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On the Dynamics of a Quasi-Strictly Non-Volterra Cubic Stochastic Operator

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Abstract. In this study, we examine cubic stochastic operators, which we will refer to as quasi-strictly non-Volterra cubic operators. Firstly, the definition of a quasi-strictly non-Volterra operator is provided, and the structure of an arbitrary quasi-strictly non-Volterra cubic operator on a two-dimensional simplex S^2 is described. Secondly, the fixed and limit points of the quasi-strictly non-Volterra operator on S^2 are investigated. It is proven that there exists a repelling unique fixed point.
Key Words and Phrases: Cubic operator, simplex, fixed point, invariant set, Lyapunov function, trajectory.
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1. Introduction

One of the primary tasks in studying a dynamical system is to examine the evolution of the system's state. Typically, the "successors" of a system's state are determined by a certain law. Quadratic stochastic operators are often used to solve problems arising in mathematical genetics. These operators attract the attention of specialists in various areas of mathematics and its applications (see [3], [4], [5], [6], [8], [9], [12], [20]).

In recent years, cubic stochastic operators have begun to be studied, which differ from quadratic operators. For the motivation to study such operators, see [1], [7], [10], [11], [13], [14], [15], [16], [17], [18], [21], [22], [23], [24], [25], and the literature therein.

In this study, we examine cubic stochastic operators, which we will call quasi-strictly non-Volterra cubic operators. The structure of the work is as follows:

1. The definition of a quasi-strictly non-Volterra operator is given, and the structure of an arbitrary quasi-strictly non-Volterra cubic operator on the two-dimensional simplex S^2 is described.

2. The fixed and limit points of the quasi-strictly non-Volterra operator on S^2 are studied. It is proven that there exists a repelling unique fixed point.

The set

$$S^{n-1} = \{x = (x_1, x_2, x_3, ..., x_n) \in R^n : x_i \geq 0, i = \overline{1,n}, \sum_{i=1}^{n} x_i = 1\}$$

is called an $(n-1)$-dimensional simplex.

A cubic stochastic operator defined on a finite-dimensional simplex $W : S^{n-1} \to S^{n-1}$ has the form:

$$(Wx)_l = x'_l = \sum_{i,j,k=1}^{n} P_{ijk,l} x_i x_j x_k, \quad l = 1, 2, ..., n, \tag{1}$$

where

$$P_{ijk,l} = P_{ikj,l} = ... = P_{kij,l} \geq 0, \quad \sum_{l=1}^{n} P_{ijk,l} = 1, \quad \forall i, j, k \in \{1, 2, ..., n\} \tag{2}$$

For a given $x^{(0)} \in S^{n-1}$, the trajectory $\{x^{(n)}\}, n = 0, 1, 2, ...$, under the action of the cubic stochastic operator (1) is defined as: $x^{(n+1)} = W(x^{(n)})$ for $n = 0, 1, 2, ...$

One of the main problems in mathematical biology is the study of the asymptotic behavior of trajectories. This problem has been completely solved for certain Volterra cubic stochastic operators (see [10], [14], [15]), which are defined by equations (2) along with the additional condition $P_{ijk,l} = 0$ for all $l \in \{i, j, k\}$.

A cubic stochastic operator is called strictly non-Volterra if:

$$P_{ijk,l} = 0, \quad \forall\, l \in \{i, j, k\}. \tag{3}$$

Definition 1. *A cubic stochastic operator W, defined on S^{n-1}, is called quasi-strictly non-Volterra if condition (3) fails only for $P_{iii,i}$ and $P_{ijk,l}$, $i \neq j \neq k$, i.e $P_{iii,i} \geq 0$ and $P_{ijk,l} \geq 0$, $i \neq j \neq k$.*

In this work, we will limit ourselves to studying quasi-strictly non-Volterra cubic stochastic operators defined on S^2. In this case, an arbitrary quasi-strictly non-Volterra cubic stochastic operator W has the form:

$$W : \begin{cases} x' = P_{111,1}x^3 + P_{222,1}y^3 + P_{333,1}z^3 + 3P_{223,1}y^2z + 3P_{233,1}yz^2 + 6P_{123,1}xyz, \\ y' = P_{222,2}y^3 + P_{111,2}x^3 + P_{333,2}z^3 + 3P_{133,2}xz^2 + 3P_{113,2}x^2z + 6P_{123,2}xyz, \\ z' = P_{111,3}x^3 + P_{222,3}y^3 + P_{333,3}z^3 + 3P_{112,3}x^2y + 3P_{122,3}xy^2 + 6P_{123,3}xyz. \end{cases} \tag{4}$$

