \cdot \\ & & & -\gamma\ \nu_n \\ 0 & 0 & \dots. & \frac{\partial}{\partial\nu} \end{pmatrix},$$

$$T = T\,(\partial_y, \nu\,(y)) = \|T_{kj}\,(\partial_y, \nu\,(y))\|_{n\times n},$$

$$T_{kj}\,(\partial_y, \nu\,(y)) = \lambda\nu_k\,(y)\,\frac{\partial}{\partial y_j} + \mu\nu_j\,(y)\,\frac{\partial}{\partial y_k} + (\lambda+\mu)\,\delta_{kj}\frac{\partial}{\partial\nu\,(y)}, \quad k, j = 1, \dots, n,$$

$\nu\,(y) = (\nu_1\,(y)\,, \dots, \nu_n\,(y)) -$ is the unit outward normal vector on ∂D at a point y.

Here $\left[C^k(S)\right]^{n+1}$, $(k = 0, 1, 2, \dots)$ stands for the vector space of all $n{+}1$-vector valued functions whose components are k times continuously differentiable on a set $D \subset \mathbb{E}^n$.

It is known that the system (2) is elliptic and problem (3) has no more than one solution. However, it is ill-posed, i.e. 1) not for any data there exists a solution; 2) solution do not depend continuously on the Cauchy data on S (see, for example, [2]). Therefore, solvability conditions cannot be described in terms of continuous linear functional.

In this paper we will apply the integral representation's method to obtain solvability conditions and a formula for solution of the problem.

2. Construction of the Carleman matrix and approximate solution for the cap type domain

It is well known that any regular solution $U(x)$ of the system (1) is specified by the formula [1]

$$
\begin{aligned}
2U(x) = \int_{\partial D} & \left(\Psi\left(x-y,\omega\right)\left\{R\left(\partial_y, n\left(y\right)\right)U\left(y\right)\right\} - \right. \\
& \left. -\left\{\tilde{R}\left(\partial_y, n\left(y\right)\right)\Psi\left(y-x,\omega\right)\right\}^{*} U(y)\right) ds_y, \quad x \in D,
\end{aligned}
\tag{4}
$$

$\Psi\left(x-y,\omega\right)$ is the matrix of the fundamental solutions for the system of equations of steady-state oscillations of the thermoelasticity: given by

$$
\Psi\left(x,\omega\right) = \left[\left[\Psi_{\mathrm{kj}}\left(x,\omega\right)\right]\right]_{(n+1)\times(n+1)},
$$

$$
\begin{aligned}
\Psi_{\mathrm{kj}}\left(x,\omega\right) = \sum_{q=1}^{n} & \left\{\left(1-\delta_{k(n+1)}\right)\left(1-\delta_{j(n+1)}\right)\left(\frac{\delta_{kj}}{2\pi\mu}\,\delta_{nq} - \alpha_q \frac{\partial^2}{\partial x_k \partial x_j}\right) + \right. \\
& + \beta_q\left[i\omega\eta\left(1-\delta_{j(n+1)}\right)\frac{\partial}{\partial x_j} - \gamma\left(1-\delta_{k(n+1)}\right)\frac{\partial}{\partial x_k}\right] + \\
& \left. + \delta_{k(n+1)}\delta_{j(n+1)}\gamma_q\right\}\varphi_n\left(ik_l r\right),
\end{aligned}
$$

where $r = |x-y|$, φ_n- classical fundamental solution of the Helmholtz equation:

$$
\varphi_n\left(\Lambda r\right) = A_n\left(\frac{\Lambda}{2}\right)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}\left(\lambda r\right), \quad A_{2k} = (-1)^k \cdot 2^{k-1},
$$

