Journal of Contemporary Applied Mathematics
V. 15, No 1, 2025, June
ISSN 2222-5498, E-ISSN 3006-3183
https://doi.org/10.62476/jcam.151.4

OPERATORS INTERTWINING WITH CONVOLUTION OPERATORS ON HYPERGROUPS

Kouakou Germain Brou *, Ibrahima Toure

Abstract. Let G be a locally compact hypergroup and let K be a compact subhypergroup such that (G, K) is a Gelfand pair. Let μ be a bounded complex-valued Borel measure on G , and let $T_\mu^\natural$ be the corresponding convolution operator of $L^{1\natural}(G)$, the subset of $L^1(G)$ consisting of K-bi-invariant functions. Suppose that S is a bounded linear operator on a Banach space X. We prove that every linear operator $\Psi : X \longrightarrow L^{1\natural}(G)$ such that $\Psi S = T_\mu^\natural \Psi$ is continuous if and only if $(S, T_\mu^\natural)$ has no critical eigenvalues.

Key Words and Phrases: Hypergroups, Convolution operator, Translation invariant operators, Gelfand pair.

2010 Mathematics Subject Classifications: 43A62, 43A22, 20N20

1. Introduction

Hypergroups generalize locally compact groups where the convolution of two Dirac measures is a Dirac measure. A hypergroup is a locally compact Hausdorff space equipped with a convolution product which maps two Dirac measures to a probability measure with compact support. The notion of hypergroup has been sufficiently studied (see for example [3, 8, 13, 15]). Harmonic analysis and probability theory on commutative hypergroups are well-developed and many results from group theory remain valid (see [1]). When G is a commutative hypergroup, the convolution algebra $M_c(G)$ consisting of measures with compact support on G is commutative. A typical example of commutative hypergroup is the double coset $G//K$ when G is a locally compact group, K is a compact subgroup of G such that (G, K) is a Gelfand pair. The class of translation invariant operators is a fundamental object in harmonic analysis. Several authors studied translation invariant operators on hypergroups [see [9, 7, 12]].

*Corresponding author.

In their paper [10], Laursen and Neumann raised the following question: “Let μ and ν be complex-valued bounded Borel measures on a locally compact abelian group G such that the corresponding pair (T_μ, T_ν) of convolution operators on $L^1(G)$ has no critical eigenvalues. Is every linear operator $\Psi : L^1(G) \to L^1(G)$ for which $T_\mu\Psi = \Psi T_\nu$, continuous?”

Let us note that this question generalizes that of Johnson on $\mathbb{R}$ [see [9]].

In the case of hypergroups, the question above is studied by Kumar and Sarma in [6], in the context of commutative hypergroups and compact hypergroups. But in general, the hypergroups involved in applications are not commutative, thus the question whether it is possible to have an affirmative response for this question when the hypergroup is neither commutative nor compact is our main concern. We consider the case where the hypergroup G is not commutative, but admitting a compact sub-hypergroup K leading to a commutative subalgebra of $M_c(G)$. In fact, if K is a compact sub-hypergroup of a hypergroup G, the pair (G, K) is said to be a Gelfand pair if $M_c(G//K)$ the convolution algebra of measures with compact support on the double cosets space $G//K$ is commutative. The notion of Gelfand pairs for hypergroups is well-known (see [4, 16, 17]).

The goal of this paper is to study the question of Johnson for locally compact hypergroups (possibly non-abelian) admitting a compact sub-hypergroup K such that (G, K) is a Gelfand pair. In this case, the charactery used in the commutative case are replaced by the spherical functions.

In the next section, we give notations and setup useful for the remainder of this paper. In section 3, we prove our main result.

