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Improving of MOMA-Plus Hybrid approach by using some new penalty functions

A. Zoungrana, A. Tougma*, K. Somé

Abstract. In this work, we study the effect of four penalty functions, namely the Lagrangian, exponential, logarithmic, and logarithmic-exponential penalties, on the new hybrid MOMA-Plus method for solving multiobjective optimization problems. We conduct a theoretical study on the convergence of the proposed approach. The numerical analysis of the solutions generated on a set of test problems highlights the strengths of each penalty function.

Key Words and Phrases: Multiobjective Method, Penalty function, Alienor, MOMA-Plus.

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1. Introduction

In the field of constrained multi-objective optimization, as in constrained single-objective optimization, there are numerous methods for handling these constraints. In recent years, most of the algorithms for solving these problems have adopted an approach based on penalty functions. Penalty methods transform the constrained multi-objective optimization problem into an unconstrained one. The constraints are integrated into one of the objective functions, with a penalty coefficient. This makes it possible to obtain solutions identical to those of the original problem.

Among penalty functions, we find the exponential penalty function, developed in [1, 2, 3], the logarithmic penalty function, mentioned in [4, 5], as well as the logarithmic-exponential penalty function, discussed in [6, 7, 8].

Several methods have been proposed in the literature using penalty functions for constraint management [1, 4, 6], including the MOMA-Plus method, another version of the MOMA method [9], which is a clever combination of penalty techniques and Alienor transformation. Another version, called the hybrid MOMA-Plus, was proposed in [10].

*Corresponding author.

In this work, we use four penalty functions, namely the Lagrangian, exponential, logarithmic, and logarithmic-exponential penalties, on the hybrid MOMA-Plus method to conduct a theoretical and numerical study on the effect of these penalty functions on the hybrid MOMA-Plus method. The main contributions and highlights of this article are as follows:

- proposal of a new approach to solving multi-objective optimization problems;
- theoretical study of the convergence of solutions generated by these proposed methods;
- numerical validation of the theoretical results on ten (10) test problems;
- comparative study between these proposed methods and existing methods in the literature.

In the continuation of this work, we present in Section 2 the preliminaries, where we expose the basic concepts, properties, and definitions concerning multi-objective optimization. In Section 3, we will present the proposed approach followed by the theoretical and numerical study. We conclude in Section 4 with conclusions and some remarks on future research.

2. Preliminaries

2.1. Basic Concepts

Let us consider the multi-objective optimization problem defined as follows [11, 12]:

$$\begin{array}{l} \min F(x) = \Big(f_1(x), f_2(x), \ldots, f_m(x)\Big)\; m \geq 2 \\ s.t: \left\{ \begin{array}{l} g_j(x) \leqslant 0,\; j = 1, \ldots, p \\ x \in \mathbb{R}^n\; ; \end{array} \right. \end{array} \tag{1}$$

where:

- $x = (x_1, x_2 \ldots . x_n)$ denotes the vector of n decision variables;
- f_i, $i = \overline{1,m}$, the objective functions;
- g_j, $j = \overline{1,p}$, the constraints associated with the optimization of f_i, $i = \overline{1,m}$.

We denote $\mathbf{D} = \{\, x \in \mathbb{R}^n : g_j(x) \leq 0;\; j = \overline{1,p}\}$ as the feasible domain of problem (1).

With the following definitions, we characterize a Pareto optimal and weakly optimal solution.

Definition 1 ([9, 10, 13]). *A point $x^* \in \mathbf{D}$ is said to be a weakly efficient or weakly Pareto optimal solution of problem (1) if and only if there does not exist another $x \in \mathbf{D}$ such that:*

$$f_i(x) < f_i(x^*), \quad \forall i = \overline{1,m}.$$

Definition 2 ([9, 14, 15]). *A point $x^* \in \mathbf{D}$ is said to be an efficient or Pareto optimal solution of problem (1) if and only if there does not exist an $x \in \mathbf{D}$ such that $f_i(x) \leq f_i(x^*)$, $\forall i = \overline{1,m}$, and for at least one $k \in \{1, \cdots, m\}$, we have $f_k(x) < f_k(x^*)$.*

We note that $Y = \{(x, f(x)), x \in \mathbf{D}\}$ is called the Pareto front.

2.2. Penalty Functions

In this section, we will define the different penalty functions.

- Lagrangian-based penalty

  Definition 3 ([16, 17]). *The Lagrangian-based penalty function applied to problem (1) is defined by*

$$P_q(x) = \eta \sum_{j=1}^{p} [g_j(x) + \mid g_j(x) \mid] \tag{2}$$

  where η is a sufficiently large positive real number.

- Exponential penalty

  Definition 4 ([1, 2, 3]). *The exponential penalty function applied to problem (1) is defined by*

$$P_q(x) = \frac{1}{\rho_q} \sum_{j=1}^{p} \vartheta[\rho_q g_j(x)] \tag{3}$$

  where g_j, $j = \overline{1,p}$, are the constraints of problem (1), and ρ_q is the penalty coefficient satisfying

$$\lim_{q \longmapsto +\infty} \rho_q = +\infty$$

  and ϑ is a real-valued function defined by:

$$\vartheta(t) = exp(t) - 1, \;\; t \in \mathbb{R}$$

  Using this penalty function, problem (1) is reformulated as follows:

$$\begin{cases} \min\{f_i(x) + P_q(x)\}, \; i = 1, 2, ..., m \\ x \in \mathbb{R}^n \; ; \end{cases} \tag{4}$$

- Logarithmic penalty

Definition 5 ([4, 5]). *The logarithmic penalty function related to problem (1) is defined by*

$$P_q(x) = \sigma_q \sum_{j=1}^{m} [\ln((g_j(x))^2 + 1^j)], \quad j = \overline{1,m} \tag{5}$$

where g_j, $j = \overline{1,p}$, are the constraints of problem (1), with σ_q being the penalty coefficient satisfying

$$\lim_{q \longmapsto +\infty} \sigma_q = +\infty$$

Applying this penalty function to problem (1), we obtain the following formulation:

$$\begin{cases} \min\{f_i(x) + P_q(x)\}, \ i = 1,2,...,m \\ x \in \mathbb{R}^n \ ; \end{cases} \tag{6}$$

Definition 6 ([4, 5]). *A feasible solution $x^* \in \mathbf{D}$ is said to be an optimal solution to the penalized optimization problem (6) if there does not exist any $x \in \mathbf{D}$ such that $f_i(x) + P_q(x) < f_i(x) + P_q(x^*) \forall i = \overline{1,m}$*

- Logarithmic-exponential penalty

Definition 7 ([6, 7, 18]). *The logarithmic-exponential penalty function related to problem (1) is defined by*

$$P_q(x) = \frac{2}{\mu_q} \sum_{j=1}^{p} \ln \left[1 + \exp[\mu_q \lambda_j g_j(x)] \right] \tag{7}$$

where g_j, $j = \overline{1,m}$ are the constraints of problem (1) with μ_q being the penalty coefficient satisfying

$$\lim_{q \longmapsto +\infty} \mu_q = +\infty$$

By applying this penalty function to problem (1), we obtain the following formulation:

$$\begin{cases} \min\{f_i(x) + P_q(x)\}, \ i = 1,2,...,m \\ x \in \mathbb{R}^n \ ; \end{cases} \tag{8}$$

2.3. Aliénor transformation

If we consider a function with n continuous variables defined on $\mathbb{R}^n$, a reducing transformation (called Aliénor transformation) allows expressing all the variables as a function of a unique variable θ such that

$$x_i = h_i(\theta)\ \theta \in \mathbb{R}_+.$$

This transformation was invented by professors Yves Cherruault, Arthur Guillez, and Blaise Somé [9]. It reduces any multi-variable function into a single-variable function via the Archimedes spiral.

Definition 8 ([19]). *Let f be a function with n variables. A reducing transformation associated with the function f is any transformation that reduces the function f to a single-variable function given by the following definition:*

Definition 9 ([14]). *A subset $\mathsf{S} \in \mathbb{R}^n$ is said to be α-dense in $\mathbb{R}^n$ if*

$$\forall M \in \mathbb{R}^n,\ \exists M' \in \mathsf{S} \text{ such that } d(M,M') \leq \alpha.$$

Thus, if we consider a function with n variables $f(x_1,x_2,...,x_n)$, continuous and defined on $\mathbb{R}^n$, a reducing transformation allows expressing all the variables as a function of a unique variable θ.

$$x_i = h_i(\theta); i = \overline{1,m} and \theta \in [0,\theta_{\max}]$$

hence $f(x_1,x_2,...,x_n)$ becomes $f(h_1(\theta),h_2(\theta),...,h_n(\theta))$. This reducing transformation has several variants, and the one we are interested in is the Konfé-Cherruault transformation [17].
This reducing transformation is defined by $h(\theta) = \big(h_1(\theta),h_2(\theta),...,h_n(\theta)\big)$ where $h_i(\theta)$ are defined as follows:

$$x_i = h_i(\theta) = \frac{1}{2}\big[(b_i - a_i)\cos(\omega_i\theta + \varphi_i) + b_i + a_i\big], i = \overline{1,n}; \tag{9}$$

where $(\omega_i)_{i=\overline{1,n}}$ and $(\varphi_i)_{i=\overline{1,n}}$ are slowly increasing sequences, $x_i \in [a_i,b_i]$ and $\theta \in [0,\theta_{\max}]$ with $\theta^1 = \dfrac{2\pi - \varphi_1}{\omega_1}$ and $\theta_{\max} = \dfrac{(b_1 - a_1)\theta^1 + (b_1 + a_i)}{2}$.

2.4. Performance measurement

Performance measures are used to study the efficiency of a new multi-objective optimization method compared to existing solution methods. In our work, we will use two performance measures [15], namely the convergence metric $\overline{\gamma}$ and the average diversity metric $\overline{\Delta}$, since our test problems have an analytical front.

2.4.1. Convergence metric $\overline{\gamma}$

The convergence metric $\overline{\gamma}$ measures the average distance between the approximate solutions and those on the Pareto front (analytical front) [20, 21]. It is defined as follows [15, 22]:

$$\overline{\gamma} = \frac{1}{n}(\sum_{i=1}^{n} d_i^p)^{\frac{1}{p}} \tag{10}$$

The parameter d_i is the Euclidean distance (in the objective space) between the i-th solution and the closest point on the Pareto front:

$$d_i = \min_{k=1}^{|P|} \sqrt{\sum_{m=1}^{M} (f_m^{(i)} - f_m^{*(k)})^2}$$

where $f_m^{*(k)}$ is the m-th objective function value of the k-th solution on the Pareto front.

2.4.2. Average diversity metric $\overline{\Delta}$

The average diversity metric $\overline{\Delta}$ measures the dispersion of the obtained solutions over the analytical front. It is defined by [23, 24]:

$$\overline{\Delta} = \frac{\sum_{m=1}^{M} d_m^e + \sum_{i=1}^{n-1} |d_i - \overline{d}|}{\sum_{m=1}^{M} d_m^e + (n-1)\overline{d}}. \tag{11}$$

Where the parameter d_i is the Euclidean distance between neighboring solutions with the average value $\overline{d}$, and the parameter d_m^e is the distance between the extreme solutions of the Pareto analytical front and the approximate front.

2.4.3. Performance Profiles

The performance of different algorithms can be compared using performance profiles and data. Performance profiles are represented by a graph showing the cumulative distribution function $\rho(\alpha)$, which captures the performance ratios of different algorithms. Given the previously defined sets, let $t_{p,s}$ denote the performance measure (e.g., computation time) of algorithm $s \in S$ on problem $p \in P$. The performance ratio $r_{p,s}$ is defined as:

$$r_{p,s} = \frac{t_{p,s}}{\min\left\{t_{p,s'} \mid s' \in S\right\}},$$

where s' iterates over all algorithms in S.

The performance profile ([21, 15, 29]) of an algorithm $s \in S$ corresponds to the proportion of problems where its performance ratio does not exceed α ($\alpha \geq 1$):

$$\rho(\alpha) = \frac{\left|\left\{p \in P \mid r_{p,s} \leq \alpha\right\}\right|}{|P|}. \tag{12}$$

For sufficiently large α, $\rho(\alpha)$ represents the proportion of problems where algorithm s satisfies the convergence criterion.