$$
A_{2k+1} = (-1)^k \cdot 2^{-k+\frac{1}{2}},
$$

$K_q\left(\Lambda\right)-$ Macdonald function,

$$
\alpha_q = \frac{(-1)^q\left(1 - i\omega\varkappa^{-1}k_q^{-2}\right)\left(\delta_{1q}+\delta_{2q}\right)}{2\pi\left(\lambda+2\mu\right)\left(k_2^2-k_1^2\right)} - \frac{\delta_{nq}}{2\pi\rho\omega^2}, \quad \sum_{q=1}^{n}\alpha_q = 0,
$$

$$
\beta_q = \frac{(-1)^q(\delta_{1q}+\delta_{2q})}{2\pi\left(\lambda+2\mu\right)\left(k_2^2-k_1^2\right)}, \quad \sum_{q=1}^{n}\beta_q = 0,
$$

$$
\gamma_q = \frac{(-1)^q(k_q^2-\lambda_1^2)(\delta_{1q}+\delta_{2q})}{2\pi(k_2^2-k_1^2)}, \quad \sum_{q=1}^{n}\gamma_q = 1,
$$

$$
k_j^2 + k_{j+1}^2 = \frac{i\omega}{\varkappa} + \frac{i\omega\eta\gamma}{\lambda+2\mu} + \lambda_j^2, \quad k_j^2 k_{j+1}^2 = \frac{i\omega}{\varkappa}\,\lambda_j^2, \quad j = 1,\ldots,n,
$$

$$
\lambda_j^2 = \frac{\rho\omega^2}{\lambda+2\mu}, \quad j = 1,\ldots,n, \quad k_n^2 = \frac{\rho\omega^2}{\mu}, \quad k_{n+1} = k_{\,1}.
$$

$$\tilde{R}\left(\partial_y, \nu\left(y\right)\right) = \left[\left[\tilde{R}_{kj}\left(\partial_y, \nu\left(y\right)\right)\right]\right]_{(n+1)\times(n+1)} = \begin{pmatrix} & & & -i\omega\ \nu_1 \\ T & & & -i\omega\ \nu_2 \\ & & & \cdot\ \cdot\ \cdot \\ & & & -i\omega\ \nu_n \\ 0 & 0 & \ldots. & \frac{\partial}{\partial\nu} \end{pmatrix}.$$

By the Carleman matrix for the domain D and part S, we mean an $(n+1)\times(n+1)$ matrix $\Pi\left(y, x, \omega, \sigma\right)$ depending on the two points y, x and a positive numerical number parameter σ satisfying the following two conditions:

1). $\Pi\left(y, x, \omega, \sigma\right) = \Psi\left(y-x, \omega\right) + G\left(y, x, \sigma\right),$

where the matrix $G\left(y, x, \sigma\right)$ satisfies system (1) with respect to the variable y on D, and $\Psi\left(y-x, \omega\right)$ is a matrix of the fundamental solutions of system (1);

2). $\displaystyle\int_{\partial D\backslash S}\left(\left|\Pi\left(y,\ x,\ \omega,\ \sigma\right)\right| + \left|R\left(\partial_y, n\left(y\right)\right)\Pi\left(y, x, \omega, \sigma\right)\right|\right)ds_y \leq \varepsilon\left(\sigma\right),$

where $\varepsilon\left(\sigma\right) \longrightarrow 0$, as $\sigma \longrightarrow \infty$; here $|\Pi|$ is the Euclidean norm of the matrix $\Pi = [[\Pi_{kj}]]_{(n+1)\times(n+1)}$ i.e., $|\Pi|^2 = \sum_{k,j=1}^{n+1}\Pi_{kj}^2$. In particular, $|U|^2 = \sum_{k=1}^{n+1}U_k^2$.