2. Notations and preliminaries

Let G be a locally compact space. We denote by:
- $C(G)$ (resp. $M(G)$) the space of continuous complex-valued functions (resp. the space of Radon measures) on G,
- $C_b(G)$ (resp. $M_b(G)$) the space of bounded continuous functions (resp. the space of bounded Radon measures) on G,
- $\mathcal{K}(G)$ (resp. $M_c(G)$) the space of continuous functions (resp. the space of Radon measures) with compact support on G,
- $\mathfrak{C}(G)$ the space of compact sub-space of G,
- δ_x the point measure at $x \in G$,
- $\operatorname{supp}(\mu)$, the support of the measure μ,
- $cl(A)$, the closure of the subset A in G.

For any linear space X, I_X denote the identity operator on X.

Let us note that the topology on $M(G)$ is the cône topology [8] and the topology on $\mathfrak{C}(G)$ is the topology of Michael [11].

Definition 1. *G is said to be a hypergroup if the following assumptions are satisfied.*

(H1) There is a binary operator $*$ named convolution on $M_b(G)$ under which $M_b(G)$ is an associative algebra such that:
i) the mapping $(\mu,\nu) \longmapsto \mu*\nu$ is continuous from $M_b(G)\times M_b(G)$ in $M_b(G)$.
ii) $\forall x, y \in G$, $\delta_x * \delta_y$ is a measure of probability with compact support.
iii) the mapping: $(x, y) \longmapsto supp(\delta_x * \delta_y)$ is continuous from $G\times G$ in $\mathfrak{C}(G)$.

(H2) There is a unique element e (called neutral element) in G such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x, \forall x \in G$.

(H3) There is an involutive homeomorphism: $x \longmapsto \overline{x}$ from G in G, named involution, such that:
i) $(\delta_x * \delta_y)^- = \delta_{\overline{y}} * \delta_{\overline{x}}, \forall x, y \in G$ with $\mu^-(f) = \mu(f^-)$ where $f^-(x) = f(\overline{x}), \forall f \in C(G)$ and $\mu \in M(G)$.
ii) $\forall x, y, z \in G$, $z \in supp(\delta_x * \delta_y)$ if and only if $x \in supp(\delta_z * \delta_{\overline{y}})$.

For two subset A and B of G, $A * B = \underset{x\in A, y\in B}{\cup} \text{supp}(\delta_x * \delta_y)$. For $x, y \in G$ $\{x\} * \{y\}$ is denoted by $x * y$.

The hypergroup G is commutative if $\delta_x * \delta_y = \delta_y * \delta_x, \forall x, y \in G$.

For $x, y \in G$ and for $f \in C(G)$, $(\delta_x * \delta_y)(f)$ is denoted by $f(x * y)$. Thus, we have

$$f(x * y) = \int_G f(z)d(\delta_x * \delta_y)(z).$$

The convolution of two measures μ, ν in $M_b(G)$ is defined by:

$$(\mu*\nu)(f) = \int_G\int_G (\delta_x * \delta_y)(f)d\mu(x)d\nu(y) = \int_G\int_G f(x*y)d\mu(x)d\nu(y), \forall f \in C(G)$$

For μ in $M_b(G)$, $\mu^* = (\overline{\mu})^-$. So $M_b(G)$ is a $*$-Banach algebra.

Definition 2. *$H \subset G$ is a sub-hypergroup of G if the following conditions are satisfied.*

1. *H is non empty and closed in G,*
2. *$\forall x \in H, \overline{x} \in H$,*
3. *$\forall x, y \in H$, $supp(\delta_x * \delta_y) \subset H$.*

