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On Homogeneous Limit Integral Equations of Fredholm Type in Spaces of Bounded Functions

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Abstract. In this work, we consider homogeneous limit integral equations of Fredholm-type in the class of bounded continuous functions. We develop a corresponding theory that establishes fundamental results analogous to Fredholm theory, similar to those in the theory of limit integral equations in Bohr spaces. The main difference compared to the Bohr space case lies in the fact that, in the present setting, it is not possible to reduce the problem to ordinary integral equations on the unit cube.

Key Words and Phrases: Fredholm equation, limit integral equation, homogeneous equation, characteristic numbers

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1. Introduction

In [1, 2], the question on Fredholm-type limit integral equations in the Bohr class of almost periodic functions was studied. In these works, the main results of Fredholm theory are established for the class of almost periodic functions. This program was successfully carried out due to the possibility of reducing the problem to a family of ordinary Fredholm-type integral equations on a multidimensional unit cube. In [3], some results of Fredholm theory were studied in the context of limit integral equations, and solutions to these equations were obtained using repeated kernels. Interesting applications of the theory of limit integral equations are presented in [4, 5], where analogs of certain boundary problems are introduced and solved by constructing analogs of Green's functions, thereby reducing the problems to limit integral equations. Here, one considers a new approach consisting in obtaining solutions as a limit of approximate solutions for special ordinary equations or systems of equations. Here, the class of almost periodic functions plays an essential role, in which the problems are studied. It is well known that standard differential equations may not be solvable in Bohr space [6]. It is an interesting problem to describe the functions defined in the real line or in intervals with an arbitrary large length by equations given in bounded intervals [12],[13],[14],[15]. Special properties of Bohr space demand consideration of modified boundary problems for which it is possible to find many applications in various areas.

Despite what was said above, such a reduction does not work in the case of [7]. The main result on the existence of solutions and a formula for the solutions of limit integral equations was established in the case of bounded continuous functions on the real axis (or on the half real line). As in the previous case, we use Theorem 1 [7] as a key auxiliary result.

Consider a homogeneous limit integral equation of the form:

$$\varphi(x) = \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi)\varphi(\xi)d\xi \tag{1}$$

In the ordinary case,this equation is called a Fredholm equation of the second kind. It is clear that the equation has a trivial solution $\psi(x) = 0, x \in \mathbb{R}_+$, where we use the notation $\mathbb{R}_+ = \{x|x \geq 0\}$. As the operator on right hand side of the equation (1) is a linear operator, then the solution set of the equation forms a linear subspace in the space $\mathbb{C}B(\mathbb{R}_+)$, i. e. every linear combination of solutions with real coefficients will be solution again. In [3], the case of such λ for which $D(\lambda) \neq 0$, was studied completely. Here we consider the case $D(\lambda) = 0$, and we shall investigate only non-trivial solutions. The basic question consists in studying the existence of solutions of the equation (1). Note that here we do not discuss the uniqueness of solutions, in general, due to fact that the limit in our case is taken over some unbounded sequence of real numbers T_m, set beforehand. Variation of numbers in this sequence can change solutions.

2. Introduction of some analogues and their properties for Fredholm functions

As it was observed in [3], considering the equations of the form (1) we shall formulate it as a problem: there exists a sequence of real numbers (T_m), $1 < T_1 < T_2 < \cdots$ such that the equation

$$\varphi(x) = \lambda \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} K(x,\xi)\varphi(\xi)d\xi \tag{2}$$

has a non-zero solution $\varphi(x) \in \mathbb{C}B(\mathbb{R}_+)$.

In [1, 3], there were introduced following analogs for Fredholm functions from [8, 9] which play an essential role in investigations of the equation (1):