Let us define: $P_{111,1} = \alpha_1$, $P_{111,2} = \alpha_2$, $P_{111,3} = \alpha_3$, $P_{222,1} = \beta_1$, $P_{222,2} = \beta_2$, $P_{222,3} = \beta_3$, $P_{333,1} = \gamma_1$, $P_{333,2} = \gamma_2$, $P_{333,3} = \gamma_3$, $P_{123,1} = P_{123,2} = P_{123,3}$. Substituting into (4), we obtain:

$$W : \begin{cases} x' = \alpha_1 x^3 + \beta_1 y^3 + \gamma_1 z^3 + 3y^2 z + 3yz^2 + 2xyz, \\ y' = \alpha_2 x^3 + \beta_2 y^3 + \gamma_2 z^3 + 3xz^2 + 3x^2 z + 2xyz, \\ z' = \alpha_3 x^3 + \beta_3 y^3 + \gamma_3 z^3 + 3x^2 y + 3xy^2 + 2xyz, \end{cases} \tag{5}$$

Let $\alpha_i, \beta_i, \gamma_i \geq 0$, $i = 1, 2, 3$, and $\sum\limits_{i=1}^{3} \alpha_i = \sum\limits_{i=1}^{3} \beta_i = \sum\limits_{i=1}^{3} \gamma_i = 1$

Assume the following:

$$\alpha_1 = \beta_1 = \beta_2 = \gamma_2 = \alpha_3 = \gamma_3 = 0, \quad \alpha_2 = \beta_3 = \gamma_1 = 1,$$

Then the operator (5) takes the form:

$$W : \begin{cases} x' = z^3 + 3y^2 z + 3yz^2 + 2xyz, \\ y' = x^3 + 3xz^2 + 3x^2 z + 2xyz, \\ z' = y^3 + 3x^2 y + 3xy^2 + 2xyz. \end{cases} \tag{6}$$

2. Fixed Points

A fixed point of operator (6) is a solution $\lambda = (x, y, z)$ of the equation $W(\lambda) = \lambda$, i.e., a solution of the system:

$$\begin{cases} z^3 + 3y^2 z + 3yz^2 + 2xyz = x, \\ x^3 + 3xz^2 + 3x^2 z + 2xyz = y, \\ y^3 + 3x^2 y + 3xy^2 + 2xyz = z, \end{cases} \tag{7}$$

We denote by $\mathrm{Fix}\,(W)$ the set of all fixed points of the operator W, i.e., $\mathrm{Fix}\,(W) = \{\lambda \in S^2 : W(\lambda) = \lambda\}$.

Define: $\mathrm{int}\, S^2 = \{(x, y, z) \in S^2 : xyz > 0\}$, $\partial S^2 = S^2 \backslash \mathrm{int}\, S^2$ and

$$C = \Big(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\Big).$$

Theorem 1. *For operator (6), the following holds:*

i) $Fix\,(W) \cap \partial S^2 = \emptyset$;

ii) $Fix\,(W) \cap int\, S^2 = \big\{C\big\}$.

Proof.

i) Suppose $(x, y, z) \in \partial S^2$. For definiteness, assume $x = 0$ (the cases $y = 0$ and $z = 0$ are analyzed similarly). Then, from (7), it is straightforward to obtain that $x = y = z = 0$ is a solution of the system. However, $(0, 0, 0) \notin S^2$. Thus, we conclude that:

$$\mathrm{Fix}\,(W) \cap \partial S^2 = \emptyset.$$

ii) Let $(x, y, z) \in \mathrm{int}\, S^2$. From the first and second equations in (7), we obtain