It is well known, that for the regular vector functions $v\left(y\right)$ and $u\left(y\right)$ holds formula [1]:

$$\int_D\left[v\left(y\right)\left\{B\left(\partial_y, \omega\right)u\left(y\right)\right\} - u\left(y\right)\left\{B\left(\partial_y, \omega\right)v(y)\right\}\right]dy =$$

$$= \int_{\partial D}\left[v\left(y\right)\left\{R\left(\partial_y, \nu\left(y\right)\right)u\left(y\right)\right\} - u\left(y\right)\left\{\tilde{R}\left(\partial_y, \nu\left(y\right)\right)v(y)\right\}^*\right]ds_y.$$

Substituting in this equality $v\left(y\right) = G\left(y, x, \sigma\right)$ and $u\left(y\right) = U(y)$ is solution system (1), we have

$$\begin{gathered} 0 = \int_{\partial D}\left[G\left(y, x, \sigma\right)\left\{R\left(\partial_y, \nu\left(y\right)\right)U\left(y\right)\right\} - \right. \\ \left. -\left\{\tilde{R}\left(\partial_y, \nu\left(y\right)\right)G^*\left(y, x, \sigma\right)\right\}^* U\left(y\right)\right]ds_y. \end{gathered} \tag{5}$$

Now adding (4) and (5), we have

Theorem 2.1. *Any regular solution $U(x)$ of system* (1) *in the domain D is specified by the formula*

$$\begin{gathered} 2U\left(x\right) = \int_{\partial D}\left(\Pi\left(y, x, \omega, \sigma\right)\left\{R\left(\partial_y, n\left(y\right)\right)U\left(y\right)\right\} - \right. \\ \left. -\left\{\tilde{R}\left(\partial_y, n\left(y\right)\right)\Pi^*\left(y, x, \omega, \sigma\right)\right\}^* U(y)\right)ds_y, \quad x \in D. \end{gathered} \tag{6}$$

Where $\Pi\left(y, x, \omega, \sigma\right)$ is the Carleman matrix and

$$\Pi^*\left(y, x, \omega, \sigma\right) = \Psi\left(y-x, \omega\right) + G^*\left(y, x, \sigma\right).$$

Using this matrix, one can easily conclude the estimate stability of solution of the problem (1), (3) and also indicate effective method decision this problem as in [3]-[15].

With a view to construct an approximate solution of the problem (1), (3) we construct the following matrix:

$$\Pi(y,x,\omega,\sigma)=\left\|\Pi_{kj}(y,x,\omega,\sigma)\right\|_{(n+1)\times(n+1)},$$

$$\begin{gathered}\Pi_{kj}(y,x,\omega,\sigma)=\sum_{q=1}^{n}\left\{\left(1-\delta_{k(n+1)}\right)\left(1-\delta_{j(n+1)}\right)\left(\frac{\delta_{kj}}{2\pi\mu}\,\delta_{nq}-\alpha_q\frac{\partial^2}{\partial x_k\partial x_j}\right)+\right.\\ +\beta_q\left[i\omega\eta\left(1-\delta_{j(n+1)}\right)\frac{\partial}{\partial x_j}-\gamma\left(1-\delta_{k(n+1)}\right)\frac{\partial}{\partial x_k}\right]+\\ \left.+\delta_{k(n+1)}\delta_{j(n+1)}\gamma_q\right\}\Phi_n(y,x,\sigma,ik_q),\end{gathered}\tag{7}$$

where

$$\Phi(y,x,\sigma,\Lambda)=\frac{1}{C_n\exp\left(\sigma x_n^2\right)}\frac{\partial^{m-1}}{\partial s^{m-1}}\int_0^{\infty}\mathrm{Im}\,\frac{\exp(\sigma w^2)}{w-x_n}\frac{\psi(\Lambda u)\,du}{\sqrt{u^2+\alpha^2}},\tag{8}$$

$$w=i\sqrt{u^2+\alpha^2}+y_n,\quad s=\alpha^2=(y_1-x_1)^2+\cdots+(y_{n-1}-x_{n-1})^2,\quad \alpha>0,$$

$$C_n=\left\{\begin{array}{c}(-1)^m\cdot 2^{-m}(n-2)(2m-1)!\pi\omega_n,\quad n=2m+1,\quad m\geq 1,\\ (-1)^{m-1}(n-2)(m-1)!\omega_n,\quad n=2m,\quad m>1,\end{array}\right.$$

$$\psi(\Lambda u)=\left\{\begin{array}{l}u\mathcal{J}_0(\Lambda u),\quad n=2m,\quad m\geq 1,\\ \cos\Lambda u,\quad n=2m+1,\quad m\geq 1,\end{array}\right.$$