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{b_n \lambda^n}{n!},$$

with

$$b_n = (-1)^n \lim_{m\to\infty} \frac{1}{T_m^n} \times$$

$$\times \int_0^{T_m} \cdots \int_0^{T_m} \begin{vmatrix} K(\xi_1,\xi_1) & K(\xi_1,\xi_2) & \cdots & K(\xi_1,\xi_n) \\ K(\xi_2,\xi_1) & K(\xi_2,\xi_2) & \cdots & K(\xi_2,\xi_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(\xi_n,\xi_1) & K(\xi_n,\xi_2) & \cdots & K(\xi_n,\xi_n) \end{vmatrix} d\xi_1 d\xi_2 \cdots d\xi_n; \qquad (3)$$

also

$$D(x,y;\lambda) = \lambda K(x,y) + \sum_{n=1}^{\infty} (-1)^n \frac{Q_n(x,y)\lambda^{n+1}}{n!}; x, y \in \boldsymbol{R},$$

where

$$Q_n(x,\xi) = \lim_{m\to\infty} \frac{1}{T_m^n} \times$$

$$\times \int_0^{T_m} \cdots \int_0^{T_m} \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & \cdots & K(x,\xi_n) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(\xi_n,\xi) & K(\xi_n,\xi_1) & \cdots & K(\xi_n,\xi_n) \end{vmatrix} d\xi_1 \cdots d\xi_n. \qquad (4)$$

Here $1 < T_1 < T_2 < \cdots$ is some unbounded sequence of real numbers which, in general, could be set freely.

Let us briefly remind some features of defining of the functions $D(\lambda)$ and $D(x,y;\lambda)$. In [3], for this purpose, taking some sequence (T_m), one shows that first function is entire, and the series defining it converges uniformly. At the same time, the series for $D(x,y;\lambda)$ also converges uniformly for every compact set of pairs (x,y). Defined sequence (T_m) or its any subsequence can be taken as an initial sequence used above.

Denoting by $D_m(\lambda)$ and $D_m(x,y;\lambda)$ sums of corresponding series without passing to the limit in (3) and (4), we find Fredholm functions in ordinary meaning. To avoid the influence of the parameter T, we transform the given equation by following way. Taking some λ, being not a root of the equation $D(\lambda) = 0$, consider the equation

$$\rho(x) = h(x) + \lambda \frac{1}{T} \int_0^T K(x,\xi)\rho(\xi) d\xi =$$

$$= h(uT) + \lambda \int_0^1 K(Tu, T\xi)\rho(T\xi)d\xi = g(u) + \lambda \int_0^1 K_T(u,\xi)\rho(T\xi)d\xi;$$

here in the last chain of the previous equality we have denoted $K_T(u,\xi) = K(Tu, T\xi)$. So, we have a new equation

$$\phi(u) = g(u) + \lambda \int_0^1 K_T(u,\xi)\phi(\xi)d\xi,$$

where $\phi(u) = \rho(Tu)$ and $g(u) = h(Tu)$. This is a Fredholm-type equation which has unique solution, if λ is not a root of the Fredholm determinant corresponding to equation found above. In [3], it was established that this function, for $T = T_m$, coincides with $D_m(\lambda)$, defined above.

Analizing definition and construction of analogs for Fredholm functions in [3, p.293], we see that at first one must take some unbounded sequence of positive real numbers $M_1 < M_2 < \cdots$. Then, it required to define a sequence of positive real numbers tending to zero $\varepsilon_1 > \varepsilon_2 > \cdots$. Now, we apply the theorem of Hurwitz [10, p.128]. Note that every bounded domain can include finite number of roots of entire function, only, and zeroes of such function have not any finite limit points. Suppose that λ' is not a root of the function $D(\lambda)$. So, $|\lambda' - \lambda_0| \geq \varepsilon_N$, where λ_0 is nearist to λ' root of the function $D(\lambda)$. In accordance with this theorem, for every natural $n > N$, such that $\varepsilon_N > 3\varepsilon_n$, in the neighborhood $|\lambda - \lambda_0| \leq \varepsilon_n$ of any root λ_1 of the function $D_n(\lambda)$, placed in the closed disc $|\lambda| \leq M_n$, we have $|\lambda' - \lambda_1| \geq 2\varepsilon_n$. So, for all $n \geq N$ we have $D_r(\lambda') \neq 0$. Applying now results of ordinary Fredholm theory, we obtain the sequence of Theorem 1 of the work [3], to find the solution of the equation (1.2) [2], for $\lambda = \lambda'$. Essential role, for completing of the proof of this theorem, plays the theorem from [1] on existing of uniformly converging subsequences for equicontinuous sequence of functions.

Let us establish now analogs for some relations from ordinary Fredholm theory.