Let us now consider a hypergroup G provided with a left Haar measure μ_G and K a compact sub-hypergroup of G with a normalized Haar measure ω_K. For $x \in G$, the double coset of x with respect to K is $K * \{x\} * K = \{k_1 * x * k_2; k_1, k_2 \in K\}$. We write simply KxK for a double coset and recall that $KxK = \bigcup_{k_1,k_2 \in K} \operatorname{supp}(\delta_{k_1} * \delta_x * \delta_{k_2})$. All double cosets form a partition of G and the quotient topology with respect to the corresponding equivalence relation equips the double cosets space $G//K$ with a locally topology ([1], page 53). The natural mapping $p_K : G \longrightarrow G//K$ defined by

$$p_K(x) = KxK, x \in G$$

is an open surjective continuous mapping. A function $f \in C(G)$ is said to be K-*bi-invariant* if $f(k_1 * x * k_2) = f(x)$ for all $x \in G$ and for all $k_1, k_2 \in K$. We denote by $C^\natural(G)$, (resp. $\mathcal{K}^\natural(G)$) the space of continuous functions (resp. continuous functions with compact support) which are K-bi-invariant. We consider the K-projection $f \longmapsto f^\natural$ from $C(G)$ into $C^\natural(G)$ where for $x \in G$,

$$f^\natural(x) = \int_K \int_K f(k_1 * x * k_2) d\omega_K(k_1) d\omega_K(k_2).$$

If $f \in \mathcal{K}(G)$, then $f^\natural \in \mathcal{K}^\natural(G)$. For a measure $\mu \in M(G)$, one defines $\mu^\natural$ by

$$\mu^\natural(f) = \mu(f^\natural) \text{ for } f \in \mathcal{K}(G).$$

μ is said to be K-bi-invariant if $\mu^\natural = \mu$ and we denote by $M^\natural(G)$ the set of all those measures. Considering these properties, one defines a hypergroup operation on $G//K$ by

$$\delta_{KxK} * \delta_{KyK}(\widetilde{f}) = \int_K f(x * k * y) d\omega_K(k)$$

where $\widetilde{f} \in \mathcal{K}(G//K)$ and $f \in \mathcal{K}^\natural(G)$ such that $f = \widetilde{f} \circ p_K$ (see [16] and [1]). This defines uniquely the convolution $(KxK) * (KyK)$ on $G//K$. The involution is defined by: $\overline{KxK} = K\overline{x}K$ and the neutral element is K. Let us put $m = \int_G \delta_{KxK} d\mu_G(x)$, m is a left Haar measure on $G//K$. We say that (G, K) is a Gelfand pair if the convolution algebra $M_c(G//K)$ is commutative (see [17]), that means $M_c^\natural(G)$ is commutative. Considering the convolution product on $\mathcal{K}(G)$ defined by

$$f * g(x) = \int_G f(x * \overline{y}) g(y) d\mu_G(y),$$

$\mathcal{K}(G)$ is a convolution algebra and $\mathcal{K}^\natural(G)$ is a subalgebra of $\mathcal{K}(G)$ Thus (G, K) is a Gelfand pair if and only if $\mathcal{K}^\natural(G)$ is commutative ([4], theorem 3.2.2).

In the rest of the paper, we consider a Gelfand pair (G, K).

Let $S(G, K)$ be the set of continuous, bounded function ϕ on G such that:
(i) ϕ is K-bi-invariant,
(ii) $\phi(e) = 1$,
(iii) $\int_K \phi(x * k * y) dw_K(k) = \phi(x)\phi(y)\ \forall x, y \in G$,
(iv) $\phi(\overline{x}) = \overline{\phi(x)}\ \forall x \in G$.
$S(G, K)$ is the set of spherical functions of G with respect to K.

Equipped with the topology of uniform convergence on compact sets, $S(G, K)$ is a locally compact Hausdorff space and the function $\mathbf{1} : x \longmapsto 1$ belongs to $S(G, K)$.