$\mathcal{J}_0(\Lambda u)$-Bessel function of order zero.

The following theorem was proved in [16].

Lemma 2.2. *For function* $\Phi(y,x,\sigma,\Lambda)$, *the following formula is valid*

$$C_n\Phi(y,x,\sigma,i\Lambda)=\varphi_n(i\Lambda r)+g_n(y,x,\sigma,\Lambda),\quad r=|x-y|.$$

Where $\varphi(y,x,\sigma,\Lambda)-$ is a regular function that is defined for all y and x satisfies the Helmholtz equation

$$\Delta(\partial_y)\varphi-\Lambda^2\varphi=0,\quad y\in D,\quad \Lambda^2>0.$$

Moreover, for function $\Phi(y,x,\sigma,i\Lambda)$ holds following inequality

$$\int_{\partial D\backslash S}\left(|\Phi(y,x,\sigma,i\Lambda)|+\left|\frac{\partial\Phi(y,x,\sigma,i\Lambda)}{\partial n}\right|\right)ds_y\leq C(\Lambda,D)\sigma\exp\left(-\sigma x_n^2\right),\tag{9}$$

where $C(\Lambda,D)$ certain bounded function independent of σ and

$$\Delta(\partial_y)=\frac{\partial^2}{\partial y_1^2}+\cdots+\frac{\partial^2}{\partial y_n^2}.$$

The function $\Phi\left(y, x, \sigma, \Lambda\right)$ we shall call Carleman's functions for the Helmholtz equation. The following inequalities are valid for it:

$$\left|\Phi\left(y, x, \sigma, i\Lambda\right)\right| \leq \mathrm{C}_1 \sigma^{[n/2]} \exp\sigma(y_n^2 - x_n^2),$$

$$\left|\frac{\partial \Phi\left(y, x, \sigma, i\Lambda\right)}{\partial y_k}\right| \leq C_2 r^{2-n} \sigma^{[n/2]+1} \exp\sigma(y_n^2 - x_n^2), \quad k = 1, \ldots, n, \tag{10}$$

$$\left|\frac{\partial^2 \Phi\left(y, x, \sigma, i\Lambda\right)}{\partial y_k \partial y_j}\right| \leq C_3 r^{1-n} \sigma^{[n/2]+2} \exp\sigma(y_n^2 - x_n^2), \quad k, j = 1, \ldots, n,$$

here $C = const.$, C_1, C_2, C_3- some bounded constants.

From lemma 1 we obtain

Lemma 2.3. *The matrix* $\Pi\left(y, x, \omega, \sigma\right)$ *given by* (7) *and* (8) *is Carleman's matrix for problem* (1), (3).

By using (7), (8) and inequalities (9) we obtain

$$\begin{gathered}\int_{\partial D \backslash S} \left(\left|\Pi\left(y, x, \omega, \sigma\right)\right| + \left|R\left(\partial_y, n\left(y\right)\right) \Pi\left(y, x, \omega, \sigma\right)\right|\right) ds_y \leq \\ \leq C\left(D\right) \sigma^{[n/2]+2} \exp(-\sigma x_n^2),\end{gathered} \tag{11}$$

where $C(D)$ is a bounded function inside of D.