Lemma 1. There exists an unbounded sequence (T_m) of positive real numbers such that for real x, y the following relation is satisfied:

$$D(x,y;\lambda) = \lambda D(\lambda)K(x,y) + \lambda \lim_{m\to\infty} \frac{1}{T_m}\int_0^{T_m} K(u,y)D(x,u;\lambda)\, du; \quad x,y \in \mathbb{R}.$$

Proof. The functions $D(\lambda)$ and $D(x,y;\lambda)$ are defined by power series above. By this reason we must transform the coefficients of these power series for obtaining relations with respect to the parameter λ. For this purpose, in the right-hand side of the equality (4), take the expansion of the determinant with respect to entries of the first column:

$$\begin{vmatrix} K(x,\xi) & K(x,\xi_1) & \cdots & K(x,\xi_n) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(\xi_n,\xi) & K(\xi_n,\xi_1) & \cdots & K(\xi_n,\xi_n) \end{vmatrix} =$$

$$= K(x,\xi) \begin{vmatrix} K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_n) \\ \vdots & \ddots & \vdots \\ K(\xi_n,\xi_1) & \cdots & K(\xi_n,\xi_n) \end{vmatrix} +$$

$$+ \sum_{i=1}^{n} (-1)^i K(\xi_i,\xi) \begin{vmatrix} K(x,\xi_1) & \cdots & K(x,\xi_n) \\ K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_n) \\ \cdots & \cdots & \cdots \\ K(\xi_{i-1},\xi_1) & \cdots & K(\xi_{i-1},\xi_n) \\ K(\xi_{i+1},\xi_1) & \cdots & K(\xi_{i+1},\xi_n) \\ \cdots & \cdots & \cdots \\ K(\xi_n,\xi_1) & \cdots & K(\xi_n,\xi_n) \end{vmatrix}.$$

In every determinant of the sum over i, we move i-th column to the first place, performing sequential replacement with the column before it. After moving i-th column to the first place, the sign before the integral stands equal to $(-1)^{2i-1} = -1$. After this procedure, we find one and the same determinants. Substituting obtained relations in the expressions (3) and (4), we get:

$$\int_0^{T_m} \cdots \int_0^{T_m} \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & \cdots & K(x,\xi_n) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(\xi_n,\xi) & K(\xi_n,\xi_1) & \cdots & K(\xi_n,\xi_n) \end{vmatrix} d\xi_1 \cdots d\xi_n =$$

$$= \int_0^{T_m} \cdots \int_0^{T_m} K(x,\xi) \begin{vmatrix} K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_n) \\ \vdots & \ddots & \vdots \\ K(\xi_n,\xi_1) & \cdots & K(\xi_n,\xi_n) \end{vmatrix} d\xi_1 \cdots d\xi_n -$$

$$-n \int_0^{T_m} \cdots \int_0^{T_m} K(\xi_i,\xi) \begin{vmatrix} K(x,\xi_1) & \cdots & K(x,\xi_n) \\ K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_n) \\ \cdots & \cdots & \cdots \\ K(\xi_{i-1},\xi_1) & \cdots & K(\xi_{i-1},\xi_n) \\ K(\xi_{i+1},\xi_1) & \cdots & K(\xi_{i+1},\xi_n) \\ \cdots & \cdots & \cdots \\ K(\xi_n,\xi_1) & \cdots & K(\xi_n,\xi_n) \end{vmatrix} d\xi_1 \cdots d\xi_n.$$

So,

$$Q_n(x,\xi) = b_n K(x,\xi) - n \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} Q_{n-1}(x,t) K(t,\xi)\ dt. \tag{5}$$

Substituting this expression in the definition of the series for $D(x,y;\lambda)$, and using the relation (3), we find out:

$$D(x,y;\lambda) = \lambda K(x,y) + \sum_{n=1}^{\infty} (-1)^n \frac{Q_n(x,y)\lambda^{n+1}}{n!} =$$

$$= \lambda K(x,y) + \sum_{n=1}^{\infty} \left((-1)^n \frac{\lambda^{n+1}}{n!} b_n K(x,y) - n \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} Q_{n-1}(x,t) K(t,y) dt \right).$$

Now, we use (3). Then putting $Q_0(x,y) \equiv 0$, and using reasonings of the work [2], we obtain:

$$D(x,y;\lambda) = D(\lambda) K(x,y) + \lambda \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} D(x,t,\lambda) K(t,y)\ dt. \tag{6}$$

Proof of the Lemma 1 is finished.

