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Spectral Properties of a Differential operator with Integral Boundary condition

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Abstract. In this work, we study the second-order differential operator with integral boundary conditions $U_v(y) = \int_0^1 \varphi_v(x)y(x)dx = 0, v = 1, 2$. Such an operator is not densely defined in any space $L_p(0, 1)$. Therefore, the operator is considered not on the whole $L_p(0, 1)$, but in its subspace $L_{p,U}(0, 1) = \left\{y(x) \in L_p(0, 1) : U_v(y), v = 1, 2\right\}$, $1 < p < \infty$, which has codimension two. Under weaker conditions than previously known on the functions $\varphi_v(x), v = 1, 2$, estimates for the resolvent are obtained and a theorem on the basis property of eigen and associated functions in subspace $L_{p,U}(0, 1)$ is proved.

Key Words and Phrases: second-order differential operator, eigenvalues, eigenfunctions, basis property, integral boundary conditions.

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1. Introduction

Consider the linear differential expression

$$l(y) = -y'' + q(x)y, x \in (0, 1), \tag{1}$$

and boundary conditions

$$U_v(y) = \int_0^1 \varphi_v(x)y(x)dx = 0, v = 1, 2, \tag{2}$$

where $q(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ are given complex-valued function belonging to the space $L_1(0, 1)$. $\varphi_1(x)$ and $\varphi_2(x)$ are linearly independent functions. Differential expression (1) and boundary conditions (2) generate a differential operator L with a domain of definition $D(L)$ in some functional space. We will be interested in the problem of the behavior of eigenvalues and eigenfunctions of this differential operator. Such a problem in the case of regular boundary conditions

$$U_v(y) = a_{\nu 1} y'(0) + b_{\nu 1} y'(1) + a_{\nu 0} y(0) + b_{\nu 0} y'(1) = 0, v = 1, 2,$$

has been studied quite well (see [1- 4] and the bibliography there). The case of irregular, as well as more general regular boundary conditions, when the boundary conditions contain some integrals of the function $y(x)$ and its derivatives, was considered in [5-11]. In these works, the spectral properties of the corresponding operator were studied (spectrality, eigenfunctions, conjugate problem and mainly in the space $L_2(0,1)$). Let us also note the works [12-14], where similar problems were studied in the spaces $L_p(0,1)$. Another class of boundary conditions are degenerate boundary conditions. For such boundary conditions, the spectrum of the corresponding operator is either empty or coincides with the entire complex plane. Such problems are studied in the works [15-17]. However, as a rule, boundary forms generated an unbounded functional in the space under consideration, and in this case the operator has a dense domain of definition, which made it possible to construct a conjugate operator or assume the regularity of boundary conditions [1-4]. Here we will consider integral boundary conditions (2). These conditions are not regular in the sense of Birkhoff [1], and there is no corresponding conjugate operator for them. Such conditions were used for other purposes in [10, 11]. In [18, 19], problem (1), (2) was studied under more stringent conditions on the functions $q(x)$ and $\varphi_v(x)$, where the asymptotic behavior of eigenvalues and eigenfunctions was found, and the theorem on the Riesz basis property of a system of eigenfunctions a certain subspace of the space $L_2(0,1)$ is proved. In [20], the completeness and minimality of the eigenfunctions and associated functions of problem (1), (2) were proven, and in [21], under additional conditions of smoothness of the functions $\varphi_v(x)$, it is proved that the system of eigen and associated functions forms a basis in the corresponding subspace of the space $L_p(0,1)$, equivalent to the trigonometric system $\{\cos \pi kx\}_{k=2}^{\infty}$. In this paper, we obtain an estimate of the resolvent on some rays and investigate the basis property of eigenfunctions under weaker restrictions on the functions $\varphi_v(x)$ than in [21]. Note that differential equations with nonlocal conditions of integral form have interesting applications in mechanics [22] and in the theory of diffusion processes [23].

We present some concepts and facts that will be needed later.

Definition 1 [24]. *Let X be a Banach space and $A-$a closed linear operator with domain $D(A) \subset X$ and the values also in X. The operator A is called positive if the interval $(-\infty, 0]$ belongs to the resolvent set of A, and there exists the number $C > 0$ such that*

$$\left\| (A + tI)^{-1} \right\|_2 \le \frac{C}{1+t}, t \le 0.$$

Definition 2. *The ray $l = \{\lambda : arg\lambda \ = \phi\}$ is called a ray of minimal growth of the resolvent of the operator L, if the resolvent $R(\lambda) = (L - \lambda I)^{-1}$ exists on this ray sufficiently far from the origin and satisfies the inequality* $\|R(\lambda)\| \leq \frac{C}{|\lambda|}$.

From this definition, it follows that there exist numbers ε and h such that the operator $A = \varepsilon L + hI$ is positive.
To study the basis property of the eigenfunctions and associated functions of the operator L, we will need two well-known theorems. The first of them is Riesz's theorem on the boundedness of the Hilbert transform in L_p (see., example, [25, p. 132]). Below we present it in a convenient form. In this form, it is given in [12, Theorem 4.1].

Theorem(M.Riesz)[25, 12]. *Let $f \in L_p(0,1)$, $1 < p < \infty$. Then the integral*

$$g(x) = \int_0^1 \frac{f(t)}{x+t} dt$$

exists almost everywhere on $[0,1]$. Moreover, there exists a constant $C > 0$ such that the following inequality holds: $\|g\|_p \leq C\|f\|_p$.

The second theorem, which will use below, is a criterion for being a basis in a Banach space.
Theorem (critery for a basis). *For a system $\{x_n\}_{n\in N}$ of a Banach space X to be a basis, it is necessary and sufficient that the following conditions be satisfied:*
i) the system $\{x_n\}_{n\in N}$ is complete and minimal in X;
ii) the projections $\{P_n\}_{n\in N}$ are uniformly bounded, where

$$P_n x = \sum_{k=1}^{n} \langle x, x_k^* \rangle x_k, \quad x \in X,$$

and $\{x_n^*\}_{n\in N} \subset X^*$ is the conjute system.