3. Main Results

3.1. Principle

The principle is to convert a multi-objective optimization problem with constraints into a single-objective optimization problem with constraints using the ε-constraint approach. This single-objective problem with constraints is then transformed into a single-objective problem without constraints by using one of the following penalty functions: Lagrangian, exponential, logarithmic, and logarithmic-exponential. Then, we use the Alienor method to transform a multi-variable, unconstrained single-objective problem into a single-variable, unconstrained problem. Finally, the Nelder-Mead simplex method is applied to solve the single-variable, unconstrained problem, and the original solutions are reconstructed.

3.2. Theoretical Description

In this section, we present the different steps of the hybrid MOMA-plus method.

Step I: Aggregation

At this step of the algorithm, one of the objective functions of problem (1) is selected to be optimized, and the others are transformed into constraints. The approach is as follows:

- choose one objective function f_k to prioritize for optimization;
- choose an initial vector of constraints ε_i, $\varepsilon_{i \geq 0}$; $i \neq k$ with $\varepsilon_i \in \left[\min\limits_{x \in \mathbf{D}} f_i(x), \max\limits_{x \in \mathbf{D}} f_i(x)\right]$, $i = \overline{1,m}$;
- transform the other objectives into inequality constraints $(f_i \leq \varepsilon_i,\ i \neq k)$, $i = \overline{1,m}$.

Applying this approach to problem (1) leads to the following single-objective optimization problem:

$$\begin{cases} \min\{f_k(x)\} \\ subject\ to: \\ \quad f_i(x) \leq \varepsilon_i, i = \overline{1,m}, i \neq k \\ \quad g_j(x) \leqslant 0, j = \overline{1,p} \\ \quad x \in \mathbb{R}^n. \end{cases} \tag{13}$$

Step II: Penalization

We use the four (4) penalty functions defined in section 2.2 to transform problem (13) into an unconstrained single-objective problem.

NB: In this subsection, we consider $\mathbf{Z}^*$ as the set of solutions to problem (13), and $\mathbf{Z}_q^*$ as the set of solutions to the problems with the different penalty functions applied.

★ **Penalty derived from the Lagrangian**

From the definition (3), applied to problem (13), we obtain:

$$\begin{cases} \min\{L(x)\} = f_k(x) + \eta\left[\sum_{j=1}^{p}(g_j(x) + |g_j(x)| + \sum_{i=1}^{m}(f_i(x) - \varepsilon_i) + |f_i(x) - \varepsilon_i|)\right] \\ subject\ to: \\ \quad x \in \mathbb{R}^n \qquad \eta \geq \frac{M - f_k(x)}{\sum_{j=1}^{p} g_j(x) + \sum_{i=1}^{m}(f_i(x) - \varepsilon_i)} \\ \quad M = \max_{x \in \mathbf{D}} f_k(x) \end{cases} \tag{14}$$

Theorem 1. *All solutions to problem 14 are solutions to problem 1 and vice versa.*

★ **Exponential Penalties**

We use the definition (4) to obtain:

$$\begin{cases} \min\left\{L(x) = f_k(x) + \frac{1}{\rho_q}\left[\sum_{j=1}^{p}\vartheta[\rho_q g_j(x)] + \sum_{\substack{i=1 \\ i\neq k}}^{m}\vartheta[\rho_q(f_i(x) - \varepsilon_i)]\right]\right\} \\ \text{subject to:} \\ \quad x \in \mathbb{R}^n \end{cases} \tag{15}$$

where ρ_q is the penalty coefficient satisfying

$$\lim_{q \longmapsto +\infty} \rho_q = +\infty$$

and ϑ is a real-valued function defined by:

$$\vartheta(t) = exp(t) - 1, \ \ t \in \mathbb{R}$$

Lemma 1. *Let* $\mathbf{D}$ *be the set of admissible solutions of problem* (1). *For any* $\varepsilon_i > 0$ *where* $i = \overline{1,m}$, *we have:*

1. *If* $x \in \mathbf{D}$, *then*

$$\lim_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q}\left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(x)] + \sum_{i=1}^{m} \vartheta[\rho_q(f_i(x) - \varepsilon_i)]\right] = 0,\ \forall\ \varepsilon_i > 0.$$

2. *If* $x \notin \mathbf{D}$ *then*

$$\lim_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q}\left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(x)] + \sum_{i=1}^{m} \vartheta[\rho_q(f_i(x) - \varepsilon_i)]\right] = +\infty,\ \forall\ \varepsilon_i > 0.$$

Proof.
Let $x \in \mathbf{D}$. We have $g_j(x) < 0$ and $f_i(x) - \varepsilon_i \leq 0, \forall\ \varepsilon_i > 0, i \neq k, i = \overline{1,m}, j = \overline{1,p}$, which implies $\lim\limits_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q}\sum\limits_{j=1}^{p}\left(e^{\rho_q g_j(x)} - 1\right) = 0$ and $\lim\limits_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q}\sum\limits_{i=1}^{m}\left(e^{\rho_q(f_i(x)-\varepsilon_i)} - 1\right) = 0$.
Thus,

$$\lim_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q}\left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(x)] + \sum_{i=1}^{m} \vartheta[\rho_q(f_i(x) - \varepsilon_i)]\right] = 0,\ \forall\ \varepsilon_i > 0.$$

If $x \notin \mathbf{D}$, we have $g_j(x) > 0$ and $f_i(x) - \varepsilon_i > 0$ for $i \neq k$, $i = \overline{i,m}$ and $j = \overline{1,p}$, then $\lim\limits_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q}\sum\limits_{j=1}^{p}\left(e^{\rho_q g_j(x)} - 1\right) = +\infty$ and $\lim\limits_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q}\sum\limits_{i=1}^{m}\left(e^{\rho_q(f_i(x)-\varepsilon_i)} - 1\right) = +\infty, \forall \varepsilon_i > 0$ since $\lim\limits_{\rho_q \longmapsto +\infty} \sum\limits_{j=1}^{p}\left(e^{\rho_q g_j(x)} - 1\right) = +\infty$ and $\lim\limits_{\rho_q \longmapsto +\infty} \sum\limits_{i=1}^{m}\left(e^{\rho_q(f_i(x)-\varepsilon_i)} - 1\right) = +\infty,\ \forall\ \varepsilon_i > 0$.
Therefore,

$$\lim_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q}\left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(x)] + \sum_{i=1}^{m} \vartheta[\rho_q(f_i(x) - \varepsilon_i)]\right] = +\infty,\ \forall\ \ \varepsilon_i > 0.$$

Theorem 2. *Let* S_q *be a sequence of numbers such that* $S_q \subset \mathbb{R}^n$ *where* $q \in \mathbb{N}$.
If
$\overline{\lim\limits_{k \longmapsto +\infty}} S_q = \left\{x \in \mathbb{R}^n : x \in S_q \text{ for an infinite number of } q \in \mathbb{N}\right\}$ *then* $\overline{\lim\limits_{q \longmapsto +\infty}} (\mathbf{Z}_q^* \backslash \mathbf{Z}^*) = \emptyset$

Proof.
Assuming by contradiction that $\overline{\lim\limits_{q \longmapsto +\infty}} (\mathbf{Z}_q^* \backslash \mathbf{Z}^*) \neq \emptyset$, then there exists a subset $\mathbf{Z}_{q_r}^*$, $r = 1, 2, \ldots$ such that $x' \in \overline{\lim\limits_{q \longmapsto +\infty}} (\mathbf{Z}_{q_r}^* \backslash \mathbf{Z}^*)$.

This implies $\exists q_0 \geq 0$ such that for $q_r \geq q_0$ we have $x^{'} \in \mathbf{Z}_q^* \backslash \mathbf{Z}^*$.
Since $x^{'} \in \mathbf{Z}^*$, then

$$L(x^{'}) < L(y) \; \forall y \in \mathbf{Z}^*. \tag{16}$$

If $x^{'} \in \mathbf{D}$ and since $x^{'} \notin \mathbf{Z}^*$, then $\exists y \in \mathbf{D}$ such that $f_k(y) < f_k(x^{'}) \; \forall \; k \neq i, \; i = \overline{1,m}$.
Since $x^{'}, y \in \mathbf{D}$ then according to Lemma 1 we have:
$\overline{\lim}_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q} \left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(x^{'})] + \sum_{i=1}^{m} \vartheta[\rho_q(f_i(x^{'}) - \varepsilon_i)] \right] = 0, \; \forall \; \varepsilon_i > 0$ and

$$\overline{\lim}_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q} \left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(y)] + \sum_{i=1}^{m} \vartheta[\rho_q(f_i(y) - \varepsilon_i)] \right] = 0, \; \forall \; \varepsilon_i > 0$$

$\Rightarrow \exists \; q_0 \in \mathbb{N}, \; q_r > q_0,$

$$f_k(y) + \frac{1}{\rho_q} \left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(y)] + \sum_{i=1}^{m} \vartheta[\rho_q(f_i(y) - \varepsilon_i)] \right] < f_k(x^{'}) + \frac{1}{\rho_q} \left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(x^{'})] + \sum_{i=1}^{m} \vartheta[\rho_q(f_i(x^{'}) - \varepsilon)] \right], \; \forall \; \varepsilon_i > 0$$

$$\Rightarrow L(y) < L(x^{'})$$

Which contradicts equation (16).
If $x^{'} \notin \mathbf{D}$, then $\exists y \in \mathbf{Z}^*$ such that $f_k(y) < f_k(x^{'}) \; \forall \; k \neq i, \; i = \overline{1,m}$.
Since $x^{'}, y \notin \mathbf{D}$ then according to Lemma 1 we have:

$$\overline{\lim}_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q} \left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(x^{'})] + \sum_{i=1}^{m} \vartheta[\rho_q(f_i(x^{'}) - \varepsilon_i)] \right] = +\infty, \; \forall \; \varepsilon_i > 0 \text{ and}$$

$$\overline{\lim}_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q} \left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(y)] + \sum_{i=1}^{m} \vartheta[\rho_q(f_i(y) - \varepsilon)] \right] = +\infty, \; \forall \; \varepsilon_i > 0.$$

There exists $q_0 \in \mathbb{N}, \; q > q_0$

$$f_k(y) + \frac{1}{\rho_q} \left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(y)] + \sum_{i=1}^{m} \vartheta[\rho_q(f_i(y) - \varepsilon)] \right] < f_k(x^{'}) + \frac{1}{\rho_q} \left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(x^{'})] + \sum_{i=1}^{m} \vartheta[\rho_q(f_i(x^{'}) - \varepsilon_i)] \right]$$

$$\Rightarrow L(y) < L(x^{'})$$

Which is absurd since $x^{'} \in \mathbf{Z}_q^*$ hence $\overline{\lim}_{q \longmapsto +\infty} (\mathbf{Z}_q^* \backslash \mathbf{Z}^*) = \emptyset$.