Let's denote:

$$k(x,y,\lambda) = \frac{D(x,y,\lambda)}{\lambda D(\lambda)}.$$

Then the previous relation acquires the form:

$$k(x,y;\lambda) - K(x,y) = \lambda \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} K(u,y)\ k(x,u;\lambda)\ du; \quad x,y \in \mathbb{R}. \tag{7}$$

The last equality shows that the ratio of two entire functions $k(x,u;\lambda)$ serves as a solution of the equation (3) of the work [2], with the function $f(x) = K(x,y)$ at the right-hand side.

Definition 1. We call the function $k(x,u;\lambda)$ as the function relative to $-K(x,y)$, and we call the function

$$r(x,u;\lambda) = \lambda k(x,u;\lambda) = \frac{D(x,y,\lambda)}{D(\lambda)}$$

to be resolvent for the equation (1).

In contrary to the theorem 2 of the work [7], we can not state, if λ is not a root of the function $D(\lambda)$, that the relative function is unique. It is clear that the values of T sets up some sequence (T_m) which is defined from the equation, the construction of which is clarified above. Clearly, the limit can be different depending on the taken subsequence.

Now we can formulate the analog of the first theorem of Fredholm from [1].

Lemma 2. Let λ be a real number such that $D(\lambda) \neq 0$. Then, there exists a sequence (T_m) such that the equation (1) has a solution belonging to $\mathbb{C}B(\mathbb{R}_+)$ given by the equality

$$\varphi(x) = f(x) + \lambda \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} f(\xi) k(x, u; \lambda) d\xi. \tag{8}$$

Theorem 1. Let a real number λ be distinct from any roots of the function $D(\lambda)$, and the function $K(x, y)$ be some symmetric bounded function from the class $\mathbb{C}B(\mathbb{R}_+ \times \mathbb{R}_+)$, and $k(x, u; \lambda)$ be relative to him. If the function $\varphi(x)$ is a solution of the equation

$$\varphi(x) - f(x) = \lambda \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} K(x, \xi)\varphi(\xi) d\xi$$

then the function $f(x)$ is a solution of the equation

$$f(\xi) - \varphi(\xi) = \lambda \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} k(\xi, u; \lambda) f(u) du.$$

Proof. Proving of the theorem is based on the scheme of the proof for corresponding theorem from [2]. Multiplying the first equation in Theorem 1 by the function $k(x, u; \lambda)$, take mean value with respect x:

$$\lambda \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} k(x, u; \lambda) f(x) dx = \lambda \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} k(\xi, x; \lambda)\varphi(x) dx -$$

$$-\lambda^2 \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} k(x, u; \lambda) dx \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} K(x, \xi)\varphi(\xi) d\xi.$$

Changing the order of integration and limiting processes, we can rewrite the second integral in the right-hand side of the last equality as follows

$$\lambda \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} \varphi(\xi) d\xi \lim_{m\to\infty} \frac{\lambda}{T_m} \int_0^{T_m} K(x, \xi) k(x, u; \lambda) dx.$$

The inner mean value in the right hand-side can be substituted by the left hand-side of the equation (8). Now, using conditions of the theorem, represent the last expression as below

$$\lambda \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} \varphi(\xi)(k(u, \xi; \lambda) - K(u, \xi)) d\xi =$$

$$= \lambda \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} \varphi(\xi)(k(u,\xi;\lambda))d\xi - (\varphi(u) - f(u)).$$

So, we have

$$\lambda \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} k(x,u;\lambda) f(x)dx = f(u) - \varphi(u),$$

as stated in the theorem. Theorem 1 is proven.

Definition 2. Roots of the equation $D(\lambda) = 0$ are called characteristic numbers of the equation or the kernel $K(x,y)$.

As the function $D(\lambda)$ is an entire function, then it has no more than countable set of complex roots with finite multiplicities. Relation, defined by the formula of Theorem 2 of the work [1], cannot be applicable when the number λ is a characteristic number.

Theorem 2. Every characteristic number of the limit integral equation is a pole of the resolvent function $r(x,u;\lambda)$.