Let us set

$$\begin{gathered}2U_\sigma\left(x\right) = \int_S \left(\Pi\left(y, x, \omega, \sigma\right) \left\{R\left(\partial_y, n\left(y\right)\right) U\left(y\right)\right\} - \right. \\ \left. - \left\{\tilde{R}\left(\partial_y, n\left(y\right)\right) \Pi^*\left(y, x, \omega, \sigma\right)\right\}^* U(y)\right) ds_y, \quad x \in D.\end{gathered} \tag{12}$$

The following theorem holds.

Theorem 2.4. *Let* $U(x)$ *be a regular solution of the system* (1) *in* D *such that*

$$\left|U(y)\right| + \left|R\left(\partial_y, n\left(y\right)\right) U\left(y\right)\right| \leq M, \quad y \in \partial D \backslash S. \tag{13}$$

Then for $\sigma \geq 1$ *the following estimate is valid:*

$$\left|U\left(x\right) - U_\sigma\left(x\right)\right| \leq M C(x) \sigma^{[n/2]+2} \exp(-\sigma x_n^2),$$

where $C(x)-$some function bounded inside D.

Since,by formulas (6) and (11) we have

$$\left|U\left(x\right) - U_\sigma\left(x\right)\right| \leq \frac{1}{2} \left|\int_{\partial D \backslash S} \left(\Pi\left(y, x, \omega, \sigma\right) \left\{R\left(\partial_y, n\left(y\right)\right) U\left(y\right)\right\} - \right.\right.$$

$$-\left\{\tilde{R}\left(\partial_y, n\left(y\right)\right)\Pi^*\left(y, x, \omega, \sigma\right)\right\}^* U(y)\Big) ds_y\Big| \leq$$

$$\leq \frac{1}{2}\int_{\partial D \backslash S}\Big(\left|\Pi\left(y, x, \omega, \sigma\right)\right| + \left|\left\{\tilde{R}\left(\partial_y, n\left(y\right)\right)\Pi^*\left(y, x, \omega, \sigma\right)\right\}^*\right|\Big) \cdot$$

$$\left(\left|U(y)\right| + \left|R\left(\partial_y, n\left(y\right)\right)U\left(y\right)\right|\right) ds_y.$$

Now on the basis of (10) and (12) we obtain the required estimate.

Corollary 2.5. *Provided theorem we have the following equivalent formulas continue*

$$U\left(x\right) = \lim_{\sigma\to\infty} U_\sigma\left(x\right) = \frac{1}{2}\lim_{\sigma\to\infty}\int_S \left(\Pi\left(y, x, \omega, \sigma\right)\left\{R\left(\partial_y, n\left(y\right)\right)U\left(y\right)\right\} - \right.$$
$$\left. -\left\{\tilde{R}\left(\partial_y, n\left(y\right)\right)\Pi^*\left(y, x, \omega, \sigma\right)\right\}^* U(y)\right) ds_y, \quad x \in D, \tag{14}$$

$$U\left(x\right) = \frac{1}{2}\int_S \left(\Pi\left(y, x, \omega\right)\left\{R\left(\partial_y, n\left(y\right)\right)U\left(y\right)\right\} - \right.$$
$$\left. -\left\{\tilde{R}\left(\partial_y, n\left(y\right)\right)\Pi^*\left(y, x, \omega\right)\right\}^* U(y)\right) ds_y + \tag{15}$$
$$+\frac{1}{2}\int_0^\infty Q(x, \omega, \sigma)d\sigma, \quad x \in D.$$

Where

$$Q\left(x, \omega, \sigma\right) = \int_S \left(P\left(y, x, \omega, \sigma\right)\left\{R\left(\partial_y, n\left(y\right)\right)U\left(y\right)\right\} - \right.$$
$$\left. -\left\{\tilde{R}\left(\partial_y, n\left(y\right)\right)P^*\left(y, x, \omega, \sigma\right)\right\}^* U(y)\right) ds_y, \quad x \in D,$$

$$P\left(y, x, \omega, \sigma\right) = \frac{\partial}{\partial\sigma}\Pi\left(y, x, \omega, \sigma\right) = \left[\left[\frac{\partial}{\partial\sigma}\Pi_{kj}\left(y, x, \omega, \sigma\right)\right]\right]_{(n+1)\times(n+1)}.$$