Theorem 3. *Let S_q be a sequence of numbers such that $S_q \subset \mathbb{R}^n$ where $q \in \mathbb{N}$. If* $\underline{\lim}_{q \longmapsto +\infty} S_q = \left\{ x \in \mathbb{R}^n : x \in S_q \;\; for\ a\ finite\ number\ q \in \mathbb{N}, \right\}$ *then* $\underline{\lim}_{q \longmapsto +\infty} (\mathbf{Z}_q^* \backslash \mathbf{Z}^*) = \emptyset$

Proof.
Suppose by contradiction that $\underline{\lim}_{q \longmapsto +\infty} (\mathbf{Z}_q^* \backslash \mathbf{Z}^*) \neq \emptyset$.
In other words, $\exists x' \in \underline{\lim}_{q \longmapsto +\infty} (\mathbf{Z}_q^* \backslash \mathbf{Z}^*)$, and there exists an index q_0 such that for $q \geq q_0$, $x' \in \mathbf{Z}_q^* \backslash \mathbf{Z}^*$.
This implies $x' \in \mathbf{Z}^*$ and $x' \notin \mathbf{Z}^*$ starting from index n_0.
Thus,

$$\forall y \in \mathbf{Z}^*,\ L(x') < L(y). \tag{17}$$

If $x' \in \mathbf{D}$, as $x' \notin \mathbf{Z}^*$, then $\exists y \in \mathbf{D}$ such that $f_k(y) < f_k(x')\ \forall\ k \neq i,\ i = \overline{1,m}$
Since $x', y \in \mathbf{D}$, by Lemma 1, we have:
$\underline{\lim}_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q} \left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(x')] + \sum_{i=1}^{m} \vartheta[\rho_q (f_i(x') - \varepsilon_i)] \right] = 0$ and
$\underline{\lim}_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q} \left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(y)] + \sum_{i=1}^{m} \vartheta[\rho_q (f_i(y) - \varepsilon)] \right] = 0$
$\Rightarrow \exists\ q_0 \in \mathbb{N},\ q_r > q_0,$

$$f_k(y) + \frac{1}{\rho_q} \left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(y)] + \sum_{i=1}^{m} \vartheta[\rho_q (f_i(y) - \varepsilon)] \right] < f_k(x') + \frac{1}{\rho_q} \left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(x')] + \sum_{i=1}^{m} \vartheta[\rho_q (f_i(x') - \varepsilon_i)] \right]$$

$$\Rightarrow L(y) < L(x')$$

which contradicts equation (17).
If $x' \notin \mathbf{D}$, $\exists y \in \mathbf{D}$, $f_k(y) < f_k(x')\ \forall\ k \neq i,\ i = \overline{1,m}$
Since $x' \notin \mathbf{D}$, by Lemma 1 we have:
$\underline{\lim}_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q} \left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(x')] + \sum_{i=1}^{m} \vartheta[\rho_q (f_i(x') - \varepsilon_i)] \right] = +\infty$ and
$\underline{\lim}_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q} \left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(y)] + \sum_{i=1}^{m} \vartheta[\rho_q (f_i(y) - \varepsilon_i)] \right] = 0$
$\Rightarrow$there exists $n_0 \in \mathbb{N}$, $n \geq n_0$
$\frac{1}{\rho_q} \left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(y)] + \sum_{i=1}^{m} \vartheta[\rho_q (f_i(y) - \varepsilon_i)] \right] < \frac{1}{\rho_q} \left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(x')] + \sum_{i=1}^{m} \vartheta[\rho_q (f_i(x') - \varepsilon)] \right]$

$$\Rightarrow f_k(y)+\frac{1}{\rho_q}\left[\sum_{j=1}^{p}\vartheta[\rho_q g_j(y)]+\sum_{i=1}^{m}\vartheta[\rho_q(f_i(y)-\varepsilon_i)]\right]<f_k(x^{'})+\frac{1}{\rho_q}\left[\sum_{j=1}^{p}\vartheta[\rho_q g_j(x^{'})]+\sum_{i=1}^{m}\vartheta[\rho_q(f_i(x^{'})-\varepsilon_i)]\right]$$

$$\Rightarrow \; L(y)<L(x^{'})$$

which is absurd since $x^{'}$ is the optimal solution of problem (17), hence $\underline{\lim}_{q\longmapsto+\infty}(\mathbf{Z}_q^*\backslash\mathbf{Z}^*)=\emptyset$.

★ Logarithmic Penalties

Using Equation (13) and Definition (4), we have:

$$\begin{cases} \min\left\{L(x)=f_k(x)+\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(x))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(x)-\varepsilon_i))^2+1^i)]\right]\right\} \\ subject\ to: \\ \quad i\neq k \\ \quad x\in\mathbb{R}^n. \end{cases} \tag{18}$$

where σ_q is the penalty coefficient satisfying

$$\lim_{q\longmapsto+\infty}\sigma_q=+\infty$$

Lemma 2. *Let* $\mathbf{D}$ *be the set of admissible solutions of problem* (1), $\forall\ \varepsilon_i>0$, $i=\overline{1,m}$, *we have:*

1. *If* $x\in\mathbf{D}$, *then*
$$\lim_{\sigma_q\longmapsto+\infty}\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(x))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(x)-\varepsilon_i))^2+1^i)]\right]=0,\ \forall\varepsilon_i>0.$$
2. *If* $x\notin\mathbf{D}$ *then*
$$\lim_{\sigma_q\longmapsto+\infty}\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(x))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(x)-\varepsilon_i))^2+1^i)]\right]=+\infty,\ \forall\varepsilon_i>0.$$

Proof.
Let $x\in\mathbf{D}$, we have $g_j(x)\leq 0$ $j=\overline{1,p}$ and $f_i(x)-\varepsilon_i\leq 0$ $\forall\ i=\overline{1,m}$ *and* $\varepsilon_i>0$, hence

$$\lim_{\sigma_q \longmapsto +\infty} \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(x))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(x) - \varepsilon_i))^2 + 1^i)] \right] = 0 \; \forall \varepsilon_i > 0$$

since $j\ln(1) = 0, \; \forall \; j = \overline{1,p}$ and $i\ln(1) = 0 \; \forall \; i = \overline{1,m}$ now assume $x \notin \mathbf{D} \Rightarrow g_j(x) > 0$ $j = \overline{1,p}$ and $f_i(x) - \varepsilon_i > 0 \; \forall \; i = \overline{1,m}$ *and* $\varepsilon_i > 0$ thus

$$\lim_{\sigma_q \longmapsto +\infty} \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(x))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(x) - \varepsilon_i))^2 + 1^i)] \right] = +\infty, \; \forall \varepsilon_i > 0.$$

Theorem 4. *Let us suppose that S_q is a sequence of numbers such that $S_q \subset \mathbb{R}^n$ where $q \in \mathbb{N}$.*
If $\overline{\lim}_{k \longmapsto +\infty} S_q = \left\{ x \in \mathbb{R}^n : x \in S_q \quad for \; an \; infinite \; number \; of \; q \in \mathbb{N}, \right\}$ *then* $\overline{\lim}_{q \longmapsto +\infty} (\mathbf{Z}_q^* \backslash \mathbf{Z}^*) = \emptyset$

Proof.
Assuming by contradiction that $\overline{\lim}_{q \longmapsto +\infty} (\mathbf{Z}_q^* \backslash \mathbf{Z}^*) \neq \emptyset$, then there exists a subset $\mathbf{Z}_{q_r}^*$, $r = 1, 2, \ldots$ such that $x' \in \overline{\lim}_{q \longmapsto +\infty} (\mathbf{Z}_{q_r}^* \backslash \mathbf{Z}^*)$.
Thus, $\exists q_0 \geq 0$ such that for $q_r \geq q_0$ we have $x' \in \mathbf{Z}_q^* \backslash \mathbf{Z}^*$.
Since $x' \in \mathbf{Z}^*$, then

$$L(x') < L(y) \; \forall y \in \mathbf{Z}^* \tag{19}$$

If $x' \in \mathbf{D}$, since $x' \notin \mathbf{Z}^*$, then $\exists y \in \mathbf{D}$ such that $f_k(y) < f_k(x') \; \forall \; k \neq i, \; i = \overline{1,m}$
Since $x', y \in \mathbf{D}$, then by Lemma 2:

$$\overline{\lim}_{\sigma_q \longmapsto +\infty} \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(x'))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(x') - \varepsilon_i))^2 + 1^i)] \right] = 0, \; \forall \varepsilon_i > 0 \text{ and}$$
$$\overline{\lim}_{\sigma_q \longmapsto +\infty} \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(y))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(y) - \varepsilon_i))^2 + 1^i)] \right] = 0, \; \forall \varepsilon_i > 0.$$

There exists $q_0 \in \mathbb{N}$, $q > q_0$

$$\Rightarrow \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(y))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(y) - \varepsilon_i))^2 + 1^i)] \right] < \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(x'))^2 + 1^j)] \right.$$
$$\left. + \sum_{i=1}^{m} [\ln(((f_i(x') - \varepsilon_i))^2 + 1^i)] \right], \; \forall \; \varepsilon_i > 0$$

$$f_k(y)+\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(y))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(y)-\varepsilon_i))^2+1^i)]\right]<f_k(x^{'})+\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(x^{'}))^2+1^j)]\right.$$
$$\left.+\sum_{i=1}^{m}[\ln(((f_i(x^{'})-\varepsilon_i))^2+1^i)]\right],\forall\varepsilon_i>0$$

$$\Rightarrow L(y)<L(x^{'})$$

which contradicts Equation (19)
If $x^{'}\notin\mathbf{D}$, then $\exists y\in\mathbf{Z}$ such that $f_k(y)<f_k(x^{'})\ \forall\ k\neq i,\ i=\overline{1,m}$
Since $x^{'},y\notin\mathbf{D}$, then by Lemma 2 we have:
$\overline{\lim}_{\sigma_q\longmapsto+\infty}\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(x^{'}))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(x^{'})-\varepsilon_i))^2+1^i)]\right]=+\infty,\ \forall\varepsilon_i>0$ and
$\overline{\lim}_{\sigma_q\longmapsto+\infty}\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(y))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(y)-\varepsilon_i))^2+1^i)]\right]=+\infty,\ \forall\varepsilon_i>0.$
There exists $q_0\in\mathbb{N}$, $q>q_0$

$$\Rightarrow\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(y))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(y)-\varepsilon_i))^2+1^i)]\right]<\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(x^{'}))^2+1^j)]+\right.$$
$$\left.\sum_{i=1}^{m}[\ln(((f_i(x^{'})-\varepsilon_i))^2+1^i)]\right],\ \forall\ \varepsilon_i>0$$

$$f_k(y)+\sigma_q\left[\sum_{j=1}^{p}\ln\big((g_j(y))^2+1\big)+\sum_{i=1}^{m}\ln\big((f_i(y)-\varepsilon_i)^2+1\big)\right]<f_k(x')+\sigma_q\left[\sum_{j=1}^{p}\ln\big((g_j(x'))^2+1\big)\right.$$
$$\left.+\sum_{i=1}^{m}\ln\big((f_i(x')-\varepsilon_i)^2+1\big)\right],\quad\forall\ \varepsilon_i>0$$

$$\Rightarrow L(y)<L(x^{'})$$

which contradicts Equation (19) since $x^{'}\in\mathbf{Z}_q^*$, hence $\overline{\lim}_{q\longmapsto+\infty}(\mathbf{Z}_q^*\backslash\mathbf{Z}^*)=\emptyset$.

Theorem 5. *Let us suppose that S_q is a sequence of numbers such that $S_q\subset\mathbb{R}^n$ where $q\in\mathbb{N}$.*
If $\underline{\lim}_{q\longmapsto+\infty}S_q=\{x\in\mathbb{R}^n:x\in S_q\ for\,a\,finite\,number\,of\ q\in\mathbb{N},\}$ *then* $\underline{\lim}_{q\longmapsto+\infty}(\mathbf{Z}_q^*\backslash\mathbf{Z}^*)=\emptyset$

Proof. Suppose, by contradiction, that $\underline{\lim}_{q\longmapsto+\infty}(\mathbf{Z}_q^*\backslash\mathbf{Z}^*)\neq\emptyset$.
In other words, there exists $x^{'}\in\underline{\lim}_{q\longmapsto+\infty}(\mathbf{Z}_q^*\backslash\mathbf{Z}^*)$ and an index n_0 such that for $n\geq n_0$,

$x^{'} \in \mathbf{Z}_q^* \backslash \mathbf{Z}^*$.
This leads to $x^{'} \in \mathbf{Z}^*$ and $x^{'} \notin \mathbf{Z}^*$ from the rank n_0.
Therefore:

$$\forall y \in \mathbf{Z}^*,\ L(x^{'}) < L(y). \tag{20}$$

If $x^{'} \in \mathbf{D}$, since $x^{'} \notin \mathbf{Z}^*$, then $\exists y \in \mathbf{D}$ such that $f_k(y) < f_k(x^{'}),\ \forall\ k \neq i,\ i = \overline{1,m}$
Since $x^{'}, y \in \mathbf{D}$, then by Lemma 2 we have:

$$\underline{\lim}_{\sigma_q \longmapsto +\infty} \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(x^{'}))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(x^{'}) - \varepsilon_i))^2 + 1^i)] \right] = 0,\ \forall \varepsilon_i > 0 \text{ and}$$
$$\underline{\lim}_{\sigma_q \longmapsto +\infty} \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(y))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(y) - \varepsilon_i))^2 + 1^i)] \right] = 0,\ \forall \varepsilon_i > 0.$$