2. Estimation of the resolvent

Let us introduce in the space $L_p(0,1), 1 < p < \infty$, a differential operator L, corresponding to the differential expression $l(y)$ with the domain of definition $D(L) = \left\{y(x) \in W_p^2(0,1), l(y) \in L_p(0,1); U_v(y) = 0, v = 1,2\right\}$ and consider the problem of the eigenvalues of this operator: $Ly = \lambda y$.

Let's put $\lambda = \rho^2$. Let us denote $S_\gamma = \left\{\rho : \frac{\gamma\pi}{2} \leq \arg\rho \leq \frac{(\gamma+1)\pi}{2}\right\}$, $\gamma = 0,\ 1,\ 2,\ 3$. In each region S_γ, equation (1) has a fundamental system of solutions with asymptotics [1, p. 58]

$$y_1(x,\rho)=e^{\rho\omega_1 x}\left(1+r_1(x,\rho)\right),\quad y_2(x,\rho)=e^{\rho\omega_2 x}(1+r_2(x,\rho)), \tag{3}$$

where the numbers ω_1 and ω_2 are different square roots of (-1) (i.e. $\pm i$), numbered so that $Re\,(\rho\omega_1)\le Re\,(\rho\omega_2)$ is satisfied for $\rho\in S_\gamma$, and the functions $r_i(x,\rho)$ are continuous even for sufficiently large values of $|\rho|$ the estimate $|r_i(x,\rho)|\le\frac{c_i}{|\rho|}, i=1,2$ is satisfied, uniformly in $x\in[0,1]$.

In what follows, with respect to functions $\varphi_v(x), v=1,2$, we will assume that the following conditions are met:

A) $q\,(x)\in L_1\,(0,1)\,;\ \exists a\in(0,1)\ :\quad \varphi_v\,(x)\in L_1\,(0,1)\cap W_1^1\,(0,a)\cap W_1^1\,(1-a,1)\,,\ v=1,2;$

B) $\alpha_1\beta_2-\alpha_2\beta_1\neq 0$, where $\alpha_v=\varphi_v\,(0)\,,\ \ \beta_v=\varphi_v(1)$.

Then in some strip $|Im\rho\,|\le h$, for some $h>0$ the following relations are satisfied:

$$\int_0^1\varphi_v\,(x)\,e^{i\rho x}dx=\frac{1}{i\rho}\left(\beta_v e^{i\rho}-\alpha_v\right)+o\left(\frac{1}{\rho}\right),$$

$$\int_0^1\varphi_v(x)e^{-i\rho x}dx=\frac{1}{-i\rho}\left(\beta_v e^{-i\rho}-\alpha_v\right)+o\left(\frac{1}{\rho}\right), \tag{4}$$

These relations are obtained using integration by parts and from the Riemann-Lebesgue theorem [26, p.18]. In addition, from (3) and (4) it follows that under the same assumptions the relations are also satisfied

$$U_\nu\,(y_1)=\int_0^1\varphi_v(x)y_1(x,\rho)dx=\frac{1}{i\rho}\left(\beta_v e^{i\rho}-a_v\right)+\frac{r_{v1}(\rho)}{\rho},$$

$$U_\nu\,(y_2)=\int_0^1\varphi_v(x)y_2(x,\rho)dx=\frac{1}{-i\rho}\left(\beta_v e^{-i\rho}-\alpha_v\right)+\frac{r_{v2}(\rho)}{\rho}, \tag{5}$$

where for functions $r_{vi}(\rho)$ for large values of $|\rho|$ and $|\mathrm{Im}\rho\,|\le h$ the estimate $r_{vi}(\rho)=o(1)$ is satisfied.

The eigenvalues of the operator L are the numbers $\lambda_k=\rho_k^2$, where ρ_k are the zeros of the characteristic determinant

$$\triangle\,(\rho)=\begin{vmatrix} U_1\,(y_1) & U_1(y_2)\\ U_2\,(y_1) & U_2(y_2)\end{vmatrix}.$$

The following theorem is true regarding the function $\triangle\,(\rho)$.

Theorem 1. *Let conditions A), B) be met. Then for the characteristic determinant $\Delta(\rho)$ of the spectral problem* (1), (2) *the following are valid:*
i) any number $\delta > 0$ corresponds to a constant $m_\delta > 0$, depending on the function $\Delta(\rho)$, such that on the set obtained from the complex $\rho-$plane by throwing out the $\delta-$neighborhoods of the zeros of $\Delta(\rho)$ the inequality holds

$$|\Delta(\rho)| \geq m_\delta \frac{1}{|\rho^2|} e^{Re(\rho\omega_2)};$$

ii) the zeros of the function $\Delta(\rho)$ are asymptotically simple and separated;

iii) the function $\Delta(\rho)$ has two series of roots: the first series has an asymptotic

$$\rho_n = \pi n + o(1),$$

and the second series ${\rho'}_n$ is defined by the equality ${\rho'}_n = -\rho_n$.