$$z^3 - x^3 + 3y^2 z - 3x^2 z + 3yz^2 - 3xz^2 = x - y$$

or

$$(z - x)(z^2 + xz + x^2) = (x - y)(1 + 3z(x + y) + 3z^2). \tag{8}$$

Similarly, from the first and third equations in (7), we obtain

$$(z - y)(z^2 + yz + y^2) = (x - z)(1 + 3y^2 + 3y(x + z)). \tag{9}$$

Furthermore, from the second and third equations in (7), we obtain

$$(x - y)(x^2 + xy + y^2) = (y - z)(1 + x^2 + 3x(y + z)). \tag{10}$$

For any $(x, y, z) \in \mathrm{int}\, S^2$ it is clear that,

$$z^2 + xz + x^2 > 0, \quad z^2 + yz + y^2 > 0, \quad x^2 + xy + y^2 > 0 \ \text{ and}$$

$$1 + 3z(x + y) + 3z^2 > 0, \quad 1 + 3y^2 + 3y(x + z), \quad 1 + x^2 + 3x(y + z) > 0.$$

Suppose that $z \geq x$ (respectively $z \leq x$) then from equation (8), (9) and (10), it follows that $x \geq y$ (respectively $x \leq y$) and $y \geq z$ (respectively $y \leq z$). That is, we obtain $z \geq x \geq y \geq z$ (respectively $z \leq x \leq y \leq z$). Therefore, the system of equations (7) has a unique solution $x = y = z = 1/3$.

Thus, $C(1/3, 1/3, 1/3)$ is the unique fixed point of operator (6).

Theorem is proven.

Definition 2. *(see [2]) If the Jacobian J of operator W at a fixed point λ does not have eigenvalues on the unit circle, then the point λ is called hyperbolic.*

Definition 3. *(see [2]) A hyperbolic fixed point λ is called:*

- *attracting if all absolute values of the eigenvalues of the Jacobian matrix $J(\lambda)$ are less than 1;*
- *repelling if all absolute values of the eigenvalues of the Jacobian matrix $J(\lambda)$ are greater than 1;*

- *saddle in all other cases.*

Definition 4. *A continuous function $\varphi\colon S^{m-1} \to R$ is called a Lyapunov function for a cubic operator W if the limit $\lim\limits_{n\to\infty} \varphi(\lambda^{(n)})$ exists for any $\lambda \in S^{m-1}$.*

To determine the type of fixed point for operator (6), we rewrite it in the form

$$W : \begin{cases} x' = 1 - 3x + 3x^2 - x^3 - y^3 + 2xy - 2x^2y - 2xy^2, \\ y' = x^3 - 3x^2 + x^2y + 3x + xy^2 - 4xy, \end{cases} \tag{11}$$

Here $(x, y) \in \{(x, y) : x, y \geq 0, \quad 0 \leq x + y \leq 1\}$ where x, y-are the first two coordinates of points in the simplex S^2.

The eigenvalues of the Jacobian matrix of operator (13) at the point C have the form

$$|\lambda_{1,2}| = \left| \frac{-7 \pm \sqrt{3}i}{6} \right| > 1$$

Thus, the point is repelling for operator (6).

3. ω-limit set

Let $\lambda^0 = (\lambda_1^0, \lambda_2^0, \lambda_3^0) \in S^2$ be the initial point and let $\lambda^{(n)}, n = 0, 1, 2, ...$- denote the trajectory of the point λ^0 under the action of operator (6), i.e.,

$$\lambda^{(n)} = (\lambda_1^{(n)}, \lambda_2^{(n)}, \lambda_3^{(n)}) = W(\lambda^{(n-1)}), n = 1, 2, ...; \quad \lambda^{(0)} = \lambda^0.$$

We denote by $\omega(\lambda^0)$ the $\omega-$limit set of the trajectory $\{\lambda^{(n)}, n = 0, 1, 2...\}$.

Since $\{\lambda^{(n)}\} \subset S^2$, and S^2 compact, we have $\omega(\lambda^0) \neq \emptyset$. Note that if $\omega(\lambda^0)$ contains a single point, the trajectory converges to this point ,which is also a fixed point of the operator W, given by formula (6).

Theorem 2. *For the cubic operator (6), the following holds:*

i) $W^3(e_1) = W^2(e_2) = W(e_3) = e_1$;

ii) For any $\lambda^0 \in \partial S^2$, the ω-limit set is $\omega(\lambda^0) = \{e_1, e_2, e_3\}$;

iii) The function $\varphi(\mathbf{x}) = xyz$ is a Lyapunov function;

iv) For any $\lambda^0 \in intS^2 \backslash \{C\}$, the ω-limit set of the trajectory lies on the boundary of the simplex.