There exists $q_0 \in \mathbb{N}$, $q > q_0$

$$\Rightarrow \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(y))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(y) - \varepsilon_i))^2 + 1^i)] \right] < \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(x^{'}))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(x^{'}) - \varepsilon_i))^2 + 1^i)] \right],\ \forall\ \varepsilon_i > 0$$

$$f_k(y) + \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(y))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(y) - \varepsilon_i))^2 + 1^i)] \right] < f_k(x^{'}) + \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(x^{'}))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(x^{'}) - \varepsilon_i))^2 + 1^i)] \right],\ \forall\ \varepsilon_i > 0$$

$$\Rightarrow L(y) < L(x^{'})$$

which contradicts Equation (20)
If $x^{'} \notin \mathbf{D}$, then $\exists y \in \mathbf{Z}$ such that $f_k(y) < f_k(x^{'})\ \forall\ k \neq i,\ i = \overline{1,m}$
Since $x^{'}, y \notin \mathbf{D}$, then by Lemma 2 we have:

$$\underline{\lim}_{\sigma_q \longmapsto +\infty} \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(x^{'}))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(x^{'}) - \varepsilon_i))^2 + 1^i)] \right] = +\infty,\ \forall \varepsilon_i > 0 \text{ and}$$
$$\underline{\lim}_{\sigma_q \longmapsto +\infty} \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(y))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(y) - \varepsilon_i))^2 + 1^i)] \right] = +\infty,\ \forall \varepsilon_i > 0.$$

There exists $q_0 \in \mathbb{N}$, $q > q_0$

$$\Rightarrow \sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(y))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(y)-\varepsilon_i))^2+1^i)]\right] < \sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(x^{'}))^2+1^j)] + \sum_{i=1}^{m}[\ln(((f_i(x^{'})-\varepsilon_i))^2+1^i)]\right],\ \forall\ \varepsilon_i > 0$$

$$\Rightarrow f_k(y)+\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(y))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(y)-\varepsilon_i))^2+1^i)]\right] < f_k(x^{'})+ \sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(x^{'}))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(x^{'})-\varepsilon_i))^2+1^i)]\right],\ \forall\ \varepsilon_i > 0$$

$$\Rightarrow L(y) < L(x^{'})$$

which contradicts (20) since $x^{'} \in \mathbf{Z}_q^*$, hence $\underline{\lim}_{q\longmapsto+\infty} (\mathbf{Z}_q^* \backslash \mathbf{Z}^*) = \emptyset$.

★ **Logarithmic-Exponential Penalties**

Using equation (13) and the definition (4), we have:

$$\begin{cases} \min\Big\{ L(x) = f_k(x) + \frac{2}{\mu_q}\left[\sum_{j=1}^{p} \ln(1+\exp[\mu_q\lambda_j g_j(x)]) + \right. \\ \left. \sum_{i=1}^{m} \ln(1+\exp[\mu_q\lambda_i(f_i(x)-\varepsilon_i)])\right] \\ subject\ to: \\ \qquad i \neq k \\ \qquad x \in \mathbb{R}^n. \end{cases} \tag{21}$$

where μ_q is the penalty coefficient satisfying

$$\lim_{q\longmapsto+\infty} \mu_q = +\infty.$$

$\lambda_i > 0$, $\lambda_j > 0\ \forall\ i = \overline{1,m},\ j = \overline{1,p}$ are multiplier parameters.

Lemma 3. *Let* $\mathbf{D}$ *be the set of admissible solutions of problem* (1), $\forall\ \varepsilon_i > 0$ *for* $i = \overline{1,m}$, *we have:*

1. If $x \in \mathbf{D}$, *then*

$$\lim_{\mu_q\longmapsto+\infty} \frac{2}{\mu_q}\left[\sum_{j=1}^{p} \ln(1+\exp[\mu_q\lambda_j g_j(x)]) + \sum_{i=1}^{m} \ln(1+\exp[\mu_q\lambda_i(f_i(x)-\varepsilon_i)])\right] = 0,\ \forall\varepsilon_i > 0.$$

2. *If $x \notin \mathbf{D}$, then*

$$\lim_{\mu_q \longmapsto +\infty} \frac{2}{\mu_q}\left[\sum_{j=1}^{p} \ln\left(1+\exp[\mu_q \lambda_j g_j(x)]\right)+\sum_{i=1}^{m} \ln\left(1+\exp[\mu_q \lambda_i (f_i(x)-\varepsilon_i)]\right)\right]=+\infty,\ \forall \varepsilon_i>0.$$

Proof. Suppose $x \in \mathbf{D}$, $\Rightarrow$ $g_j(x) \leq 0$ for $j=\overline{1,p}$ and $f_i(x)-\varepsilon_i \leq 0$ for all $i=\overline{1,m}$ and $\varepsilon_i>0$, therefore

$$\lim_{\mu_q \longmapsto +\infty} \frac{2}{\mu_q}\left[\sum_{j=1}^{p} \ln\left(1+\exp[\mu_q \lambda_j g_j(x)]\right)+\sum_{i=1}^{m} \ln\left(1+\exp[\mu_q \lambda_i (f_i(x)-\varepsilon_i)]\right)\right]=0,\ \forall \varepsilon_i>0$$

since $\ln(1)=0$.
If $x \notin \mathbf{D}$, $\Rightarrow$ $g_j(x)>0$ for $j=\overline{1,p}$ and $f_i(x)-\varepsilon_i>0$ for all $i=\overline{1,m}$ and $\varepsilon_i>0$, therefore

$$\lim_{\mu_q \longmapsto +\infty} \frac{2}{\mu_q}\left[\sum_{j=1}^{p} \ln\left(1+\exp[\mu_q \lambda_j g_j(x)]\right)+\sum_{i=1}^{m} \ln\left(1+\exp[\mu_q \lambda_i (f_i(x)-\varepsilon_i)]\right)\right]=+\infty,\ \forall \varepsilon_i>0.$$

Theorem 6. *Suppose S_q is a sequence of numbers such that $S_q \subset \mathbb{R}^n$ where $q \in \mathbb{N}$.*
If
$\overline{\lim_{k \longmapsto +\infty}} S_q = \left\{x \in \mathbb{R}^n : x \in S_q \text{ for an infinite number of } q \in \mathbb{N}\right\}$
then $\overline{\lim_{q \longmapsto +\infty}} (\mathbf{Z}_q^* \backslash \mathbf{Z}^*) = \emptyset$

Proof.
Assume by contradiction that $\overline{\lim_{q \longmapsto +\infty}} (\mathbf{Z}_q^* \backslash \mathbf{Z}^*) \neq \emptyset$, then there exists a subset $\mathbf{Z}_{q_r}^*$, $r=1,2,\ldots$ such that $x' \in \overline{\lim_{q \longmapsto +\infty}} (\mathbf{Z}_{q_r}^* \backslash \mathbf{Z}^*)$.
Therefore, there exists $q_0 \geq 0$ such that for $q_r \geq q_0$, we have $x' \in \mathbf{Z}_q^* \backslash \mathbf{Z}^*$.
Since $x' \in \mathbf{Z}^*$, we have

$$L(x') < L(y)\ \forall y \in \mathbf{Z}^*. \tag{22}$$

If $x' \in \mathbf{D}$, since $x' \notin \mathbf{Z}^*$, then $\exists y \in \mathbf{D}$ such that $f_k(y) < f_k(x')\ \forall\ k \neq i$, $i=\overline{1,m}$.
Since $x', y \in \mathbf{D}$ then, according to Lemma 3, we have:

$$\overline{\lim_{\mu_q \longmapsto +\infty}} \frac{2}{\mu_q}\left[\sum_{j=1}^{p} \ln\left(1+\exp[\mu_q \lambda_j g_j(x')]\right)+\sum_{i=1}^{m} \ln\left(1+\exp[\mu_q \lambda_i (f_i(x')-\varepsilon_i)]\right)\right]=0,\ \forall \varepsilon_i>0$$

and

$$\overline{\lim_{\mu_q \longmapsto +\infty}} \frac{2}{\mu_q}\left[\sum_{j=1}^{p} \ln\left(1+\exp[\mu_q\lambda_j g_j(y)]\right)+\sum_{i=1}^{m} \ln\left(1+\exp[\mu_q\lambda_i(f_i(y)-\varepsilon_i)]\right)\right]=0,\ \forall\varepsilon_i>0.$$

There exists $q_0 \in \mathbb{N}$, $q > q_0$,

$$\Rightarrow \frac{2}{\mu_q}\left[\sum_{j=1}^{p} \ln\left(1+\exp[\mu_q\lambda_j g_j(y)]\right)+\sum_{i=1}^{m} \ln\left(1+\exp[\mu_q\lambda_i(f_i(y)-\varepsilon_i)]\right)\right] <$$
$$\frac{2}{\mu_q}\left[\sum_{j=1}^{p} \ln\left(1+\exp[\mu_q\lambda_j g_j(x^{'})]\right)+\sum_{i=1}^{m} \ln\left(1+\exp[\mu_q\lambda_i(f_i(x^{'})-\varepsilon_i)]\right)\right],\ \forall\ \varepsilon_i>0.$$

$$\Rightarrow f_k(y)+\frac{2}{\mu_q}\left[\sum_{j=1}^{p} \ln\left(1+\exp[\mu_q\lambda_j g_j(y)]\right)+\sum_{i=1}^{m} \ln\left(1+\exp[\mu_q\lambda_i(f_i(y)-\varepsilon_i)]\right)\right] <$$
$$f_k(x^{'})+\frac{2}{\mu_q}\left[\sum_{j=1}^{p} \ln\left(1+\exp[\mu_q\lambda_j g_j(x^{'})]\right)+\sum_{i=1}^{m} \ln\left(1+\exp[\mu_q\lambda_i(f_i(x^{'})-\varepsilon_i)]\right)\right],\ \forall\ \varepsilon_i>0.$$

$$\Rightarrow L(y) < L(x^{'})$$

which contradicts (22).
If $x^{'} \notin \mathbf{D}$, then $\exists y \in \mathbf{Z}$ such that $f_k(y) < f_k(x^{'})$ $\forall\ k \neq i$, $i=\overline{1,m}$.
Since $x^{'}, y \notin \mathbf{D}$, then according to Lemma 3, we have:

$$\overline{\lim_{\mu_q \longmapsto +\infty}} \frac{2}{\mu_q}\left[\sum_{j=1}^{p} \ln\left(1+\exp[\mu_q\lambda_j g_j(x^{'})]\right)+\sum_{i=1}^{m} \ln\left(1+\exp[\mu_q\lambda_i(f_i(x^{'})-\varepsilon_i)]\right)\right]=+\infty\ \forall\varepsilon_i>0.$$

and

$$\overline{\lim_{\mu_q \longmapsto +\infty}} \frac{2}{\mu_q}\left[\sum_{j=1}^{p} \ln\left(1+\exp[\mu_q\lambda_j g_j(y)]\right)+\sum_{i=1}^{m} \ln\left(1+\exp[\mu_q\lambda_i(f_i(y)-\varepsilon_i)]\right)\right]=+\infty\ \forall\varepsilon_i>0.$$

There exists $q_0 \in \mathbb{N}$, $q > q_0$,

$$\Rightarrow \frac{2}{\mu_q}\left[\sum_{j=1}^{p} \ln\left(1+\exp[\mu_q\lambda_j g_j(y)]\right)+\sum_{i=1}^{m} \ln\left(1+\exp[\mu_q\lambda_i(f_i(y)-\varepsilon_i)]\right)\right] <$$
$$\frac{2}{\mu_q}\left[\sum_{j=1}^{p} \ln\left(1+\exp[\mu_q\lambda_j g_j(x^{'})]\right)+\sum_{i=1}^{m} \ln\left(1+\exp[\mu_q\lambda_i(f_i(x^{'})-\varepsilon_i)]\right)\right]\ \forall\ \varepsilon_i>0.$$

$$\Rightarrow f_k(y)+\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(y)]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(y)-\varepsilon_i)]\right)\right]<f_k(x^{'})+$$
$$\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\Big(1+\exp[\mu_q\lambda_j g_j(x^{'})]\Big)+\sum_{i=1}^{m}\ln\Big(1+\exp[\mu_q\lambda_i(f_i(x^{'})-\varepsilon_i)]\Big)\right]\ \forall\ \varepsilon_i>0.$$

$\Rightarrow L(y)<L(x^{'})$ which contradicts (22) since $x^{'}\in\mathbf{Z}_q^*$. Thus, $\overline{\lim}_{q\longmapsto+\infty}(\mathbf{Z}_q^*\backslash\mathbf{Z}^*)=\emptyset$.