Proof. Taking into account the expression (6) and letting x=y, we obtain:

$$D(x,x;\lambda) = \lambda D(\lambda)K(x,x) + \lambda \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} K(u,x)D(x,u;\lambda)\ du; \quad x \in \boldsymbol{R}.$$

From the equality (4), it follows that if to put $x = y$ in the formula for $Q_n(x,y)$, and take the mean value, we obtain $(-1)^{n+1}\ b_{n+1}$:

$$(-1)^{n+1}\ b_{n+1} = \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} Q_n(x,x)\ dx. \tag{9}$$

Now, every coefficient of the series $D(\lambda)$ and $D(x,u;\lambda)$ may be calculated using relations (5) and (9), taking initial values $b_0 = 1, Q_0(x,y) = K(x,y)$.

For the proof of our theorem, first we shall transform the formulae obtained above. First of all, we note that

$$\lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} D(x,x,\ \lambda)\ dx = \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} (\sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{n+1} Q_n(x,t)}{n!})\ dx =$$

$$= -\sum_{n=0}^{\infty} \frac{b_{n+1}}{n!} \lambda^{n+1} = -\lambda \sum_{n=0}^{\infty} \frac{b_{n+1}}{n!} \lambda^n.$$

Take derivative of $D(\lambda)$:

$$\frac{dD(\ \lambda)}{d\ \lambda} = \sum_{n=0}^{\infty} \frac{b_{n+1}\lambda^n}{n!} = -\lambda \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} D(x,x,\ \lambda)\ dx. \tag{10}$$

Divide now every part of the last equality by $D(\lambda)$. Then, we find a logarithmic derivative of the function $D(\lambda)$ expressed by mean values for the functions $k(x,x,\lambda)$ and $r(x,x,\lambda)$, as below:

$$\frac{d\ln D(\ \lambda)}{d\ \lambda} = -\lim_{m\to\infty}\frac{1}{T_m}\int_0^{T_m} k(x,x,\ \lambda)\ dx = -\lambda\lim_{m\to\infty}\frac{1}{T_m}\int_0^{T_m} r(x,x,\lambda)\ dx.$$

Suppose that λ is not a pole for the resolvent. Then, λ must be a root of the numerator of the resolvent, moreover, order p of multiplicity of this root is not less than the order q for multiplicity of λ as a root of $D(\lambda)$. So, $q \le p$, and from (8) we find, applying term by term differentiating:

$$\frac{d^{h+1}D(\ \lambda)}{d\ \lambda^{h+1}} = -\lim_{m\to\infty}\frac{1}{T_m}\int_0^{T_m}\frac{d^h}{d\ \lambda^h}D(x,x,\ \lambda)\ dx.$$

Taking $h = 1,\ 2,\ldots,p-1$, one has

$$\frac{d^h}{d\ \lambda^h}D(x,x,\ \lambda) = 0,\ \ x\in R$$

Therefor,

$$\frac{d^{h+1}D(\ \lambda)}{d\ \lambda^{h+1}} = 0,$$

from which it follows that $q \ge p+1$. This is a contradiction. This completes the proof of Theorem 2.

3. The Case of Homogeneous Equations

In [1], it was studied the Fredholm-type limit integral equation in Bohr spaces of almost periodic functions for the case of non-homogeneous equations with values of the parameter satisfying the equation $D(\lambda) = 0$. In the case of homogeneous equations, in accordance with the results of the work [2], it is shown that the equation has trivial solutions only, when $D(\lambda) \ne 0$.

In this section, we consider homogeneous equations and find their solutions. Clearly, the equation (1) has the trivial solution $\varphi(x) = 0$, which is a bounded function. But only non-trivial solutions of this equation are of primary interest. It is obvious that when $\varphi_1(x),\ldots,\varphi_k(x)$ are bounded solutions of a homogeneous equation, then every linear combination of these functions will be a solution of the equation, also. So, the set of solutions forms a linear space (zero solution completes the set of solutions to a linear space). As in [2], we show that this is a finite-dimensional linear space, after introducing below some equivalence relation.

Theorem 3. If the number λ is a characteristic number of the kernel $K(u,x)$, then the homogenous equation (1) has nonzero solutions.