Proof.

i) It is easy to verify that $W^3(e_1) = W^2(e_2) = W(e_3) = e_1$.

ii) Let $\lambda^0 \in \partial S^2$ and for definiteness, let $\lambda^0 = (0, \varepsilon, 1-\varepsilon) \in \partial S^2$, where $\varepsilon \in [0;1]$. Then from (6), it is easy to obtain that:

$$\lambda^{(n)} = \lambda^{(n)}(\varepsilon) = \begin{cases} (0, \varepsilon^{3^n}, 1-\varepsilon^{3^n}), & \text{if } n = 3k, \\ (1-\varepsilon^{3^n}, 0, \varepsilon^{3^n}), & \text{if } n = 3k+1, \\ (\varepsilon^{3^n}, 1-\varepsilon^{3^n}, 0), & \text{if } n = 3k+2, \end{cases} \quad k = 0,1,2,...$$

Thus, $\omega(\lambda^0) = \{e_1, e_2, e_3\}$ for $\lambda^0 \in \partial S^2$.

iii) For $\lambda = (x, y, z) \in S^2$ define:

$$\varphi(\lambda) = \varphi(x,y,z) = xyz, \quad \psi(\lambda) = 4 - 3(xy + yz + xz).$$

It is easy to verify that:

$$\max_{\lambda \in S^2} \psi(\lambda) = \psi(C) = 1, \quad C = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right). \tag{12}$$

Thus, we have:

$$\varphi(W(\lambda)) = \varphi(\lambda)\psi(\lambda) \leq \varphi(\lambda). \tag{13}$$

Hence, the function $\varphi(\lambda) = xyz$ is a Lyapunov function for operator (6).

iv) Now, let $\lambda^0 \in \operatorname{int} S^2 \setminus \{C\}$. In this case , we will prove that $x^{(n)}y^{(n)}z^{(n)} \to 0$ as $n \to \infty$. From (13), we obtain:

$$\varphi(\lambda^{(n+1)}) \leq \varphi(\lambda^{(n)}) \tag{14}$$

and from where we get $\lim\limits_{n\to\infty} \varphi(\lambda^{(n)}) = \mu$.

Let us prove that $\mu = 0$. Suppose $\mu > 0$, then:

$$1 = \lim_{n\to\infty} \frac{\varphi(\lambda^{(n+1)})}{\varphi(\lambda^{(n)})} = \lim_{n\to\infty} \psi(\lambda^{(n)}). \tag{15}$$

Using $\psi(\lambda) = 4 - 3(xy + yz + xz)$, from (15) we have :

$$\lim_{n\to\infty} \left(x^{(n)}y^{(n)} + y^{(n)}z^{(n)} + x^{(n)}z^{(n)}\right) = \frac{1}{3}.$$

Since the expression $xy + yz + xz$ achieves its conditional minimum of $1/3$ on the two- dimensional simplex only at the point C, it follows that:

$$\lim_{n\to\infty} \lambda^{(n)} = C.$$

Assume the opposite, i.e., suppose there exists a subsequence $\{n_k\}_{k=1,2,\dots}$ such that

$$\lim_{k\to\infty} \lambda^{(n_k)} = \nu \neq C. \tag{16}$$

Due to the continuity of ψ from (12), (15) and (16), we get:

$$\lim_{k\to\infty} \psi(\lambda^{(n_k)}) = \psi(\nu) < 1. \tag{17}$$

Since $\nu \neq C$, inequality (17) contradicts equality (15).Thus we conclude:

$$\lim_{n\to\infty} x^{(n)} = \lim_{n\to\infty} y^{(n)} = \lim_{n\to\infty} z^{(n)} = \frac{1}{3}.$$

This is impossible, as the unique fixed point C of the cubic stochastic operator (6) is repelling.

Therefore:

$$\lim_{n\to\infty} x^{(n)}y^{(n)}z^{(n)} = 0 \quad \Leftrightarrow \quad \omega\big(\lambda^0\big) \subset \partial S^2, \;\; \forall\, \lambda^0 \in \operatorname{int} S^2 \backslash \{C\}.$$

Theorem is proven.

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