Theorem 7. *Suppose that S_q is a sequence of numbers such that $S_q\subset\mathbb{R}^n$ where $q\in\mathbb{N}$. If* $\underline{\lim}_{q\longmapsto+\infty} S_q=\left\{x\in\mathbb{R}^n : x\in S_q\ \ for\ a\ finite\ number\ of\ q\in\mathbb{N},\ \right\}$ *then* $\underline{\lim}_{q\longmapsto+\infty}(\mathbf{Z}_q^*\backslash\mathbf{Z}^*)=\emptyset$.

Proof.
Suppose by contradiction that $\underline{\lim}_{q\longmapsto+\infty}(\mathbf{Z}_q^*\backslash\mathbf{Z}^*)\neq\emptyset$, then there exists a subset $\mathbf{Z}_{q_r}^*$, $r=1,2,\ldots$ such that $x^{'}\in\underline{\lim}_{q\longmapsto+\infty}(\mathbf{Z}_{q_r}^*\backslash\mathbf{Z}^*)$.
Thus, there exists $q_0\geq 0$ such that for $q_r\geq q_0$, we have $x^{'}\in\mathbf{Z}_q^*\backslash\mathbf{Z}^*$.
Since $x^{'}\in\mathbf{Z}^*$, then
$$L(x^{'})<L(y),\ \forall y\in\mathbf{Z}^*. \tag{23}$$
If $x^{'}\in\mathbf{D}$, since $x^{'}\notin\mathbf{Z}^*$, then $\exists y\in\mathbf{D}$ such that $f_k(y)<f_k(x^{'})\ \forall\ k\neq i,\ i=\overline{1,m}$.
Since $x^{'},y\in\mathbf{D}$, then according to Lemma 3, we have:

$$\underline{\lim}_{\mu_q\longmapsto+\infty}\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\Big(1+\exp[\mu_q\lambda_j g_j(x^{'})]\Big)+\sum_{i=1}^{m}\ln\Big(1+\exp[\mu_q\lambda_i(f_i(x^{'})-\varepsilon_i)]\Big)\right]=0\ \forall\varepsilon_i>0$$

and

$$\underline{\lim}_{\mu_q\longmapsto+\infty}\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(y)]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(y)-\varepsilon_i)]\right)\right]=0,\ \forall\varepsilon_i>0.$$

There exists $q_0\in\mathbb{N}$, $q>q_0$

$$\Rightarrow\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(y)]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(y)-\varepsilon_i)]\right)\right]<$$
$$\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\Big(1+\exp[\mu_q\lambda_j g_j(x^{'})]\Big)+\sum_{i=1}^{m}\ln\Big(1+\exp[\mu_q\lambda_i(f_i(x^{'})-\varepsilon_i)]\Big)\right],\forall\varepsilon_i>0.$$

$$\Rightarrow f_k(y)+\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(y)]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(y)-\varepsilon_i)]\right)\right]<$$

$$f_k(x^{'})+\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(x^{'})]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(x^{'})-\varepsilon_i)]\right)\right],\ \forall\,\varepsilon_i>0.$$

$$\Rightarrow L(y)<L(x^{'})$$

which contradicts (23).
If $x^{'}\notin\mathbf{D}$, then $\exists y\in\mathbf{Z}$ such that $f_k(y)<f_k(x^{'})\ \forall\,k\neq i,\ i=\overline{1,m}$.
Since $x^{'},y\notin\mathbf{D}$, then according to Lemma 3, we have:

$$\underline{\lim}_{\mu_q\longmapsto+\infty}\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(y)]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(y)-\varepsilon_i)]\right)\right]=+\infty,\ \forall\varepsilon_i>0$$

and

$$\underline{\lim}_{\mu_q\longmapsto+\infty}\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(x^{'})]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(x^{'})-\varepsilon_i)]\right)\right]=+\infty,\ \forall\varepsilon_i>0.$$

There exists $q_0\in\mathbb{N}$, $q>q_0$

$$\Rightarrow\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(y)]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(y)-\varepsilon_i)]\right)\right]<$$
$$\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(x^{'})]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(x^{'})-\varepsilon_i)]\right)\right],\ \forall\,\varepsilon_i>0.$$

$$\Rightarrow\ f_k(y)+\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(y)]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(y)-\varepsilon_i)]\right)\right]<$$

$$f_k(x^{'})+\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(x^{'})]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(x^{'})-\varepsilon_i)]\right)\right],\ \forall\,\varepsilon_i>0.$$

$\Rightarrow L(y)<L(x^{'})$ which contradicts (23) because $x^{'}\in\mathbf{Z}_q^*$, thus $\underline{\lim}_{q\longmapsto+\infty}(\mathbf{Z}_q^*\backslash\mathbf{Z}^*)=\emptyset$.

Step III: Alienor Transformation

This step involves transforming the previously defined problems with penalty functions into a single-variable optimization problem. An Alienor transformation allows us to

express all variables as a function of a unique variable θ.
By setting $x_i = h_i(\theta)$, we obtain:

★ **Penalty Derived from the Lagrangian**

$$\left\{\begin{array}{l} \min\{L(h(\theta))\} = f_k(h(\theta)) + \eta\left[\sum_{j=1}^{p}(g_j(h(\theta))\right. \\ \left. +|g_j(h(\theta))| + \sum_{i=1}^{m}(f_i(h(\theta)) - \varepsilon_i) + |f_i(h(\theta)) - \varepsilon_i|)\right] \\ subject\ to: \\ \quad \theta \in [0, \theta_{max}]. \\ \quad \eta \geq \frac{M - f_k(h(\theta))}{} \sum_{j=1}^{p} g_j(h(\theta)) + \sum_{i=1}^{m}(f_i(h(\theta)) - \varepsilon_i) \\ \quad M = \max_{h(\theta)\in\mathbf{D}} f_k(h(\theta)) \end{array}\right. \tag{24}$$

Theorem 8. *If $\theta^* \in [0, \theta_{\max}]$ is an optimal solution to problem (24), then all $x_i^* = h_i(\theta^*) \in \mathbf{D}$ is an optimal solution to problem (14).*

★ **Exponential penalization**

$$\left\{\begin{array}{l} \min\left\{L(h(\theta)) = f_k(h(\theta)) + \frac{1}{\rho_q}\left[\sum_{j=1}^{p}\vartheta[\rho_q g_j(h(\theta))] + \right.\right. \\ \left.\left.\sum_{i=1}^{m}\vartheta[\rho_q(f_i(h(\theta)) - \varepsilon_i)]\right]\right\} \\ subject\ to: \\ \quad i \neq k \\ \quad \theta \in [0, \theta_{max}]. \end{array}\right. \tag{25}$$

Theorem 9. *If $\theta^* \in [0, \theta_{\max}]$ is an optimal solution of the problem (25), then all $x_i^* = h_i(\theta^*) \in \mathbf{D}$ is an optimal solution of the problem (15).*

Proof. Suppose that θ^* is an optimal solution of the problem (25) then

$$\forall\theta \in [0, \theta_{\max}],\ L(h(\theta^*)) < L(h(\theta))$$

which implies

$$f_k(h(\theta^*)) + \frac{1}{\rho_q}\left[\sum_{j=1}^{p}\vartheta[\rho_q g_j(h(\theta^*))] + \sum_{i=1}^{m}\vartheta[\rho_q(f_i(h(\theta^*)) - \varepsilon_i)]\right] < f_k(h(\theta)) +$$

$$\frac{1}{\rho_q}\left[\sum_{j=1}^{p}\vartheta[\rho_q g_j(h(\theta))]+\sum_{i=1}^{m}\vartheta[\rho_q(f_i(h(\theta))-\varepsilon_i)]\right],\ \varepsilon_i>0$$

According to lemma 1, we have:

$$\lim_{\rho_q\longmapsto+\infty}\frac{1}{\rho_q}\left[\sum_{j=1}^{p}\vartheta[\rho_q g_j(h(\theta^*))]+\sum_{i=1}^{m}\vartheta[\rho_q(f_i(h(\theta^*))-\varepsilon_i)]\right]=0,\ \varepsilon_i>0$$

$$\lim_{\rho_q\longmapsto+\infty}\frac{1}{\rho_q}\left[\sum_{j=1}^{p}\vartheta[\rho_q g_j(h(\theta))]+\sum_{i=1}^{m}\vartheta[\rho_q(f_i(h(\theta))-\varepsilon_i)]\right]=0,\ \varepsilon_i>0$$

and $\Rightarrow f_k(h(\theta^*))<f_k(h(\theta))$ or $x_i^*=h_i(\theta^*)\Rightarrow f_k(x^*)<f_k(x)$.
With Lemma 1, we have:

$$\lim_{\rho_q\longmapsto+\infty}\frac{1}{\rho_q}\left[\sum_{j=1}^{p}\vartheta[\rho_q g_j(x^*)]+\sum_{i=1}^{m}\vartheta[\rho_q(f_i(x^*)-\varepsilon_i)]\right]=0,\ \varepsilon_i>0$$

and

$$\lim_{\rho_q\longmapsto+\infty}\frac{1}{\rho_q}\left[\sum_{j=1}^{p}\vartheta[\rho_q g_j(x)]+\sum_{i=1}^{m}\vartheta[\rho_q(f_i(x)-\varepsilon_i)]\right]=0,\ \varepsilon_i>0$$

Therefore

$$f_k(x^*)+\frac{1}{\rho_q}\left[\sum_{j=1}^{p}\vartheta[\rho_q g_j(x^*)]+\sum_{i=1}^{m}\vartheta[\rho_q(f_i(x^*)-\varepsilon_i)]\right]<f_k(x)+\frac{1}{\rho_q}\left[\sum_{j=1}^{p}\vartheta[\rho_q g_j(x)]+\right.$$
$$\left.\sum_{i=1}^{m}\vartheta[\rho_q(f_i(x)-\varepsilon_i)]\right],\ \varepsilon_i>0.$$

$$\Rightarrow\forall x\in\mathbf{D},L(x^*)<L(x)$$

Consequently, $x^*\in\mathbf{D}$ is an optimal solution of the problem (15).
Conversely, if x^* is an optimal solution of the problem (14) then

$$\Rightarrow\forall x\in\mathbf{D},L(x^*)<L(x)$$

$\Rightarrow f_k(x^*)+\frac{1}{\rho_q}\left[\sum_{j=1}^{p}\vartheta[\rho_q g_j(x^*)]+\sum_{i=1}^{m}\vartheta[\rho_q(f_i(x^*)-\varepsilon_i)]\right]<$
$f_k(x)+\frac{1}{\rho_q}\left[\sum_{j=1}^{p}\vartheta[\rho_q g_j(x)]+\sum_{i=1}^{m}\vartheta[\rho_q(f_i(x)-\varepsilon_i)]\right],\ \varepsilon_i>0.$
By using lemma 1

$$\lim_{\rho_q\longmapsto+\infty}\frac{1}{\rho_q}\left[\sum_{j=1}^{p}\vartheta[\rho_q g_j(x^*)]+\sum_{i=1}^{m}\vartheta[\rho_q(f_i(x^*)-\varepsilon_i)]\right]=0,\ \varepsilon_i>0$$

and

$$\lim_{\rho_q \longmapsto +\infty} \frac{1}{\rho_q}\left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(x)] + \sum_{i=1}^{m} \vartheta[\rho_q(f_i(x)-\varepsilon_i)]\right] = 0,\ \varepsilon_i > 0$$

$\Rightarrow f_k(x^*) < f_k(x)$. Since $x^* = h_i(\theta^*)$, we then have $f_k(h(\theta^*)) < f_k(h(\theta))$

$$\Rightarrow f_k(h(\theta^*)) + \frac{1}{\rho_q}\left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(h(\theta^*))] + \sum_{i=1}^{m} \vartheta[\rho_q(f_i(h(\theta^*))-\varepsilon_i)]\right] < f_k(h(\theta)) +$$
$$\frac{1}{\rho_q}\left[\sum_{j=1}^{p} \vartheta[\rho_q g_j(h(\theta))] + \sum_{i=1}^{m} \vartheta[\rho_q(f_i(h(\theta))-\varepsilon_i)]\right],\ \varepsilon_i > 0$$

from lemma 1
which leads to

$$\forall \theta \in [0, \theta_{\max}],\ L(h(\theta^*)) < L(h(\theta))$$

Thus, $\theta^* \in [0, \theta_{\max}]$ is an optimal solution of the problem (24).