Proof. Let $y_1(x)$ and $y_2(x)$ - are the fundamental system of solutions from (3). The determinant $\Delta(\rho)$ is divided into the sum

$$\Delta(\rho) = \Delta_0(\rho) + \Delta_1(\rho) + \Delta_2(\rho) + \Delta_3(\rho), \tag{6}$$

where

$$\Delta_0(\rho) = \begin{vmatrix} \int_0^1 \varphi_1(x) e^{i\rho x} dx & \int_0^1 \varphi_1(x) e^{-i\rho x} dx \\ \int_0^1 \varphi_2(x) e^{i\rho x} dx & \int_0^1 \varphi_2(x) e^{-i\rho x} dx \end{vmatrix}, \tag{7}$$

$\Delta_1(\rho)$ is obtained from $\Delta_0(\rho)$ by replacing the second row with the elements $\int_0^1 \varphi_2(x)e^{i\rho x}r_1(x,\rho)dx$, $\int_0^1 \varphi_2(x)e^{-i\rho x}r_2(x,\rho)dx$, and $\Delta_2(\rho)-$ from $\Delta_0(\rho)$ by replacing the first row with the elements $\int_0^1 \varphi_1(x)e^{i\rho x}r_1(x,\rho)dx$, $\int_0^1 \varphi_1(x)e^{-i\rho x}r_2(x,\rho)dx$, finally, $\Delta_3(\rho)$ - replacing both lines with the specified elements (the first with $\varphi_1(x)$, the second with $\varphi_2(x)$). Let's consider the determinant $\Delta_1(\rho)$ By virtue of formulas (4), (5) we have

$$\Delta_1(\rho) = \frac{R_1(\rho)}{\rho^2}, \Delta_2(\rho) = \frac{R_2(\rho)}{\rho^2}, \ \Delta_3(\rho) = \frac{R_3(\rho)}{\rho^3}$$

where $R_i(\rho) = o(1), i = 1, 2, 3$, for $\rho \to \infty$ and $|Im\rho| \leq h$. Thus, in expansion (6) the main role as $\rho \to \infty$ is played by the term $\Delta_0(p)$, therefore, taking into account formulas (6), (7) we have

$$\Delta(\rho) = \frac{1}{\rho^2} \begin{vmatrix} \beta_1 e^{i\rho} - \alpha_1 & \beta_1 e^{-i\rho} - \alpha_1 \\ \beta_2 e^{i\rho} - \alpha_2 & \beta_2 e^{-i\rho} - \alpha_2 \end{vmatrix} + \frac{R(\rho)}{\rho^2}, \tag{8}$$

where $R(\rho) = o(1)$ for $\rho \to \infty$ and $|Im\rho| \leq h$. From equality (8) it follows that

$$\Delta(\rho) = \frac{1}{\rho^2}(a_1\beta_2 - a_2\beta_1)\left(e^{i\rho} - e^{-i\rho}\right) + \frac{R(\rho)}{\rho^2}.$$

Now all the statements of the theorem are obtained by similar reasoning carried out in [1, pg. 77, 78].

The operator L constructed above does not have a dense domain of definition in the space $L_p(0,1)$ and therefore the eigenfunctions of the operator L cannot be complete in this space. To eliminate this drawback, consider the operator L not on the whole space $L_p(0,1)$, but in its closed subspace

$$L_{p,U}\,(0,1) = \{f(x) \in L_p(0,1) : U_v(f) = 0, v = 1,2\}.$$

It is obvious that codim $L_{p,U} = 2$. Similarly we define the space

$$W^2_{p,U}\,(0,1) = \left\{f(x) \in W^2_p(0,1) : U_v(f) = 0, v = 1,2\right\}.$$

Let us define the operator L in the space $L_{p,U}\,(0,1)$ as follows:
$D(L) = \left\{y \in W^2_{p,U}\,(0,1) : l(y) \in X_p\right\}$ and for $y \in D(L) : Ly = l(y)$.

The operator L thus defined has an everywhere dense domain of definition in $L_{p,U}\,(0,1)$ [27, Lemma 2.2]. To study the question of completeness of eigenfunctions of the operator L in the space $L_{p,U}\,(0,1)$ we construct and estimate the resolvent of the operator L. It is known (see [1, p. 47]) that the Green's function of the operator $L - \rho^2 I$ has the form

$$G\,(x,\xi,\rho) = \frac{1}{\Delta\,(\rho)} \begin{vmatrix} g\,(x,\xi,\rho) & y_1\,(x,\rho) & y_2(x,\rho) \\ U_1\,(g) & U_1\,(y_1) & U_1(y_2) \\ U_2\,(g) & U_2(y_1) & U_2(y_2) \end{vmatrix}, \tag{9}$$

where

$$g(x,\xi,\rho) = \begin{cases} -y_1(x,\rho)z_1(\xi,\rho), x \geq \xi, \\ \;\; y_2(x,\rho)z_2(\xi,\rho), \;\; x < \xi, \end{cases}$$

$$z_1(\xi,\rho) = \frac{y_2(\xi,\rho)}{W(\rho)}, z_2(\xi,\rho) = \frac{y_1(\xi,\rho)}{W(\rho)}, W(\rho) = \begin{vmatrix} y_1(\xi,\rho) & y_2(\xi,\rho) \\ y'_1(\xi,\rho) & y'_2(\xi,\rho) \end{vmatrix}.$$

Consider in the complex ρ -plane the region $\Omega_\delta = \cap_n \{\rho : |\rho - \rho_n| \geq \delta\}$, where $\{\rho_n\}$- is the set of zeros of the function $\Delta\,(\rho)$. Let K_δ denote the region of the complex $\lambda-$plane, which is the image of Ω_δ under the mapping $\lambda = \rho^2$.