For μ belongs to $M_b(G)$, the Fourier transform of μ, is defined by

$$\widehat{\mu}(\phi) = \int_G \phi(\overline{x}) d\mu(x),\ \phi \in S(G, K).$$

$\widehat{\mu} \in C_b(S(G, K))$ and the map:

$$\begin{array}{ll} M_b(G) & \longrightarrow C_b(S(G, K)) \\ \mu & \longmapsto \widehat{\mu} \end{array}$$

is a continuous linear operator. For $\beta \in M_b(S(G, K))$, the inverse Fourier transform of β is defined by

$$\overset{\vee}{\beta}(x) = \int_{S(G,K)} \phi(x) d\beta(\phi),\ x \in G.$$

By identifying, first $f \in L^1(G)$ with $f\mu_G$, we have $\widehat{f}(\phi) = \int_G \phi(\overline{x}) f(x) d\mu_G(x)$ $\forall \phi \in S(G, K)$, and then $\varphi \in L^1(S(G, K))$ with $\varphi\pi$ (where π is the Plancherel measure [5]), we have $\overset{\vee}{\varphi}(x) = \int_{S(G,K)} \phi(x)\varphi(\phi) d\pi(\phi)$, $x \in G$.

Remark 1. *By the commutativity of* $M_c^{\natural}(G)$ *and the characteristics of the elements of* $S(G, K)$*, it have been proven that*
a) $\widehat{\mu} = \widehat{\mu^{\natural}}$ *for any* $\mu \in M_b(G)$
b) $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$ *for* $\mu \in M_b^{\natural}(G)$ *and* $\nu \in M_b(G)$
c) If $f \in L^{1\natural}(G)$ *and* $\widehat{f} \in L^{1\natural}(S(G, K))$*, then* $\left(\widehat{f}\right)^{\vee} = f$.

For the properties above and more details on the Fourier and the inverse Fourier transform, see [2] and [5]

3. Operators commuting with convolution operators

In this part, after recalling some definitions and properties, we specify the convolution operator we will use and establish our result.

Definition 3. *Let X and Y be Banach spaces. Let S and T be continuous linear operators on X and Y respectively. A linear operator $\Psi : X \to Y$ is said to intertwine the pair (S, T) if $\Psi S = T\Psi$.*
A complex number λ is called a critical eigenvalue of (S, T) if λ is an eigenvalue of T and $(\lambda I_X - S)(X)$ is a subspace with infinite co-dimension in X.

Lemma 1. *(See [14],Lemma 3.2). If (S, T) has a critical eigenvalue then there exists a discontinuous operator which intertwines the pair (S, T).*

Definition 4. *Let $\Psi : X \longrightarrow Y$ be a linear operator from the Banach space X into the Banach space Y. The separating space of Ψ is the subspace*
$\mathfrak{S}(\Psi) = \{y \in Y : \exists$ *a sequence* $(x_n)_n$ *such that* $x_n \longrightarrow 0$ *in* X *and* $\Psi(x_n) \longrightarrow y$ *in* $Y\}$.

Remark 2. *It is clear that Ψ is continuous if and only if $\mathfrak{S}(\Psi) = \{0_Y\}$.*

Lemma 2. *(See [14], Lemma 1.6). Let X and Y be two Banach spaces and let $(T_n)_n$ and $(R_n)_n$ be sequences of continuous linear operators on X and Y respectively. If S is a linear operator from X into Y satisfying $ST_n = R_nS$ for all n, then there exists an integer N such that $cl(R_1...R_n(\mathfrak{S}(\Psi)) = cl(R_1...R_N(\mathfrak{S}(\Psi))$ for $n \geq N$.*

Let $\mu \in M_b(G)$ and let $f \in L^1(G)$. The convolution of μ by f is defined by

$$\mu * f(x) = \int_G f(\overline{y} * x) d\mu(y),\ x \in G.$$

It is well known that $\mu * f \in L^1(G)$ and the convolution operator $T_\mu : f \longmapsto \mu * f$ is a bounded linear operator on $L^1(G)$ (see [1]).