★ **Logarithmic penalty**

$$\begin{cases} \min\Big\{ L(h(\theta)) = f_k(h(\theta)) + \sigma_q \Big[\sum_{j=1}^{p}[\ln((g_j(h(\theta)))^2 \\ +1^j)] + \sum_{i=1}^{m}[\ln(((f_i(h(\theta))-\varepsilon_i))^2 + 1^j)]\Big]\Big\} \\ subject\ to: \\ \qquad i \neq k \\ \qquad \theta \in [0, \theta_{max}]. \end{cases} \tag{26}$$

Theorem 10. *If $\theta^* \in [0, \theta_{\max}]$ is an optimal solution of the problem (26), then all $x_i^* = h_i(\theta^*) \in \mathbf{D}$ is an optimal solution of the problem (18).*

Proof. Suppose that θ^* is an optimal solution of the problem (26) then

$$\forall \theta \in [0, \theta_{\max}],\ L(h(\theta^*)) < L(h(\theta))$$

hence

$$f_k(h(\theta^*)) + \sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(h(\theta^*)))^2 + 1^j)] + \sum_{i=1}^{m}[\ln(((f_i(h(\theta^*))-\varepsilon_i))^2 + 1^j)]\right] < f_k(h(\theta)) +$$
$$\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(h(\theta)))^2 + 1^j)] + \sum_{i=1}^{m}[\ln(((f_i(h(\theta))-\varepsilon_i))^2 + 1^j)]\right],\ \varepsilon_i > 0$$

According to lemma 2, we have:

$$\lim_{\sigma_q \longmapsto +\infty} \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(h(\theta^*)))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(h(\theta^*)) - \varepsilon_i))^2 + 1^j)] \right] = 0,\ \varepsilon_i > 0$$

and

$$\lim_{\sigma_q \longmapsto +\infty} \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(h(\theta)))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(h(\theta)) - \varepsilon_i))^2 + 1^j)] \right] = 0,\ \varepsilon_i > 0$$

$\Rightarrow f_k(h(\theta^*)) < f_k(h(\theta))$ or $x_i^* = h_i(\theta^*) \Rightarrow f_k(x^*) < f_k(x)$.
With lemma 2, we have:

$$\lim_{\sigma_q \longmapsto +\infty} \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(x^*))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(x^*) - \varepsilon_i))^2 + 1^j)] \right] = 0,\ \ \varepsilon_i > 0$$

and

$$\lim_{\sigma_q \longmapsto +\infty} \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(x))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(x) - \varepsilon_i))^2 + 1^j)] \right] = 0,\ \ \varepsilon_i > 0$$

Thus,

$$f_k(x^*) + \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(x^*))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(x^*) - \varepsilon_i))^2 + 1^j)] \right] < f_k(x) +$$
$$\sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(x))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(x) - \varepsilon_i))^2 + 1^j)] \right], \varepsilon_i > 0.$$

$$\Rightarrow \forall x \in \mathbf{D}, L(x^*) < L(x)$$

Consequently, $x^* \in \mathbf{D}$ is an optimal solution of the problem (18).

Now suppose that x^* is an optimal solution of the problem (18), then

$$\Rightarrow \forall x \in \mathbf{D}, L(x^*) < L(x)$$

$$\Rightarrow f_k(x^*) + \sigma_q \left[\sum_{j=1}^{p} [\ln((g_j(x^*))^2 + 1^j)] + \sum_{i=1}^{m} [\ln(((f_i(x^*) - \varepsilon_i))^2 + 1^j)] \right] < f_k(x) +$$

$$\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(x))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(x)-\varepsilon_i))^2+1^j)]\right],\varepsilon_i>0.$$

Using lemma 2, we have

$$\lim_{\sigma_q\longmapsto+\infty}\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(x^*))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(x^*)-\varepsilon_i))^2+1^j)]\right]=0,\ \ \varepsilon_i>0$$

and

$$\lim_{\sigma_q\longmapsto+\infty}\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(x))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(x)-\varepsilon_i))^2+1^j)]\right]=0,\ \ \varepsilon_i>0$$

$\Rightarrow f_k(x^*)<f_k(x)$, and since $x^*=h_i(\theta^*)$, we have $f_k(h(\theta^*))<f_k(h(\theta))$

$$\Rightarrow f_k(h(\theta^*))+\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(h(\theta^*)))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(h(\theta^*))-\varepsilon_i))^2+1^j)]\right]<$$

$$f_k(h(\theta))+\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(h(\theta)))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(h(\theta))-\varepsilon_i))^2+1^j)]\right],\ \varepsilon_i>0$$

from lemma 2
which gives that

$$\forall\theta\in[0,\theta_{\max}],\ L(h(\theta^*))<L(h(\theta))$$

hence, $\theta^*\in[0,\theta_{\max}]$ is an optimal solution of the problem (26).

★ **Logarithmic-exponential Penalty**

$$\begin{cases}\min\Big\{L(h(\theta))=f_k(h(\theta))+\frac{2}{\mu_q}\Big[\sum_{j=1}^{p}\ln(1+\exp[\mu_q\lambda_j g_j(h(\theta))])\\ +\sum_{i=1}^{m}\ln(1+\exp[\mu_q\lambda_i(f_i(h(\theta))-\varepsilon_i)])\Big]\\ subject\ to:\\ \qquad i\neq k\\ \qquad \theta\in[0,\theta_{max}].\end{cases}\tag{27}$$

Theorem 11. *If $\theta^*\in[0,\theta_{\max}]$ is an optimal solution of problem (27), then all $x_i^*=h_i(\theta^*)\in\mathbf{D}$ is an optimal solution of problem (21).*

Proof. Suppose that θ^* is an optimal solution of problem (27), then

$$\forall\theta\in[0,\theta_{\max}],\ L(h(\theta^*))<L(h(\theta))$$

hence

$$f_k(h(\theta^*))+\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(h(\theta^*))]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(h(\theta^*))-\varepsilon_i)]\right)\right]$$
$$<f_k(h(\theta))+\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(h(\theta))]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(h(\theta))-\varepsilon_i)]\right)\right],\ \varepsilon_i>0$$

From lemma 3

$$\lim_{\mu_q\longmapsto+\infty}\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(h(\theta^*))]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(h(\theta^*))-\varepsilon_i)]\right)\right]=0,\ \varepsilon_i>0$$
$$\lim_{\mu_q\longmapsto+\infty}\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(h(\theta))]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(h(\theta))-\varepsilon_i)]\right)\right]=0,\ \varepsilon_i>0$$

Thus, $\Rightarrow f_k(h(\theta^*))<f_k(h(\theta))$ or $x_i^*=h_i(\theta^*)\Rightarrow f_k(x^*)<f_k(x)$.
With Lemma 3 we have:

$$\lim_{\mu_q\longmapsto+\infty}\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(x^*)]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(x^*)-\varepsilon_i)]\right)\right]=0,\ \varepsilon_i>0$$

and $\displaystyle\lim_{\mu_q\longmapsto+\infty}\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(x)]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(x)-\varepsilon_i)]\right)\right]=0,\ \varepsilon_i>0$

Therefore,

$$f_k(x^*)+\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(x^*)]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(x^*)-\varepsilon_i)]\right)\right]<f_k(x)+$$
$$\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(x)]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(x)-\varepsilon_i)]\right)\right],\ \varepsilon_i>0.$$

$$\Rightarrow\forall x\in\mathbf{D},L(x^*)<L(x)$$

Consequently, $x^*\in\mathbf{D}$ is an optimal solution of problem (15).
Now suppose that x^* is an optimal solution of problem (15), then

$$\Rightarrow\forall x\in\mathbf{D},L(x^*)<L(x)$$

$$\Rightarrow f_k(x^*)+\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(x^*))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(x^*)-\varepsilon_i))^2+1^j)]\right]<f_k(x)+$$
$$\sigma_q\left[\sum_{j=1}^{p}[\ln((g_j(x))^2+1^j)]+\sum_{i=1}^{m}[\ln(((f_i(x)-\varepsilon_i))^2+1^j)]\right],\ \varepsilon_i>0.$$

Using Lemma 3

$$\lim_{\mu_q\longmapsto+\infty}\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(x^*)]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(x^*)-\varepsilon_i)]\right)\right]=0,\ \varepsilon_i>0$$

$$\text{and}\ \lim_{\mu_q\longmapsto+\infty}\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(x)]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(x)-\varepsilon_i)]\right)\right]=0,\ \varepsilon_i>0 \tag{28}$$

$\Rightarrow f_k(x^*)<f_k(x)$ as $x^*=h_i(\theta^*)$, we have

$$f_k(h(\theta^*))<f_k(h(\theta))$$

$$\Rightarrow f_k(h(\theta^*))+\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(h(\theta^*))]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(h(\theta^*))-\varepsilon_i)]\right)\right]<$$
$$f_k(h(\theta))+\frac{2}{\mu_q}\left[\sum_{j=1}^{p}\ln\left(1+\exp[\mu_q\lambda_j g_j(h(\theta))]\right)+\sum_{i=1}^{m}\ln\left(1+\exp[\mu_q\lambda_i(f_i(h(\theta))-\varepsilon_i)]\right)\right],\ \varepsilon_i>0$$

From lemma 3
this leads to

$$\forall\theta\in[0,\theta_{\max}],\ L(h(\theta^*))<L(h(\theta))$$

Thus, $\theta^*\in[0,\theta_{\max}]$ is an optimal solution of problem (27).

Step IV: Resolution
With the problem having a single variable and an unconstrained objective, we use the Nelder-Mead algorithm to find the $\theta^*=\arg\min L(\theta)$.

Step V: Configuration.
This involves determining the values of the variables of the initial problem through the relationship:

$$x_i=h_i(\theta^*)\ i=\overline{1,m} \tag{29}$$

Algorithm 1: Algorithme

Input: functions g_i defining the feasible set
$\mathscr{X} = \{x \in \mathbb{R}^n : g_j(x) \leq 0, j = 1, \ldots, p\}$, and objective function
$f(x) = (f_1(x), f_2(x), \cdots, f_m(x))^\top$.

Output: A discrete approximation $\mathscr{S}$ of the complete Pareto set of (1).

select a function f_k from f_i as the priority function;
Compute $\underline{\varepsilon_i} = \min\{f_i(x) : x \in \mathscr{X}\}$ for each $i = 1,2,3,\ i \neq k$;
Compute $\overline{\varepsilon_i} = \max\{f_i(x) : x \in \mathscr{X}\}$ for each $i = 1,2,3,\ i \neq k$;
choose the type of penalty function to have either equation (14), (15), (18) or (21);
if *we use equation* (14) **then**
  Choose η such that $\eta \geq \dfrac{M - f_k(x)}{\sum_{j=1}^{p} g_j(x) + \sum_{i=1}^{m}(f_i(x) - \varepsilon_i)}$ with $M = \max_{x \in \mathbf{D}} f_k(x)$;
else if *we use equation* (15) **then**
  Choose ρ_q such that $\lim_{q \longmapsto +\infty} \rho_q = +\infty$;
else if *we use equation* (18) **then**
  Choose σ_q such that $\lim_{q \longmapsto +\infty} \sigma_q = +\infty$;
else if *we use equation* (21) **then**
  Choose μ_q such that $\lim_{q \longmapsto +\infty} \mu_q = +\infty$;
Set $\mathscr{S} \leftarrow 0$;
for *i=1:n* **do**
  Set $h_i(\theta) = \frac{1}{2}\left[(b_i - a_i)\cos(\omega_i\theta + \varphi_i) + b_i + a_i\right]$;
  Set $x_i = h_i(\theta)$;
Compute $\overline{f}(\theta) \longleftarrow L\big(h_1(\theta), h_2(\theta), \ldots, h_n(\theta)\big)$;
for $\varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}]$ **do**
  Set $\theta^* = \arg\min\big(f^{\varepsilon}(\theta)\big)$;
  for *i=1:n* **do**
    Set $x_i = h_i(\theta^*)$;
  Set $\bar{x} = x$ and update $\mathscr{S} \leftarrow \mathscr{S} \cup \{f(\bar{x})\}$;
return $\mathscr{S}$ as a discrete approximation of the complete Pareto set of (1).