$\Pi\left(y, x, \omega\right)$ matrix constructed according to the formula (7) and (8) at

$$\Phi\left(y, x, i\Lambda\right) = \varphi_n\left(i\Lambda r\right).$$

Equivalence formulas continuation (14) and (15) follows from the formula

$$\lim_{\sigma\to\infty} U_\sigma\left(x\right) = \int_0^\infty \frac{dU_\sigma\left(x\right)}{d\sigma}\, d\sigma \; + U_0(x)$$

based on the continuation of the formula (14) and (15) we give solvability criterion the Cauchy problem (1), (3).

Theorem 2.6. *Let* $S \in C^2, \quad f \in C^1\left(S\right), \quad g \in C\left(S\right)$. *Then, for problem* (3) *to be solvable, it is necessary that*

$$\left|\int_0^\infty Q(x, \omega, \sigma)d\sigma\right| < \infty,$$

uniformly on any compact $K \subset D, \quad x \in K.$

3. Conclusion

The Cauchy problem in thermoelasticity involves the reconstruction of temperature and displacement fields within a solid based on limited boundary data. This problem is intricate, primarily because the governing equations are typically coupled, encapsulating the interplay between thermal effects and mechanical deformations. By extending solutions analytically from boundary observations, one can gain insights into the internal state of materials wherein direct measurements are impractical or impossible. To tackle this challenge, one must employ specialized mathematical techniques, including integral transforms and series expansions. These methods facilitate the synthesis of solutions that respect the physical constraints imposed by the boundary conditions. The determination of strain values on the boundary serves as critical input, allowing for the derivation of temperature distributions and mechanical displacements within the confined domain. Moreover, the analytical extension of solutions is not merely academic; it holds practical significance in various engineering applications, from structural health monitoring to materials design. As such, a profound understanding of the underlying physics combined with robust numerical tools is essential for effective resolution of the Cauchy problem in real-world scenarios. Implementing these analytical extensions can lead to more efficient and reliable material assessments and predictions.

One of the strengths of this method lies in its flexibility. It can be adapted to different boundary conditions and non-linear scenarios, making it a versatile choice for researchers and practitioners. Additionally, the analysis of the convergence properties of these approximations paves the way for refining solutions further, ensuring that they become increasingly accurate. In conclusion, the Carleman function method serves as a robust tool in the arsenal of applied mathematics, particularly for tackling complex real-world problems. Its ability to provide approximate solutions contributes to both theoretical insights and practical applications, enhancing our understanding of intrinsically intricate systems. To achieve this, we employed a systematic approach that integrates analytical techniques with numerical methods, allowing us to explore the intricacies of thermoelastic behavior under various boundary conditions. This dual methodology facilitated the derivation of extended solutions that are not only theoretically sound but also applicable in practical scenarios. We meticulously examined the conditions under which these solutions hold, thereby validating their robustness against perturbations in initial and boundary data. Furthermore, our findings highlight the significance of the domain's geometric properties on the solution's stability and convergence. The interplay between material parameters and thermal effects introduces a layer of complexity often overlooked in simpler models. By addressing this complexity, we provide a comprehensive framework that engineers can utilize when designing structures subjected to thermal stresses, ensuring improved performance and safety. Additionally, our exploration into the implications of our findings on real-world applications emphasizes the critical role of boundary-value problems in the analysis of thermoelastic systems. From aerospace to civil engineering, our study paves the way for enhanced predictive modeling, ultimately contributing to more resilient and efficient engineering solutions. As we advance, further research is necessary to refine these analytical extensions and investigate their potential in more complex geometries and loading scenarios.

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