Theorem 2. *If conditions A), B) are satisfied, then for the resolvent $R_\lambda\,(L) = (L - \lambda I)^{-1}$ of the operator L generated by the differential expression $l\,(y)$ and the boundary conditions* (2) *in the domain K_δ for large values of $|\lambda|$ the following estimate is correct:*

$$\|R_\lambda(L)\| \leq \frac{c}{|\lambda|^{\frac{1}{2}}}. \tag{10}$$

Proof. It is known [1] that for the derivatives of the functions $y_1(x,\rho)$ and $y_2(x,\rho)$ in the region S_0 asymptotic estimates are valid

$$y_1'(x,\rho) = i\rho e^{i\rho x}(1+r_3(x,\rho)), y_2'(x,\rho) = -i\rho e^{-i\rho x}(1+r_4(x,\rho)),$$

where$|r_i(x,\rho)| \le \frac{c_i}{|\rho|}, i = 3,4,$ uniformly along$x \in [0,1]$. From the last relations, taking into account (4) for the Wronskian $W(\rho)$, we have

$$W(\rho) = \begin{vmatrix} e^{i\rho x}(1+r_1(x,\rho)) & e^{-i\rho x}(1+r_2(x,\rho)) \\ ie^{i\rho x}\left(1+r_3\left(x,\rho\right)\right) & -ie^{-i\rho x}(1+r_3(x,\rho)) \end{vmatrix} = -2i\rho\left(1+0\left(\frac{1}{\rho}\right)\right).$$

From here, for the functions $z_1(\xi,\rho)$ and $z_2(\xi,\rho)$ we obtain

$$z_1\left(\xi,\rho\right) = -\frac{1}{2i\rho}e^{-i\rho\xi}\left(1+0\left(\frac{1}{\rho}\right)\right), z_2\left(\xi,\rho\right) = -\frac{1}{2i\rho}e^{i\rho\xi}\left(1+0\left(\frac{1}{\rho}\right)\right).$$

Therefore for $g(x,\xi,\rho)$ we have

$$g(x,\xi,\rho) = \begin{cases} \frac{1}{2i\rho}e^{i\rho(x-\xi)}\left(1+0\left(\frac{1}{\rho}\right)\right), x \ge \xi, \\ -\frac{1}{2i\rho}e^{-i\rho(x-\xi)}\left(1+0\left(\frac{1}{\rho}\right)\right), x < \xi. \end{cases}$$

Taking into account formulas (4), as well as the last relations, we estimate $U_1(g)$ and $U_2(g)$:

$$U_1(g) = \int_0^1 \varphi_1(x)g(x,\xi,\rho)dx = -\frac{1}{2i\rho}e^{ip\xi}\int_0^\xi \varphi_1(x)e^{-i\rho x}\left(1+0\left(\frac{1}{\rho}\right)\right)dx+$$

$$+\frac{1}{2i\rho}e^{-i\rho\xi}\int_\xi^1 \varphi_1(x)e^{i\rho x}\left(1+0\left(\frac{1}{\rho}\right)\right)dx = \frac{1}{2\rho^2}\left(\alpha_1 e^{i\rho\xi} - \beta_1 e^{i\rho(1-\xi)}\right) + o\left(\frac{1}{\rho^2}\right),$$

$$U_2(g) = \int_0^1 \varphi_2(x)g(x,\xi,\rho)dx = -\frac{1}{2i\rho}e^{ip\xi}\int_0^\xi \varphi_2(x)e^{-i\rho x}\left(1+0\left(\frac{1}{\rho}\right)\right)dx+$$

$$+\frac{1}{2ip}e^{-i\rho\xi}\int_\xi^1 \varphi_2(x)e^{i\rho x}\left(1+0\left(\frac{1}{\rho}\right)\right)dx = \frac{1}{2\rho^2}\left(\alpha_2 e^{i\rho\xi} - \beta_2 e^{i\rho(1-\xi)}\right) + o\left(\frac{1}{\rho^2}\right).$$

Finally, we replace all the functions included in (9) with their asymptotic expressions. Then for $x \ge \xi$ we have

$$G(x,\xi,\rho) = \frac{1}{\xi(\rho)}\begin{vmatrix} g\left(x,\xi,\rho\right) & y_1\left(x,\rho\right) & y_2(x,\rho) \\ U_1\left(g\right) & U_1\left(y_1\right) & U_1(y_2) \\ U_2(g) & U_2(y_1) & U_2(y_2) \end{vmatrix} =$$

$$= \frac{1}{2\rho(\alpha_1\beta_2 - \alpha_2\beta_1)(e^{i\rho} - e^{-i\rho})[1]} \times$$

$$\times \begin{vmatrix} e^{i\rho(x-\xi)}\,[1] & e^{i\rho x}\,[1] & e^{-i\rho x}[1] \\ \left(\alpha_1 e^{i\rho\xi} - \beta_1 e^{i\rho(1-\xi)}\right)[1] & \left(\alpha_1 - \beta_1 e^{i\rho}\right)[1] & (\beta_1 e^{-i\rho} - \alpha_1)[1] \\ \left(a_2 e^{i\rho\xi} - \beta_2 e^{i\rho(1-\xi)}\right)[1] & \left(a_2 - \beta_2 e^{i\rho}\right)[1] & (\beta_2 e^{-i\rho} - a_2)[1] \end{vmatrix} =$$

$$= \frac{1}{2\rho(\alpha_1\beta_2 - a_2\beta_1)(e^{2i\rho} - 1)[1]} \times$$

$$\times \begin{vmatrix} e^{i\rho(x-\xi)}\,[1] & e^{i\rho x}[1] & e^{i\rho(1-x)}[1] \\ \left(\alpha_1 e^{i\rho\xi} - \beta_1 e^{i\rho(1-\xi)}\right)[1] & \left(\alpha_1 - \beta_1 e^{i\rho}\right)[1] & (\beta_1 - \alpha_1 e^{i\rho})[1] \\ \left(a_2 e^{i\rho\xi} - \beta_2 e^{i\rho(1-\xi)}\right)[1] & \left(a_2 - \beta_2 e^{i\rho}\right)[1] & (\beta_2 - a_2 e^{i\rho})[1] \end{vmatrix}. \tag{11}$$