3.3. Algorithm

The algorithm of the improved hybrid method is presented as follows:

3.4. Numerical Experiments

We implement Algorithm 1 with the following parameters: $\eta = 10000$, $\rho = 100000$, $\sigma = 1000000$, $\mu = 100000$, and $\lambda = 1$. We chose ten (10) multi-objective test problems from the literature that have Pareto fronts.

The specifications of the computer used for the experimentation are as follows: ASUS Processor 11th Gen Intel(R) Core(TM) i3-1115G4 @ 3.00GHz; RAM Memory 8 GB; Operating System Windows 11 / 64 bits. The test problems used are extracted from the works of Zitzler [24] and also from the works of Deb [25, 27, 28]. The various problems are presented in Table 1.

Table 1: List of multiobjective optimization problems

Function	Sources	Types	m	n	Parameters bounds
SCH	[13, 23, 25]	Convex, Continuous	2	1	$x \in [-5, 5]$
$Min-ex$	[13, 23, 26]	Convex, Continuous	2	2	$x \in [0.1, 1] \times [0, 0.5]$
$Max-ex$	[13, 23, 27]	Concave, Continuous	2	2	$x \in [0.1, 1] \times [0, 0.5]$
$ZDT1$	[13, 23, 28]	Convex, Continuous	2	30	$x \in [0, 1]^{30}$
$ZDT2$	[13, 23, 24]	Convex, Continuous	2	30	$x \in [0, 1]^{30}$
$ZDT-n-c$	[13, 17, 23]	Convex, Continuous	2	30	$x \in [0, 1]^{30}$
$ZDT6$	[10, 13, 23]	Convex, Continuous	2	30	$x \in [0, 1]^{30}$
SOP	[13, 22, 23]	Concave, Continuous	2	30	$x \in [0, 1]^{30}$
$ZDT-linear$	[13, 22, 23]	Convex, Continuous	2	30	$x \in [0, 1]^{30}$
$ZDT3$	[13, 23, 24]	Nonconvex, Discontinuous	2	30	$x \in [0, 1]^{30}$

3.4.1. Graphical representation

The graphical results of the various problems are presented in Figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, where (a) represents the Lagrangian penalty, (b) corresponds to the exponential, (c) is for logarithmic, and (d) is for logarithmic-exponential.

3.5. Performance measurement

We compare our algorithm with that of NSGA-II. To do this, we use the convergence metric $\overline{\gamma}$ and the distribution metric $\overline{\Delta}$. The results are reported in Tables 2 and 3.

3.5.1. Discutions

The results in Table 2 show good convergence of the proposed algorithm for the ten test problems. The metric values close to zero confirm this convergence. Furthermore, the

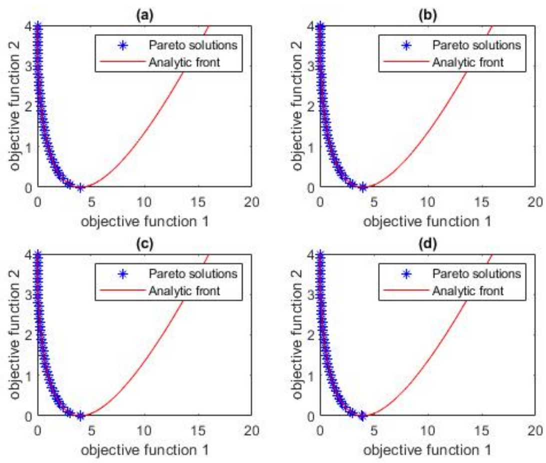

Figure 1: SCH problem Pareto front

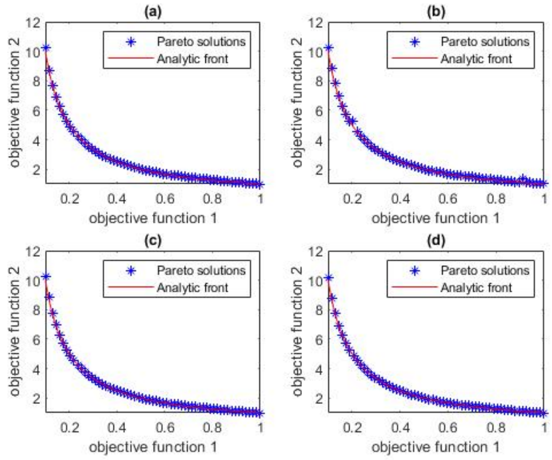

Figure 2: Min-ex problem Pareto front

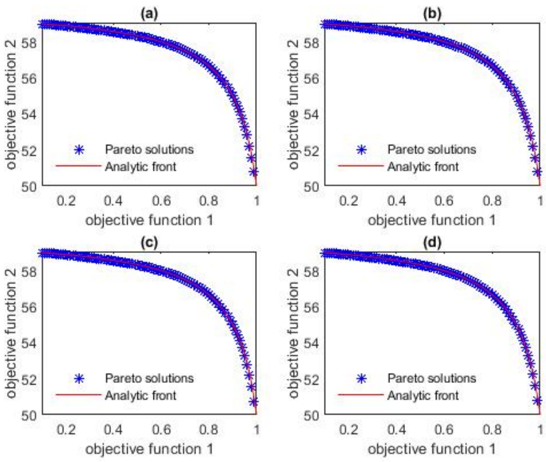

Figure 3: Front de Pareto du problème Max-ex

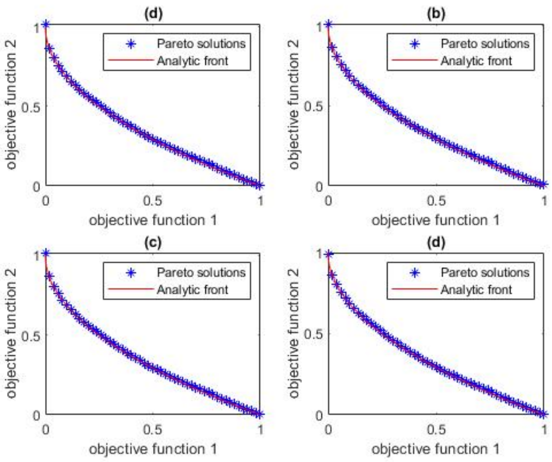

Figure 4: ZDT1 problem Pareto front

Lagrangian and exponential penalty functions yield higher values than the logarithmic, logarithmic-exponential penalties, and the NSGA-II method for all problems.

The results in Table 3 also demonstrate the good diversity of the generated solutions, with all values within the interval $]0,1[$. The exponential and logarithmic-exponential functions offer better performance in terms of diversity compared to the Lagrangian, logarithmic penalties, and the NSGA-II method.

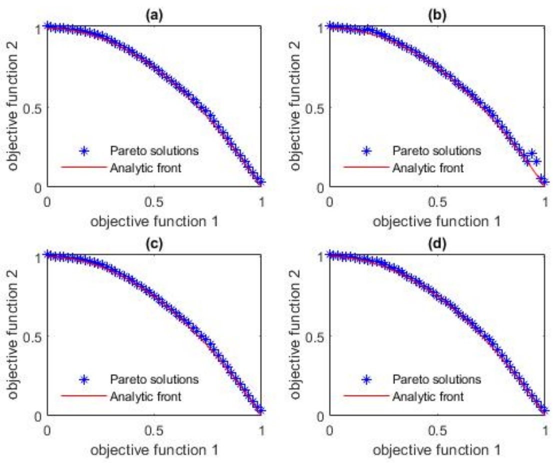

Figure 5: ZDT2 problem Pareto front

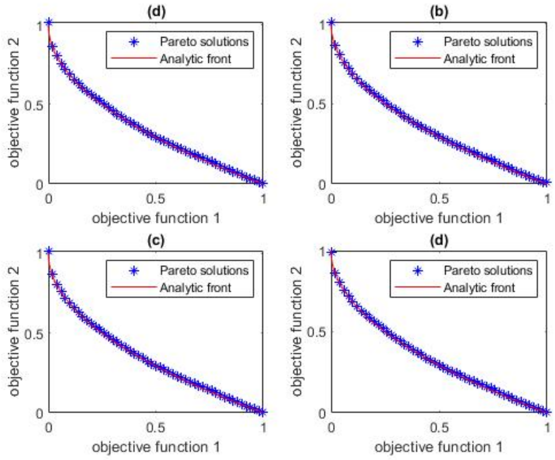

Figure 6: ZDT-n-c problem Pareto front

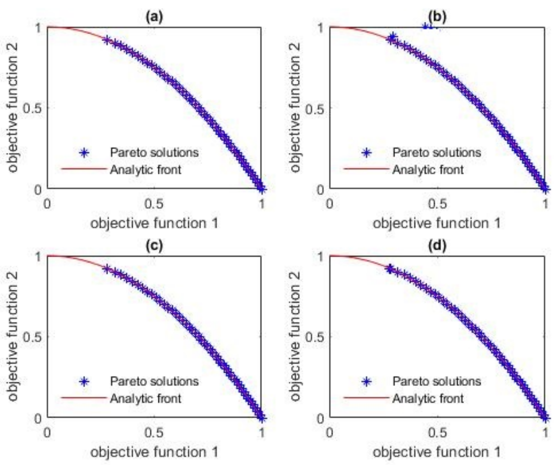

Figure 7: ZDT6 problem Pareto front

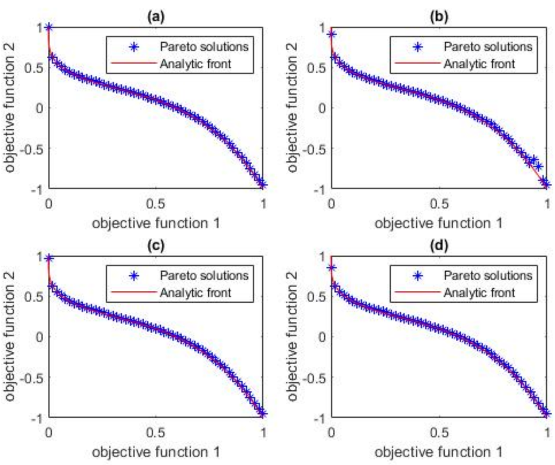

Figure 8: SOP problem Pareto front

In summary, the use of Lagrangian and exponential penalties in the hybrid MOMA-plus approach improves convergence compared to the logarithmic, logarithmic-exponential penalties and NSGA-II. However, for diversity, the exponential and logarithmic-exponential penalties outperform the Lagrangian, logarithmic penalties, and NSGA-II.

We also compare the performance profiles of the two metrics, namely the metrics γ and Δ.

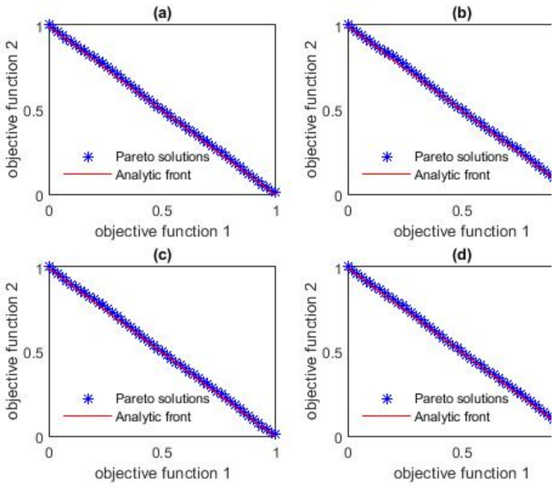

Figure 9: ZDT-linear problem Pareto front

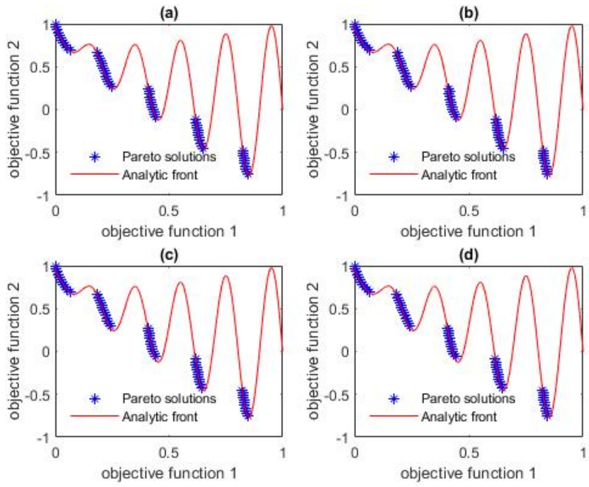

Figure 10: ZDT3 problem Pareto front

Figure 11 illustrates the performance of the five optimization methods based on the convergence criterion γ. The **Lagrangian** and **Exponential** methods stand out clearly: they dominate the other methods with a probability of 0.8 for an interest factor $\tau < 2$, with a slight dominance of the **Exponential** method.