Let μ belongs to $M_b^\natural(G)$. We call by *convolution operator* on $L^{1\natural}(G)$, corresponding to μ, the operator

$$T_\mu^\natural : f \longmapsto \mu * f, f \in L^{1\natural}(G).$$

Remark 3. *(i) $T_\mu^\natural$ is a bounded linear operator on $L^{1\natural}(G)$.*
(ii) $\forall f \in L^{1\natural}(G)$, $\Big((T_\mu^\natural)^n(f)\Big)^\wedge = \underbrace{\widehat{\mu}...\widehat{\mu}}_{n\ times}.\widehat{f}, \forall n \geq 1$. In fact, for $n \geq 1$, we have

$(T^{\natural}_{\mu})^n(f) = \underbrace{\mu * ... * \mu}_{n\ times} * f$. *It then follows by Remark 1 b) that* $\Big((T^{\natural}_{\mu})^n(f)\Big)^{\wedge} = \underbrace{\widehat{\mu}...\widehat{\mu}}_{n\ times} .\widehat{f}$.

We have the following result which is our main theorem.

Theorem 1. *Let (G, K) be a Gelfand pair and $T^{\natural}_{\mu}$ be the convolution operator on $L^{1\natural}(G)$ corresponding to $\mu \in M^{\natural}_b(G)$. Let S be a continuous linear operator on a Banach space X. If the pair $(S, T^{\natural}_{\mu})$ has no critical eigenvalue, then every linear operator $\Psi : X \longrightarrow L^{1\natural}(G)$ which intertwines $(S, T^{\natural}_{\mu})$ is continuous.*

Proof. Suppose that $\Psi : X \longrightarrow L^{1\natural}(G)$ is a discontinuous operator intertwining the pair $(S, T^{\natural}_{\mu})$. So $\mathfrak{G}(\Psi) \neq \{0\}$. Let us consider the space $\widehat{\mu}(S(G, K))_{\Psi} = \Big\{\widehat{\mu}(\phi) : \phi \in S(G, K) \text{ and } \widehat{f}(\phi) \neq 0 \text{ for some } f \in \mathfrak{G}(\Psi)\Big\}$.

$\widehat{\mu}(S(G, K))_{\Psi}$ is finite. In fact, if we assume that $\widehat{\mu}(S(G, K))_{\Psi}$ is infinite, we can have a sequence $(\phi_n)_n$ with $\phi_n \in S(G, K)$ such that $\widehat{\mu}(\phi_n) \neq \widehat{\mu}(\phi_m)$ for $n \neq m$ and for any $n \in \mathbb{N}$ there exists $f_n \in \mathfrak{G}(\Psi)$ such that $\widehat{f_n}(\phi_n) \neq 0$. For $k \in \mathbb{N}$, let us put $\lambda_k = \widehat{\mu}(\phi_k)$. Let I_{L^1} and I_X denote the identity operators on $L^{1\natural}(G)$ and X respectively.
For any ϕ belongs to $S(G, K)$ and any $f \in L^{1\natural}(G)$, let us prove by recurrence that

$$\Big[(\lambda_1 I_{L^1} - T^{\natural}_{\mu})...(\lambda_n I_{L^1} - T^{\natural}_{\mu})f\Big]^{\wedge}(\phi) = (\lambda_1 - \widehat{\mu}(\phi))...(\lambda_n - \widehat{\mu}(\phi))\widehat{f}(\phi)\ \forall n \in \mathbb{N}^*. \tag{1}$$

Indeed, $\Big[(\lambda_1 I_{L^1} - T^{\natural}_{\mu})f\Big]^{\wedge}(\phi) = [\lambda_1 f - \mu * f]^{\wedge}(\phi) = (\lambda_1 - \widehat{\mu}(\phi))\widehat{f}(\phi)$ (see Remark 3 (ii)), and suppose that

$$\Big[(\lambda_1 I_{L^1} - T^{\natural}_{\mu})...(\lambda_k I_{L^1} - T^{\natural}_{\mu})f\Big]^{\wedge}(\phi) = (\lambda_1 - \widehat{\mu}(\phi))...(\lambda_k - \widehat{\mu}(\phi))\widehat{f}(\phi) \text{ for } k \geq 1$$