Here we use the notation $[a] = a + o(1)$. Note that if ρ belongs to the domain, $S_0 \cap \Omega_\delta$ then $Re(\ i\rho) \le 0$ and therefore there is a number $m_\delta > 0$ such that $\left|\left(e^{2i\rho} - 1\right)[1]\right| \ge m_\delta$. In addition, in the last determinant all components are bounded, since all exponents present there in the indicator have a negative real part. In the case of $x < \xi$ in the last determinant, the first element of the first row must be replaced by $e^{-i\rho(x-\xi)}\,[1]$.
If ρ belongs to the domain, $S_3 \cap \Omega_\delta$ then $Re(-\ i\rho) \le 0$ and therefore there is a number $m_\delta > 0$ such that $\left|\left(e^{-2i\rho} - 1\right)[1]\right| \ge m_\delta$. Then for $x \ge \xi$ we have

$$G(x, \xi, \rho) = \frac{1}{2\rho(\alpha_1\beta_2 - a_2\beta_1)(1 - e^{-2i\rho})[1]} \times$$

$$\times \begin{vmatrix} e^{-i\rho(x-\xi)}\,[1] & e^{-i\rho(1-x)}[1] & e^{-i\rho x}[1] \\ \left(\alpha_1 e^{-i\rho\xi} - \beta_1 e^{-i\rho(1-\xi)}\right)[1] & \left(\alpha_1 e^{-i\rho} - \beta_1\right)[1] & (\beta_1 e^{-i\rho} - \alpha_1)[1] \\ \left(a_2 e^{-i\rho\xi} - \beta_2 e^{-i\rho(1-\xi)}\right)[1] & \left(a_2 e^{-i\rho} - \beta_2\right)[1] & (\beta_2 e^{-i\rho} - a_2)[1] \end{vmatrix}. \tag{12}$$

In the case of $x < \xi$ in the last determinant, the first element of the first row must be replaced by $e^{i\rho(x-\xi)}\,[1]$.
Taking these considerations into account and expanding the last determinants in (11) and (12), we obtain the following representation for the Green's function

$$G\,(x, \xi, \rho) = \frac{1}{\rho} A_{00}\,(\rho, x, \xi)\, E_0\,(\rho, x, \xi) + \frac{1}{\rho} \sum_{i,k=1}^{2} A_{ik}\,(\rho, x, \xi)\, E_i\,(\rho, x)\, E_k\,(\rho, \xi), \tag{13}$$

$$A_{ik}(\rho,x,\xi)=a_{ik}(\rho)(1+u_i(\rho,x))(1+v_k(\rho,\xi)),\ i,k=0,1,2, \tag{14}$$

where

$$E_0(x,\xi,\rho)=\begin{cases} e^{i\rho(x-\xi)}, & x>\xi,\\ e^{i\rho(\xi-x)}, & x<\xi,\end{cases} \quad E_1(\rho,x)=e^{i\rho x},\ E_2(\rho,x)=e^{i\rho(1-x)}, \tag{15}$$

if $\rho\in S_0\cup\Omega_\delta$ and

$$E_0(x,\xi,\rho)=\begin{cases} e^{-i\rho(x-\xi)}, & x>\xi,\\ e^{-i\rho(\xi-x)}, & x<\xi,\end{cases} \quad E_1(\rho,x)=e^{-i\rho x},\ E_2(\rho,x)=e^{-i\rho(1-x)} \tag{16}$$

if $\rho\in S_3\cup\Omega_\delta$. In addition, the following relations are fulfilled:

$$|a_{ik}(\rho)|\le c,\quad i,k=0,1,2;\quad \rho\in(S_0\cup S_3)\cup\Omega_\delta; \tag{17}$$

$$u_i(\rho,x)\to 0,\ v_k(\rho,\xi)\to 0\ as\rho\to 0\ uniformlyinx,\xi\in[0,1]. \tag{18}$$

From what has been said it immediately follows that the estimate

$$|G(x,\xi,\rho)|=\frac{c}{|\rho|},\rho\in S_0\cup\Omega_\delta,x,\xi\in[0,1]$$

from which we directly obtain (10). The theorem is proved.

Using estimate (10), in [20] the completeness of the eigenfunctions and associated functions of the operator L in the space $L_{p,U}(0,1)$, $1<p<\infty$, is proved. However, in many questions this estimate turns out to be rough and we will now show that on some rays it can be improved.

Theorem 3. *Let ρ take values on a ray on which $Re(\pm i\rho)\neq 0$, and let the operator L be generated by the differential expression $l(y)$ and boundary conditions (2), where conditions A), B) are satisfied. Then for the resolvent of the operator L, for sufficiently large values of $|\rho|$, the following estimate holds:*

$$\left\|R\left(\rho^2\right)\right\|_{L_{p,U}\to L_{p,U}}\le\frac{C}{|\rho|^2}. \tag{19}$$

Proof. Let $f\in L_p(0,1)$. Then using representation (13), for $x\in[0,1]$ we can write

$$\left(R\left(\rho^2\right)f\right)(x)=\int_0^1 G(x,\xi,\rho)f(\xi)\,d\xi=$$

$$= \frac{1}{\rho} \int_0^1 A_{00}(\rho, x, \xi) E_0(\rho, x, \xi) f(\xi) d\xi +$$

$$+ \frac{1}{\rho} \sum_{i,k=1}^{2} \int_0^1 A_{ik}(\rho, x, \xi) E_i(\rho, x) E_k(\rho, \xi) f(\xi) d\xi =$$

$$= \frac{1}{\rho} I_0(\rho) f(x) + \frac{1}{\rho} \sum_{i,k=1}^{2} I_{ik}(\rho) f(x), \tag{20}$$

where denoted by

$$I_0(\rho) f(x) = \int_0^1 A_{00}(\rho, x, \xi) E_0(\rho, x, \xi) f(\xi) d\xi,$$

$$I_{ik}(\rho) f(x) = \int_0^1 A_{ik}(\rho, x, \xi) E_i(\rho, x) E_k(\rho, \xi) f(\xi) d\xi, \ i, k = 1, 2. \tag{21}$$