Figure 12 highlights the performance in terms of diversity (Δ-spread). Here, **Expo-Log**, **Exponential**, and **Logarithmic** dominate: they ensure an optimal distribution of solutions on the Pareto front starting from $\tau \approx 1.2$, with over 90 % success rate. For $\tau > 7$, methods such as **Expo-Log**, **Exponential**, **Lagrangian**, and **Logarithmic** show no significant differences.

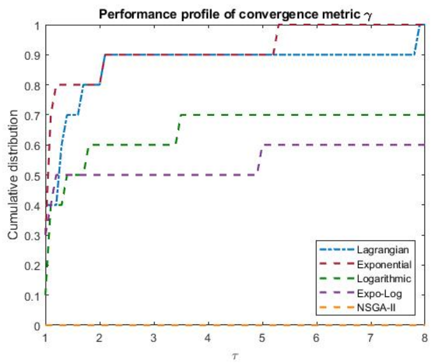

Figure 11: Performance profiles of γ

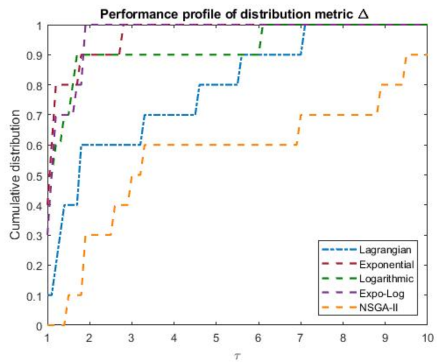

Figure 12: Performance profiles of Δ

Table 2: Performance measurement Values of the convergence metric $\overline{\gamma}$ obtained by the algorithms

Problems	Lagrangian	Exponential	Logarithmic	Expo-Log	NSGA-II
SCH	2,2104e^{-3}	**1,1000e**$^{-3}$	1,0900e^{-2}	1,090e^{-2}	$3.2700e^{-2}$
Min-ex	3,6200e^{-3}	2,4000e^{-3}	**4,6000e**$^{-4}$	2,3000e^{-3}	$8.2500e^{-2}$
ZDT1	5,4732e^{-4}	3,4066e^{-4}	5,7311e^{-4}	**3,3066e**$^{-4}$	$2.3600e^{-2}$
Max-ex	**2,2114e**$^{-6}$	4,6000e^{-6}	3,7000e^{-3}	2,5000e^{-3}	$2.6900e^{-2}$
ZDT2	**1,3000e**$^{-4}$	**1,3000e**$^{-4}$	7,2391e^{-2}	7,2391e^{-2}	$2.6900e^{-2}$
ZDT-n-c	7,9359e^{-4}	**5,7053e**$^{-4}$	6,27316e^{-4}	5,9432e^{-4}	$2.1100e^{-2}$
ZDT6	**3,6511e**$^{-5}$	3,6530e^{-5}	4,7596e^{-5}	5,7053e^{-3}	$2.4300e^{-2}$
SOP	**3,2008e**$^{-4}$	3,6732e^{-4}	1,1000e^{-3}	3,6205e^{-4}	$1.0260e^{-1}$
ZDT-linear	8,1037e^{-04}	**6.5037e**$^{-04}$	6,5309e^{-04}	6,4885e^{-04}	$2.5600e^{-2}$
ZDT3	2,6644e^{-4}	**2,2600e**$^{-4}$	**2,2600e**$^{-4}$	2,1598e^{-4}	$4.100e^{-3}$

Table 3: The values of the distribution metric $\overline{\Delta}$ obtained by the algorithms

Problems	Lagrangian	Exponential	Logarithmic	Expo-Log	NSGA-II
SCH	**0,4300**	**0,4300**	0,6650	0,7924	0,8169
Min-ex	0,6789	0,1289	0,7329	**0,1216**	0,8506
ZDT1	0,7106	0,2573	**0,2206**	0,2624	0,5551
Max-ex	0,6667	0,2644	**0,0946**	0,0953	0,8350
ZDT2	0,4326	0,4326	0,2525	**0,2517**	0,7320
ZDT-n-c	0,2589	0,2094	0,2410	**0,2064**	0,6628
ZDT6	0,6997	0,6997	**0,6096**	0,6967	0,8975
SOP	0,4084	**0,0899**	0,1247	0,1620	0,8489
ZDT-linear	0,0530	**0,0393**	0,0409	0,0413	0,5965
ZDT3	0,7136	**0,4110**	0,6859	0,6813	0,7479

4. Conclusion

In this article, we conducted a theoretical and numerical study on the effect of Lagrangian, logarithmic, exponential, and logarithmic-exponential penalties in the hybrid MOMA-plus method for solving multi-objective optimization problems. The analysis showed that exponential and logarithmic-exponential penalty functions perform better than Lagrangian and logarithmic penalties in terms of solution diversity. For the convergence of the generated solutions, the Lagrangian and exponential penalty functions yield better results than the logarithmic and logarithmic-exponential penalties. Finally, a comparison with the NSGA-II algorithm indicates that the latter is outperformed by the hybrid MOMA-plus method for all four penalty functions.

Future work will focus on applying the hybrid MOMA-plus method to solve constrained dynamic problems, as presented in [29].

References

[1] S. Liu, & E. Feng.(2010). The exponential penalty function method for multiobjective programming problems. Optimization Methods & Software, 25, 667-675.

[2] A. Jayswal, & S. Choudhury.(2014). Convergence of exponential penalty function method for multiobjective fractional programming problems. *Ain Shams Engineering Journal*, **5**, 1371-1376.

[3] N. Echebest, M. Sánchez, & M. Schuverdt.(2016). Convergence results of an augmented Lagrangian method using the exponential penalty function. *Journal Of Optimization Theory And Applications*, **168** pp. 92-108.

[4] M. Hassan, & A. Baharum.(2019). A new logarithmic penalty function approach for nonlinear constrained optimization problem. *Decision Science Letters*, **8**, 353-362.

[5] M. Hassan, , A. Baharum, & M. Ali.(2020). Logarithmic penalty function method for invex multi-objective fractional programming problems. *Journal Of Taibah University For Science*, **14**, 211-216.

[6] X. Sun, & D. Li.(1999). Logarithmic-exponential penalty formulation for integer programming. *Applied Mathematics Letters*, **12**, 73-77.

[7] R. Cominetti, & J. Dussault.(1994). Stable exponential-penalty algorithm with superlinear convergence. *Journal Of Optimization Theory And Applications*, **83**, 285-309.

[8] T. Antczak.(2009). Exact penalty functions method for mathematical programming problems involving invex functions. *European Journal Of Operational Research*, **198**, 29-36.

[9] K. Somé, B. Ulungu, W. Sawadogo, & B. Somé.(2013). A theoretical foundation metaheuristic method to solve some multiobjective optimization problems. *International Journal Of Applied Mathematics Research*, **2**, 464.

[10] A. Zoungrana, K. Somé, & J. Poda.(2023). Obtaining optimal pareto solutions using a hybrid approach combining ε-constraint and MOMA-Plus method. *International Journal Of Numerical Methods And Application*, **23**, 90C29.

[11] S. Mirjalili, & A. Gandomi, Seyedeh Zahra Mirjalili, Shahrzad Saremi, Hossam Faris, Seyed Mohammad Mirjalili.(2017). Salp Swarm Algorithm: A bio-inspired optimizer for engineering design problems. *Adv. Eng. Softw.*, **114** pp. 163-191.

[12] N. Khodadadi, M.Azizi, S. Talatahari, & P. Sareh. (2021). Multi-objective crystal structure algorithm (MOCryStAl): Introduction and performance evaluation. *IEEE Access*, **9** pp. 117795-117812.

[13] S. Huband, P. Hingston, L. Barone, & L. While. (2006). A review of multiobjective test problems and a scalable test problem toolkit. *IEEE Transactions On Evolutionary Computation*, **10**, 477-506.

[14] A.Compaoré , K. Somé, J. Poda, & B. Somé. (2018). Efficiency of MOMA-Plus method to solve some fully fuzzy LR triangular multiobjective linear programs. *Journal Of Mathematics Research*, **10**, 77-87.

[15] A. Tougma, A. Kaboré, & K. Somé. (2024). Hyperbolic Augmented Lagrangian algorithm for multiobjective optimization problems. *Gulf Journal Of Mathematics*, **16**, 151-170.

[16] A. Compaoré, K. Somé, & B. Somé. (2017). New approach to the resolution of triangular fuzzy linear programs: MOMA-Plus method. *International Journal Of Applied Mathematical Research*. **6**, 115-120.

[17] B. Konfé, Y. Cherruault, & T. Benneouala. (2005). A global optimization method for a large number of variables (variant of Alienor method). *Kybernetes*, **34**, 1070-1083.

[18] X. Sun, & D. Li. (2000). Asymptotic strong duality for bounded integer programming: A logarithmic-exponential dual formulation. *Mathematics Of Operations Research*. **25**, 625-644.

[19] M. Maimos, B. Konfé, Koussoube, S. & Somé, B. (2011). Alienor method for nonlinear multi-objective optimization. *Applied Mathematics*, **2**, 217-224.

[20] S. Mirjalili, P. Jangir, & S. Saremi. (2017). Multi-objective ant lion optimizer: a multi-objective optimization algorithm for solving engineering problems. *Applied Intelligence*, **46** pp. 79-95.

[21] M. Premkumar, P. Jangir, R. Sowmya, H. Alhelou, S. Mirjalili, & B. Kumar. (2022). Multi-objective equilibrium optimizer: Framework and development for solving multi-objective optimization problems. *Journal Of Computational Design And Engineering*, **9**, 24-50.

[22] M. Parvizi, E. Shadkam, & N. Jahani. (2015). A hybrid COA ε-constraint method for solving multiobjective problems. *International Journal In Foundations Of Computer Science And Technology*, **5**, 27-40.

[23] K. Deb, A. Pratap, S. Agarwal & T. Meyarivan. (2002). A fast and elitist multiobjective genetic algorithm: NSGA-II. *IEEE Transactions On Evolutionary Computation*, **6**, 182-197.

[24] E. Zitzler, & L. Thiele. (1999). Multiobjective evolutionary algorithms: a comparative case study and the strength Pareto approach. *IEEE Transactions On Evolutionary Computation*, **3**, 257-271.

[25] K. Deb, U. Rao, & S. Karthik. (2007). Dynamic multi-objective optimization and decision-making using modified NSGA-II: A case study on hydro-thermal power scheduling bi-objective optimization problems. *Proceedings Of The Fourth International Conference On Evolutionary Multi-criterion Optimization*, pp. 803-817.

[26] K. Deb. (2001). Multi-objective optimization using evolutionary algorithms. John Wiley & Sons.

[27] K. Deb, A. Sinha, & S. Kukkonen. (2006). Multi-Objective Test Problems, Linkages, and Evolutionary Methodologies. *Proceedings Of The 8th Annual Conference On Genetic And Evolutionary Computation*, pp. 1141-1148.

[28] W. Sawadogo, N. Alaa, K. Somé, & B. Somé. (2015). Application of the genetic algorithms to the identification of the hydrodynamic parameters. *International Journal Of Applied Mathematical Research*, **4**, 78-89.

[29] A. Upadhayay, D. Ghosh, Q. Ansari, & Jauny. (2023). Augmented Lagrangian cone method for multiobjective optimization problems with an application to an optimal control problem. *Optimization And Engineering*, **24**, 1633-1665.

A. Zoungrana
Département de Mathématiques, Laboratoire de Mathématiques, Informatique et Applications (L@MIA),Université Norbert Zongo, BP 376 Koudougou, Burkina Faso.
E-mail: amidzoung@gmail.com

A. Tougma
Université Lédéa Bernard Ouédraogo, Département de Mathématiques et Informatique Ouahigouya, Burkina Faso.
E-mail: appolinaire.tougma19@gmail.com

K. Somé
Département de Mathématiques, Laboratoire de Mathématiques, Informatique et Applications (L@MIA),Université Norbert Zongo, BP 376 Koudougou, Burkina Faso.
E-mail: sokous11@gmail.com

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