Proof. Consider the equality

$$D(x,y;\lambda) = \lambda D(\lambda)K(x,y) + \lambda \lim_{m\to\infty} \frac{1}{T_m}\int_0^{T_m} K(u,y)D(x,u;\lambda)\,du; \quad x,y \in \boldsymbol{R}$$

of the Lemma 1. If λ is a characteristic number, then

$$D(x,y;\lambda) = \lambda \lim_{m\to\infty} \frac{1}{T_m}\int_0^{T_m} K(u,y)D(x,u;\lambda)\,du.$$

So, the function $D(x,y;\lambda)$ is a solution of the equation (1). But this solution is not of interest, if this function equals to zero identically. By this reason, we must find non-zero solutions. Suppose that $D(\lambda_0) = 0$ for some real λ_0. Since $D(\lambda)$ is an entire function, then we can expand it into power series:

$$D(\lambda) = d_s(\lambda-\lambda_0)^s + d_{s+1}(\lambda-\lambda_0)^{s+1} + \cdots;$$

here s is a natural number. The function $D(x,y;\lambda)$ is an entire function, and therefore, can be expanded into power series:

$$D(x,\ y,\lambda) = a_r(x,y)(\lambda-\lambda_0)^r + a_{r+1}(x,y)(\lambda-\lambda_0)^{r+1} + \cdots; r \geq 0,$$

for which the number λ_0 can also be a root.

Now from (10) we deduce:

$$-\frac{dD(\ \lambda)}{d\ \lambda} = c_r(\lambda-\lambda_0)^r + c_{r+1}(\lambda-\lambda_0)^{r+1} + \cdots,$$

where

$$c_n = \lim_{m\to\infty} \frac{1}{T_m}\int_0^{T_m} a_n(u,u)\,du; n = 0,\ 1,\ldots.$$

The multiplicity of the root λ_0 in the left hand-side is equal to $s-1$ (we suppose that $\lambda \neq 0$, if not then the integral equation has a trivial solution). Since the right hand-side has zero λ_0 of multiplicity r, then $s-1 \geq r$.

Using reasonings of the work [10], we arrive at the equation:

$$a_r(x,y) = \ \lambda \lim_{m\to\infty} \frac{1}{T_m}\int_0^{T_m} K(u,y)a_r(x,y)du\ .$$

The function $a_r(x,x)$ at some x is distinct from zero. So, the function $a_r(x,y)$ is not identically zero, for some $y = y_0$. The proof of the theorem 3 is finished.

Consequence. The equation (1) either has not any solutions or it has infinitely many solutions, when $\lambda = \lambda_0$.

If $D'(\lambda_0) \neq 0$, then the function $D(x,y;\lambda_0)$ is not identically zero for some $y{=}y_0$. Therefore, in this case the function $D(x,y;\lambda)$ is a solution of the homogenous equation (1), for $\lambda = \lambda_0$.

4. Orthogonal Systems in the Space of Bounded Functions

Definition 3. We say that bounded functions $f_1(x)$ and $f_2(x)$ are orthogonal if there exists a sequence of positive numbers $0 < T_1 < T_2 < \cdots$ such that

$$\lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} f_1(x) f_2(x) dx = 0.$$

System of bounded functions $f_1(x)$, $f_2(x)$,..., is called orthogonal, if there exists a sequence of positive numbers of the form $0 < T_1 < T_2 < \cdots$ such that for every pair of different indices $i \neq j$ following relation is valid:

$$\lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} f_i(x) f_j(x) dx = 0.$$

We call the expression

$$\|f\| = \left(\lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} |f(x)|^2 \, dx \right)^{1/2}$$

to be the norm of the bounded function f. If the norm of bounded function is equal to 1, then this function is called normalized.

From vanishing of the norm, in contrary with [2], we can not state that the function is identically zero. The set of bounded functions

$$\{g(x) \in \mathbb{C}B(\mathbb{R}_+) : ||f(x) - g(x)|| = 0\}$$

we call as an equivalence class of the function $f(x)$. If the pair of functions are orthogonal, then equivalent to them functions are orthogonal, also. This follows from the reasonings below. Let the functions $f_1(x)$ and $f_2(x)$ be equivalent and $f_1(x)$ is orthogonal to the function $f(x)$:

$$\lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} f(x) f_1(x) dx = 0,$$

$$\lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} |f_1(x) - f_2(x)|^2 dx = 0.$$

Then,

$$\lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} f(x) f_2(x) dx =$$

$$\lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} f(x) f_1(x) dx + \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} f(x)(f_2(x) - f_1(x)) dx.$$

Since

$$\left|\frac{1}{T_m}\int_0^{T_m} f(x)(f_2(x)-f_1(x))dx\right| \leq$$

$$\leq \frac{1}{T_m}\left(\int_0^{T_m} |f(x)|^2 dx\right)^{1/2}\left(\int_0^{T_m} |f_2(x)-f_1(x)|^2 dx\right)^{1/2},$$

then our statement is true.