Let ρ belong to the ray $arg\rho = \theta$, on which $Re(\pm i\rho) \neq 0$. Since $Re(\pm i\rho) = |\rho| \cos\left(\theta \pm \frac{\pi}{2}\right)$, we get $\cos\left(\theta \pm \frac{\pi}{2}\right) \neq 0$. We denote

$$M^{-1} = \min\left\{\cos\left(\theta + \frac{\pi}{2}\right), \cos\left(\theta - \frac{\pi}{2}\right)\right\}.$$

Then we have

$$\frac{1}{|Re(\pm i\rho)|} \leq M \frac{1}{|\rho|}. \tag{22}$$

First, we will estimate the integral operator $I_0(\rho)$. Taking into account (15), we represent it in the form $I_0(\rho) = I_{01}(\rho) + I_{02}(\rho)$, where

$$I_{01}(\rho) f(x) = \int_0^x f(\xi) A_{00}(\rho, x, \xi) e^{i\rho(x-\xi)} d\xi,$$

$$I_{02}(\rho) f(x) = \int_x^1 f(\xi) A_{00}(\rho, x, \xi) e^{i\rho(\xi-x)} d\xi.$$

Using the representation (14) of the function $A_{00}(\rho, x, \xi)$ and estimates (17), (18), we obtain

$$|I_{01}(\rho) f(x)| = \left| \int_0^x f(\xi) A_{00}(\rho, x, \xi) e^{i\rho(x-\xi)} d\xi \right| \leq$$

$$\leq C \int_0^x |f(\xi)| |A_{00}(\rho, x, \xi)| e^{(x-\xi)Re(i\rho)} d\xi \leq$$

$$= C \int_0^x |f(\xi)| e^{\frac{1}{p}(x-\xi)Re(i\rho)} e^{\frac{1}{q}(x-\xi)Re(i\rho)} d\xi \leq$$

$$\leq C\left(\int_0^x |f(\xi)|^p e^{(x-\xi)Re(i\rho)}d\xi\right)^{\frac{1}{p}}\left(\int_0^x e^{(x-\xi)Re(i\rho)}d\xi\right)^{\frac{1}{q}}\leq$$

$$\leq C\left(\int_0^x |f(\xi)|^p e^{(x-\xi)Re(i\rho)}d\xi\right)^{\frac{1}{p}}\left(\frac{1}{|Re(i\xi)|}\left(1-e^{xRe(i\xi)}\right)\right)^{\frac{1}{q}}. \qquad (23)$$

Here and in what follows, we denoted by C positive constants, different in different places, independent of the function $f(x)$ and the variables ρ, x, ξ. Taking into account (22) and the inequality $1-e^{xRe(i\rho)}\leq 1$, which is true for $Re(i\rho)\leq 0$, from (23) we obtain

$$|I_{01}(\rho) f(x)|\leq \frac{\mathrm{C}}{|\rho|^{\frac{1}{q}}}\left(\int_0^x |f(\xi)|^p e^{(x-\xi)Re(i\rho)}d\xi\right)^{\frac{1}{p}}.$$

From here we have

$$\|I_{01}(\rho) f\|_{L_p}^p \leq \frac{C}{|\rho|^{\frac{p}{q}}}\int_0^1 dx\int_0^x |f(\xi)|^p e^{(x-\xi)Re(i\rho)}d\xi =$$

$$= \frac{C}{|\rho|^{\frac{p}{q}}}\int_0^1 |f(\xi)|^p\left(\int_\xi^1 e^{(x-\xi)Re(i\rho)}dx\right)d\xi =$$

$$= \frac{C}{|\rho|^{\frac{p}{q}}}\int_0^1 |f(\xi)|^p\frac{1}{-Re(i\rho)}\left(1-e^{(1-\xi)Re(i\rho)}\right)d\xi\leq$$

$$\leq \frac{C}{|\rho|^{\frac{p}{q}+1}}\int_0^1 |f(\xi)|^p d\xi.$$

Therefore

$$\|I_{01}(\rho) f\|_{L_p}\leq \frac{\mathrm{C}}{|\rho|}\|f\|_{L_p}. \qquad (24)$$

Similarly, we obtain an estimate for the operator $I_{02}(\rho)$:

$$|I_{02}(\rho) f(x)| = \left|\int_x^1 f(\xi) A_{00}(\rho,x,\xi) e^{i\rho(\xi-x)}d\xi\right|\leq$$

$$\leq C\int_x^1 |f(\xi)|\,|A_{00}(\rho,x,\xi)|\, e^{(\xi-x)Re(i\rho)}d\xi\leq$$

$$\leq C\int_x^1 |f(\xi)|\, e^{\frac{1}{p}(\xi-x)Re(i\rho)}e^{\frac{1}{q}(\xi-x)Re(i\rho)}d\xi\leq$$

$$\leq C\left(\int_x^1 |f(\xi)|^p e^{(\xi-x)Re(i\rho)}d\xi\right)^{\frac{1}{p}}\left(\int_x^1 e^{(\xi-x)Re(i\rho)}d\xi\right)^{\frac{1}{q}} \leq$$

$$\leq C\left(\int_x^1 |f(\xi)|^p e^{(\xi-x)Re(i\rho)}d\xi\right)^{\frac{1}{p}}\left(\frac{1}{-Re(i\rho)}\left(1-e^{(1-x)Re(i\rho)}\right)\right)^{\frac{1}{q}} \leq$$

$$\leq \frac{C}{|\rho|^{\frac{1}{q}}}\left(\int_x^1 |f(\xi)|^p e^{(\xi-x)Re(i\rho)}d\xi\right)^{\frac{1}{p}}.$$