We have

$$\begin{aligned}\Big[(\lambda_1 I_{L^1} - T^{\natural}_{\mu})...(\lambda_{k+1} I_{L^1} - T^{\natural}_{\mu})f\Big]^{\wedge}(\phi) &= \Big[(\lambda_1 I_{L^1} - T^{\natural}_{\mu})...(\lambda_k I_{L^1} - T^{\natural}_{\mu})(\lambda_{k+1} f - \mu * f)\Big]^{\wedge}(\phi)\\ &= (\lambda_1 - \widehat{\mu}(\phi))...(\lambda_k - \widehat{\mu}(\phi))(\lambda_{k+1} f - \mu * f)^{\wedge}(\phi)\\ &= (\lambda_1 - \widehat{\mu}(\phi))...(\lambda_k - \widehat{\mu}(\phi))(\lambda_{k+1} - \widehat{\mu}(\phi))\widehat{f}(\phi),\end{aligned}$$

hence we have the assertion.
Otherwise, since Ψ intertwines $(S, T^{\natural}_{\mu})$, we have

$$(\lambda_n I_{L^1} - T^{\natural}_{\mu})\Psi = \Psi(\lambda_n I_X - S),\ \forall n \geq 1.$$

Then by lemma 3.4, there exists $n \geq 1$ such that

$$cl((\lambda_1 I_{L^1} - T^{\natural}_{\mu})...(\lambda_n I_{L^1} - T^{\natural}_{\mu})\mathfrak{G}(\Psi)) = cl((\lambda_1 I_{L^1} - T^{\natural}_{\mu})...(\lambda_{n+1} I_{L^1} - T^{\natural}_{\mu})\mathfrak{G}(\Psi)) \quad (2)$$

Using (1) we have

$$\begin{aligned}\left[(\lambda_1 I_{L^1} - T^{\natural}_{\mu})...(\lambda_{n+1} I_{L^1} - T^{\natural}_{\mu})f\right]^{\wedge}(\phi_{n+1}) &= (\lambda_1 - \lambda_{n+1})...(\lambda_{n+1} - \lambda_{n+1})\widehat{f}(\phi_{n+1}) \\ &= 0.\end{aligned}$$

So the Fourier transform of any function belongs to $(\lambda_1 I_{L^1} - T^{\natural}_{\mu})...(\lambda_{n+1} I_{L^1} - T^{\natural}_{\mu})\mathfrak{G}(\Psi)$ vanishes at ϕ_{n+1}. Then by (2), the Fourier transform of any function belongs to $cl((\lambda_1 I_{L^1} - T^{\natural}_{\mu})...(\lambda_n I_{L^1} - T^{\natural}_{\mu})\mathfrak{G}(\Psi))$ vanishes at ϕ_{n+1}, that is

$$\left[(\lambda_1 I_{L^1} - T^{\natural}_{\mu})...(\lambda_n I_{L^1} - T^{\natural}_{\mu})f\right]^{\wedge}(\phi_{n+1}) = 0$$

for all $f \in \mathfrak{G}(\Psi)$ and consequently

$$(\lambda_1 - \lambda_{n+1})...(\lambda_n - \lambda_{n+1})\widehat{f}(\phi_{n+1}) = 0.$$

Since $(\lambda_1 - \lambda_{n+1})...(\lambda_n - \lambda_{n+1}) \neq 0$, then $\widehat{f}_{n+1}(\phi_{n+1}) = 0$, this contradicts the choice of f_{n+1}, and $\left\{\widehat{\mu}(\phi) : \phi \in S(G,K) \text{ and } \widehat{f}(\phi) \neq 0 \text{ for some } f \in \mathfrak{G}(\Psi)\right\}$ is finite.