Let us consider a homogenous equation

$$f(x)=\lambda \lim_{m\to\infty}\frac{1}{T_m}\int_0^{T_m} K(x,\xi)f(\xi)d\xi,$$

Let $\lambda=\lambda_0$ be some characteristic number for which some system of orthogonal solutions $f_1(x), f_2(x), \dots$ is given. We call these solutions as characteristic functions. In accordance with the said above, a set of solutions form class of equivalence. We may suppose all of these functions are normalized, also.

Theorem 4. The number of various classes of normalized characteristic functions corresponding to a given characteristic number λ_0, satisfies the inequality:

$$n \leq \lambda_0^2 \lim_{m\to\infty}\frac{1}{T_m^2}\int_0^{T_m}\int_0^{T_m} (K(x.y))^2 dxdy.$$

Proof. Consider the limit

$$\lim_{m\to\infty}\frac{1}{T_m}\int_0^{T_m}\left|\sum_{n=1}^{k} f_n(y)\lim_{m\to\infty}\frac{1}{T_m}\int_0^{T_m} K(x,\xi)f_n(\xi)d\xi\right|^2 dy=$$

$$\lim_{m\to\infty}\frac{1}{T_m}\int_0^{T_m}\sum_{r=1}^{k} f_r(y)\lim_{m\to\infty}\frac{1}{T_m}\int_0^{T_m} K(x,\xi)f_r(\xi)d\xi\sum_{s=1}^{k} f_s(y)\times$$

$$\times\lim_{m\to\infty}\frac{1}{T_m}\int_0^{T_m} K(x,\theta)f_s(\theta)d\theta dy.$$

Performing transformations analogical to made in [2], we complete the proof of Theorem 4.

Theorem 5. Characteristic functions related to various characteristic numbers are orthogonal.

Proof. Let functions $f_1(x)$, $f_2(x)$ be characteristic functions relative to characteristic numbers λ_1, λ_2, correspondingly. Then we have

$$f_1(x)f_2(x)=\lambda_1 \lim_{m\to\infty}\frac{1}{T_m}\int_0^{T_m} K(x,\xi)f_1(\xi)f_2(x)d\xi.$$

Repeating the proof of Theorem 5 of [2] we complete the proof.

Theorem 6. If the kernel of the equation is symmetric, then all characteristic numbers are real.

Proof. Consider the equation

$$f(x) = \lambda \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} K(x,\xi) f(\xi) d\xi,$$

with real symmetric kernel. Taking complex conjugate we get

$$\overline{f(x)} = \overline{\lambda} \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} K(x,\xi) \overline{f(\xi)} d\xi;$$

so, the characteristic function relative to the characteristic number $\overline{\lambda}$ is complex conjugate to $f(x)$. Supposing that λ is not real, we see that $\lambda \neq \overline{\lambda}$. By Theorem 5

$$\lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} \overline{f(x)} f(x) dx = \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} \lceil f(x) \rceil^2 dx = 0,$$

which is not true. The got contradiction completes the proof of Theorem 6.

As in the case of ordinary integral equations, in the case of limit integral equations with symmetric kernel, there are only real characteristic numbers. The classes of characteristic functions set up a linear space with finite dimension. In Theorem 4, one established an estimate for this dimension. The characteristic functions, relative to different characteristic numbers, are orthogonal. Since Fredholm functions are entire functions, then the set of characteristic numbers is a countable set. Therefore, the classes of characteristic functions set up a countable set. Let the sequence $f_1(x)$, $f_2(x)$,... be a complete system of orthogonal functions (that is, from every class of equivalence is taken only one solution) satisfying following homogenous limit integral equation

$$f(x) = \lambda \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} K(x,\xi) f(\xi) d\xi,$$

and λ_1, λ_2,... be sequence of all characteristic numbers. Suppose that the series

$$\sum_{m=1}^{\infty} \frac{f_m(y) f_m(x)}{\lambda_m}$$

uniformly converges in the product $\mathbb{R}_+ \times \mathbb{R}_+$; here the characteristic numbers are taken with their multiplicities.