From here we have

$$\|I_{02}(\rho) f\|_{L_p}^p \leq \frac{C}{|\rho|^{\frac{p}{q}}}\int_0^1 dx \int_x^1 |f(\xi)|^p e^{(\xi-x)Re(i\rho)}d\xi =$$

$$= \frac{C}{|\rho|^{\frac{p}{q}}}\int_0^1 |f(\xi)|^p \left(\int_0^\xi e^{(\xi-x)Re(i\rho)}dx\right) d\xi =$$

$$= \frac{C}{|\rho|^{\frac{p}{q}}}\int_0^1 |f(\xi)|^p \frac{1}{-Re(i\rho)}\left(1-e^{\xi Re(i\rho)}\right) d\xi \leq$$

$$\leq \frac{C}{|\rho|^{\frac{p}{q}+1}}\int_0^1 |f(\xi)|^p d\xi.$$

Therefore,

$$\|I_{02}(\rho) f\|_{L_p} \leq \frac{\mathrm{C}}{|\rho|}\|f\|_{L_p}. \tag{25}$$

From (24) and (25) we obtain the estimate

$$\|I_0(\rho) f\|_{L_p} \leq \frac{\mathrm{C}}{|\rho|}\|f\|_{L_p}. \tag{26}$$

Let us proceed to the evaluation of the operators $I_{ik}(\rho)$. From formula (21), taking into account (16)-(18), we have

$$|I_{11}(\rho) f(x)| = \left|\int_0^1 A_{11}(\rho, x, \xi) e^{i\rho(x+\xi)} f(\xi) d\xi\right| \leq$$

$$\leq C\int_0^1 |f(\xi)| e^{(x+\xi)Re(i\rho)}d\xi = C\int_0^1 |f(\xi)| e^{\frac{1}{p}(x+\xi)Re(i\rho)} e^{\frac{1}{q}(x+\xi)Re(i\rho)}d\xi \leq$$

$$\leq C\left(\int_0^1 |f(\xi)|^p e^{(x+\xi)Re(i\rho)}d\xi\right)^{\frac{1}{p}}\left(\int_0^1 e^{(x+\xi)Re(i\rho)}d\xi\right)^{\frac{1}{q}} =$$

$$= C\left(\int_0^1 |f(\xi)|^p e^{(x+\xi)Re(i\rho)} d\xi\right)^{\frac{1}{p}} \left(\frac{1}{-Re(i\rho)}\left(e^{(x+1)Re(i\rho)} - e^{xRe(i\rho)}\right)\right)^{\frac{1}{q}} \le$$

$$\le \frac{C}{|\rho|^{\frac{1}{q}}}\left(\int_0^1 |f(\xi)|^p e^{(x+\xi)Re(i\rho)} d\xi\right)^{\frac{1}{p}}.$$

From here we get

$$\|I_{11}(\rho) f\|_{L_p}^p \le \frac{C}{|\rho|^{\frac{p}{q}}} \int_0^1 dx \int_0^1 |f(\rho)|^p e^{(x+\rho)Re(i\rho)} d\xi =$$

$$= \frac{C}{|\rho|^{\frac{p}{q}}} \int_0^1 |f(\xi)|^p \left(\int_0^1 e^{(x+\xi)Re(i\rho)} dx\right) d\xi =$$

$$= \frac{C}{|\rho|^{\frac{p}{q}}} \int_0^1 |f(\xi)|^p \frac{1}{-Re(i\rho)}\left(e^{(1+\xi)Re(i\rho)} - e^{\xi Re(i\rho)}\right) d\xi \le$$

$$\le \frac{C}{|\rho|^{\frac{p}{q}+1}} \int_0^1 |f(\xi)|^p d\xi.$$

Therefore,

$$\|I_{11}(\rho) f\|_{L_p} \le \frac{C}{|\rho|}\|f\|_{L_p}.$$

In a completely similar way we obtain an estimate for the operators $I_{ik}(\rho)$ for other values of i and k :

$$\|I_{ik}(\rho) f\|_{L_p} \le \frac{C}{|\rho|}\|f\|_{L_p}, \; i,k = 1,2 \tag{27}$$

Now the validity of estimate (19) follows from (20), (26) and (27). The theorem is proved.

Remark 1. *From Theorem 3 it follows that, in addition to the positive real semi-axis, all rays on the $\lambda-$plane emanating from the origin are rays of minimal growth of the resolvent of the operator L.*

Remark 2. *Let $f \in L_p(0,1)$ and $g = R(\rho^2) f$. Then on the rays $Re(\pm i\rho) \neq 0$ the following estimate is valid:*

$$\|g'\|_p \le \frac{C}{|\rho|}\|f\|_p.$$

This follows from the fact that for the function $\frac{1}{\rho}\frac{\partial}{\partial x}G(x,\xi,\rho)$ an asymptotic representation similar to (20) is valid.

Remark 3. *It follows from Theorem 3 that for sufficiently large $h > 0$ the operator $A = L + hI$ is positive.*

3. Basis property of eigenfunctions in $L_{p,U}(0,1)$

The main result of the work is the following:

Theorem 4. *When conditions A), B) are satisfied, the eigenfunctions and associated functions of the operator L, generated by the differential expression $l(y)$ and the integral boundary conditions (2) form a basis in the space $L_{p,U}(0,1)$, $1 < p < \infty$.*

Proof. According to Theorem 1, for a given $h > 0$ all zeros ρ_n of the function $\Delta_0(\rho)$, with the possible exception of a finite number of them, are in the strip $\Pi(h) = \{\rho : |Im\ \rho| \leq h\}$. We describe around each point ρ_n a circle $K_n(\varepsilon) = \{\rho : |\rho - \rho_n| < \varepsilon\}$ and form the region $G(\varepsilon) = \bigcup_n K_n(\varepsilon)$. It also follows from the properties of the function $\Delta(\rho)$, described in Theorem 1 that for sufficiently small $\varepsilon > 0$ each circle $K_n(\varepsilon)$ contains one point ρ_n. Let us consider a system of contours Γ_n, with the following properties:

1) Γ_n is a part of a circle of radius R_n, located in the region $\{\rho : Re\rho \geq 0\}$;

2) the radius R_n tend to infinity as $n \to \infty$;

3) between adjacent contours Γ_n and Γ_{n+1} there is only one circle $K_n(\varepsilon)$.