Let $\beta_1, ..., \beta_m \in \mathbb{C}$ such that

$$\left\{\widehat{\mu}(\phi) : \phi \in S(G,K) \text{ and } \widehat{f}(\phi) \neq 0 \text{ for some } f \in \mathfrak{G}(\Psi)\right\} = \{\beta_1, ..., \beta_m\}.$$

$(\beta_1 I_{L^1} - T^{\natural}_{\mu})...(\beta_m I_{L^1} - T^{\natural}_{\mu})\mathfrak{G}(\Psi) = 0$. In fact, let us put

$$\widehat{G}_{\Psi} = \left\{\phi \in S(G,K) \colon \widehat{f}(\phi) \neq 0 \text{ for some } f \in \mathfrak{G}(\Psi)\right\}.$$

For $f \in \mathfrak{G}(\Psi)$ and $\phi \in S(G,K)$, we have

$$\left[(\beta_1 I_{L^1} - T^{\natural}_{\mu})...(\beta_m I_{L^1} - T^{\natural}_{\mu})f\right]^{\wedge}(\phi) = (\beta_1 - \widehat{\mu}(\phi))...(\beta_m - \widehat{\mu}(\phi))\widehat{f}(\phi).$$

It is clear that $(\beta_1 - \widehat{\mu}(\phi))...(\beta_m - \widehat{\mu}(\phi))\widehat{f}(\phi) = 0$ if $\phi \notin \widehat{G}_{\Psi}$.

If $\phi \in \widehat{G}_{\Psi}$, then there exists $k \in \{1, ..., m\}$ such that $\widehat{\mu}(\phi) = \beta_k$, hence $(\beta_1 - \widehat{\mu}(\phi))...(\beta_m - \widehat{\mu}(\phi))\widehat{f}(\phi) = 0$. So $\left[(\beta_1 I_{L^1} - T^{\natural}_{\mu})...(\beta_m I_{L^1} - T^{\natural}_{\mu})f\right]^{\wedge} = 0$.

It follows that $(\beta_1 I_{L^1} - T^{\natural}_{\mu})...(\beta_m I_{L^1} - T^{\natural}_{\mu})f = 0$ since the function $(\beta_1 I_{L^1} - T^{\natural}_{\mu})...(\beta_m I_{L^1} - T^{\natural}_{\mu})f$ belongs to $L^{1\natural}(G)$.

Thus, as all operators $\beta_k I_{L^1} - T_\mu^\natural; k \in \{1, ..., m\}$ commute among themselves, we can state that β_k is a eigenvalue of $T_\mu^\natural$ for $k \in \{1, ..., m\}$. Since $(S, T_\mu^\natural)$ has no critical eigenvalues, then the subspace $(\beta_k I_X - S)(X)$ has finite co-dimension for $k \in \{1, ..., m\}$, therefore $N = (\beta_1 I_X - S)...(\beta_m I_X - S)(X)$ has finite co-dimenson. The operator $R = (\beta_1 I_X - S)...(\beta_m I_X - S)$ is continuous, hence by [[14], Lemme 3.3], N is a closed subspace of X. We have

$$\Psi R = \Psi(\beta_1 I_X - S)...(\beta_m I_X - S) = (\beta_1 I_{L^1} - T_\mu^\natural)...(\beta_m I_{L^1} - T_\mu^\natural)\Psi.$$

$(\beta_1 I_{L^1} - T_\mu^\natural)...(\beta_m I_{L^1} - T_\mu^\natural)\mathfrak{G}(\Psi) = 0$, so $\Psi R = (\beta_1 I_{L^1} - T_\mu^\natural)...(\beta_m I_{L^1} - T_\mu^\natural)\Psi$ is continuous (see [14], Lemma 1.3 (i)). Otherwise let $(y_n)_n$ be a sequence in N such that $y_n \longrightarrow 0$. Since $R : X \longrightarrow N$ is continuous and surjective, then R is an open operator. So there exists a sequence $(x_n)_n$ in X such that $x_n \longrightarrow 0$ and $R(x_n) = y_n$. We have $\Psi(y_n) = \Psi R(x_n)$ and by the continuity of ΨR, we have $\Psi(y_n) \longrightarrow 0$. Thus the restriction of Ψ to N is continuous. Since N has finite co-dimenson, then Ψ is continuous on X, this is a contradiction, and the proof is complete.