Theorem 7. Suppose that the limit integral equation (1) has non-zero characteristic numbers $\lambda_n, n = 1, 2, ...$ with complete orthogonal system of normalized

characteristic functions $f_n(x)$. Then:
1) a new kernel

$$L(x,y) = K(x,y) - \sum_{n=1}^{\infty} \frac{f_n(y) f_n(x)}{\lambda_n}$$

has only characteristic functions equivalent to zero;
2) following relation holds true:

$$\lim_{m\to\infty} \frac{1}{T_m^2} \int_0^{T_m} \int_0^{T_m} (K(x,y))^2 dxdy \geq \sum_{n=1}^{\infty} \lambda_n^2.$$

Proof. Consider a new kernel

$$L(x,y) = K(x,y) - \sum_{m=1}^{\infty} \frac{f_m(y) f_m(x)}{\lambda_m}.$$

This is a symmetric kernel. If this function is not equal to zero, then it has non-zero characteristic number μ. Take any characteristic function

$$f(x) = \mu \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} L(x,\xi) f(\xi) d\xi.$$

Let us take mean value

$$\lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} f_m(y) f(y) dy =$$

$$= \mu \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} \left\{ K(x,y) - \sum_{n=1}^{\infty} \frac{f_n(y) f_n(x)}{\lambda_n} \right\} f_m(y) f(y) dydx.$$

Performing term by term integration, we get following expression in the right hand-side:

$$\frac{\mu}{\lambda_m} \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T} f_m(\xi) f(\xi) d\xi - \frac{\mu}{\lambda_m} \lim_{T\to\infty} \frac{1}{T} \int_0^{T_m} f_m(\xi) f(\xi) d\xi = 0.$$

Then from previous relation it follows that the function $f(x)$ is orthogonal to every function from the set of characteristic functions. So,

$$f(x) = \mu \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} L(x,\xi) f(\xi) d\xi =$$

$$= \mu \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} \left\{ K(x,y) - \sum_{n=1}^{\infty} \frac{f_n(y) f_n(x)}{\lambda_n} \right\} f(\xi) d\xi =$$

$$= \mu \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} K(x,\xi) f(\xi) d\xi.$$

These equalities show that μ is a characteristic number and the characteristic function $f(x)$ must be equivalent to a linear combination of several number of characteristic functions relative to this characteristic number:

$$\sum_{j=1}^{s} c_j f_j(x).$$

The function $f(x)$ is orthogonal to all functions. Then mean value for absolute value of this function is equal to zero:

$$\lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} f(\xi) f_j(\xi) d\xi = 0.$$

We have proved the statement 1) of the theorem.

Consider the repeated mean value:

$$\lim_{m\to\infty} \frac{1}{T_m^2} \int_0^{T_m} \int_0^{T_m} (L(x,y))^2 dxdy \geq 0.$$

Opening the square of parentheses, integrate term by term. Since

$$\lim_{m\to\infty} \frac{1}{T_m^2} \int_0^{T_m} \int_0^{T_m} \sum_{n=1}^{\infty} \lambda_n^{-1} f_n(x) K(x,y) dx f_n(y) dy =$$

$$= \lim_{m\to\infty} \frac{1}{T_m} \int_0^{T_m} \sum_{n=1}^{\infty} \lambda_n^{-2} f_n^2(x) dx = \sum_{n=1}^{\infty} \lambda_n^{-2},$$

then

$$\lim_{m\to\infty} \frac{1}{T_m^2} \int_0^{T_m} \int_0^{T_m} (L(x,y))^2 dxdy =$$

$$= \lim_{m\to\infty} \frac{1}{T_m^2} \int_0^{T_m} \int_0^{T_m} (K(x,y))^2 dxdy - \sum_{n=1}^{\infty} \lambda_n^{-2}.$$

So, the relation 2) of Theorem 7 is true.

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