To obtain such a system of contours, it is sufficient to take $R_n = \pi\left(n + \frac{1}{2}\right)$. Let $R(\lambda)$ be the resolvent of the operator L: $R(\lambda) = (L - \lambda I)^{-1}$. We denote by

$$E_n = E(D_n) = \frac{1}{2\pi}\int_{\partial D_n} R(\xi)\,d\xi, \quad S_N(f) = \sum_{n=2}^{N} E_n f.$$

where $D_n-$is the image of the domain $K_n(\varepsilon)$ under the mapping $\lambda = \rho^2$. It is obvious that

$$S_N(f) = \frac{1}{2\pi}\int_{|\lambda|=R_n^2} R(\lambda)\,f d\lambda = \frac{1}{2\pi}\int_{G_N} 2\rho \int_0^1 G(x,\xi,\rho)\,f(\xi)\,d\xi d\rho.$$

Using representation (20) and taking into account that $A_{00}(x,\xi,\rho)-$ is a regular function on the $\rho-$ plane, we obtain

$$S_N(f) = \frac{1}{\pi}\int_{G_N}\int_0^1 \rho G(x,\xi,\rho)\,f(\xi)\,d\xi d\rho =$$

$$= \frac{i}{2p}\int_0^1 f(\xi)\left(\int_{G_N} A_{00}(x,\xi,\rho)\,E_0(x,\xi,\rho)\,d\rho\right)d\xi +$$

$$+\frac{i}{2p}\sum_{i,k=1}^{2}\int_0^1 f(\xi)\left(\int_{G_N} A_{ik}(\rho,x,\xi)\,E_i(\rho,x)\,E_k(\rho,\xi)\,d\rho\right)d\xi =$$

$$= \frac{i}{2p} \sum_{i,k=1}^{2} S_{N,ik}\left(f\right)\left(x\right), \tag{28}$$

where denoted by

$$S_{N,ik}\left(f\right)\left(x\right) = \int_0^1 f\left(\xi\right)\left(\int_{\Gamma_N} A_{ik}\left(\rho, x, \xi\right) E_i\left(\rho, x\right) E_k\left(\rho, \xi\right) d\rho\right) d\xi.$$

Let us show that $S_N\left(f\right)$, $N \geq 1$, are uniformly bounded, i.e. there exists a constant $C > 0$, independent of N and f, such that

$$\left\|S_N\left(f\right)\right\|_p \leq C\|f\|_p. \tag{29}$$

We will prove (29) for each term in (28). Let this term be of the form

$$S_{N,11}\left(f\right)\left(x\right) = \int_0^1 f\left(\xi\right)\left(\int_{\Gamma_N} A_{11}\left(\rho, x, \xi\right) e^{i\rho(x+\xi)} d\rho\right) d\xi.$$

Hence, setting $\rho = R_N e^{i\theta}$, and using the inequality $|\sin\theta\,| \geq \frac{2}{\pi}\left|\theta\right|$, we obtain

$$\left|S_{N,11}\left(f\right)\left(x\right)\right| = C \int_0^1 \left|f\left(\xi\right)\right| \left(\int_{\Gamma_N} e^{(x+\xi)Re(i\rho)}\ \left|d\rho\right|\right) d\xi =$$

$$= CR_N \int_0^1 \left|f\left(\xi\right)\right| \left\{\int_{-\frac{\pi}{2}}^0 e^{(x+\xi)R_N \cos\left(\theta - \frac{\pi}{2}\right)}\ d\theta + \int_0^{\frac{\pi}{2}} e^{(x+\xi)R_N \cos\left(\theta + \frac{\pi}{2}\right)}\ d\theta\right\} d\xi =$$

$$= CR_N \int_0^1 \left|f\left(\xi\right)\right| \left\{\int_{-\frac{\pi}{2}}^0 e^{(x+\xi)R_N \sin\theta}\ d\theta + \int_0^{\frac{\pi}{2}} e^{-(x+\xi)R_N \sin\theta}\ d\theta\right\} d\xi =$$

$$= 2CR_N \int_0^1 \left|f\left(\xi\right)\right| \int_0^{\frac{\pi}{2}} e^{-(x+\xi)R_N \sin\theta}\ d\theta d\xi \leq$$

$$\leq 2CR_N \int_0^1 \left|f\left(\xi\right)\right| \int_0^{\frac{\pi}{2}} e^{-(x+\xi)R_N \frac{2}{\pi}\theta}\ d\theta d\xi =$$

$$= 2CR_N \int_0^1 \left|f\left(\xi\right)\right| \frac{\pi}{2\left(x+\xi\right) R_N} \left(1 - e^{-(x+\xi)R_N}\right) d\xi \leq$$

$$\leq \pi C \int_0^1 \frac{\left|f\left(\xi\right)\right|}{x+\xi} d\xi.$$

Now, applying the Riesz theorem on the boundedness of the Hilbert transform [25, 12], from the last inequality we obtain

$$\|S_{N,11}(f)\|_{L_p} \leq C\|f\|_{L_p}$$

Inequality (29) is proved similarly for the remaining terms in (28). Since the eigenfunctions and associated functions of the operator L form a complete and minimal system in the space $L_{p,U}(0,1)$ [20], the assertion of the theorem follows from the basis criterion.

The theorem is proved.

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