Remark 4. *The above result is a generalization of the one established in the case of commutative hypergroups (see [6]). In fact, if G is a commutative hypergroup, then $(G, \{e\})$ is a Gelfand pair.*

References

[1] W. R. Bloom and H. Heyer, Harmonic analysis of probability measures on hypergroups. De Gruyter, Berlin, (1995).

[2] K. G. Brou, I. Toure, and K. Kangni, A Bochner theorem on a noncommutative hypergroup, Palestine Journal of Mathematics, Vol 13(2),8–17,(2024).

[3] C. F. Dunkl, The Measure Algebra of a Locally Compact Hypergroup, Trans. Amer. Math. Soc., Volume 179,331-348, (1973).

[4] B. K. Germain and K. Kangni, On Gelfand pair over Hypergroups, Far East J. Math, Vol. 132, Number 1, 63-76, (2021).

[5] B. K. Germain, I. Toure, K. Kangni, A Plancherel Theorem On a Noncommutative Hypergroup, Int. J. Anal. Appl.20,32,(2022)

[6] V. Kumar and R. Sarma, Continuity of operators intertwining with translation operators on hypergroups, Aequat. Math., Springer Nature Switzerland AG (2020). https://doi.org/10.1007/s00010-020-00745-y.

[7] V. Kumar, R. Sarma, N Shravan Kumar, Orlicz spaces on hypergroups. Publ. Math. Debrecen 94(1–2), 31-47 (2019).

[8] R. I. Jewett, Spaces with an Abstract Convolution of Measure, Adv. Math., Vol. 18, 1-101 (1975).

[9] B. E. Johnson, Continuity of linear operator commuting with continuous linear operators, Trans. Am. Math. Soc. 156, 88–102 (1967).

[10] K. B. Laursen, M. M. Neumann, An introduction to local spectral theory. In: London Mathematical Society Monograph, New Series, Vol. 20. The Clarendon Press, Oxford University Press, New York, (2000).

[11] L. Nachbin, On the finite dimensionality of every irreducible representation of a group, Proc. Amer, Math. Soc. 12, 11-12, (1961).

[12] L. Pavel, Multipliers for the Lp-spaces of a hypergroup. Rocky Mt. J. Math. 37(3), 987–1000 (2007).

[13] K. A. Ross, Centers of Hypergroups, Trans. Amer. Math. Soc., Vol. 243, Septembre 1978.

[14] A. M. Sinclair, Automatic continuity of linear operators. Cambridge University Press, Cambridge (1976).

[15] R. Spector, Mesures invariantes sur les Hypergroupes, Trans. Amer. Math. Soc., Vol. 239, (1978).

[16] L. Szekelyhidi, Spherical Spectral Synthetis On Hypergroups, Acta Math. Hungar (2020). https://doi.org/10.1007/s10474-020-01068-9.

[17] K. Vatti, Gelfand Pairs over Hypergroups Joins, Acta Math.Hungar., May 2019. Doi: 10.1007/s 10474-019-00946-1.

Kouakou Germain BROU
UFR of Sciences and Technology of UMAN, Man, Cote d'Ivoire
*E-mail:*germain.brou@univ-man.edu.ci

Ibrahima TOURE
UFR Mathematics and Computer Science of UFHB of Abidjan, Abidjan, Cote d'Ivoire
E-mail: toure.ibrahima@ufhb.edu.ci

Received 25 October 2024
Accepted 11 February 2025