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Lorentz boundedness characterization of commutators of maximal operators on spaces of homogeneous type

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Abstract. We study the Lorentz boundedness properties of the maximal commutator operator M_b on the space of homogeneous type and relate this property to spaces of bounded mean oscillations. We also study the Lorentz boundedness properties of the commutators of the maximal operator $[b,M]$ and the commutators of the sharp maximal operator $[b,M^{\sharp}]$ on the space of homogeneous type and relate this property for certain subclasses of spaces of bounded mean oscillations.

Key Words and Phrases: Maximal operator, commutator, Lorentz space, BMO space

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1. Introduction

The main goal of this paper is to study the Lorentz boundedness of the maximal commutator operator M_b, the commutators of the maximal operator $[b,M]$ and the commutators of the sharp maximal operator $[b,M^{\sharp}]$ on spaces (X,d,μ) of homogeneous type.

To extend traditional Euclidean space and build a general basic structure for real harmonic analysis, Coifman and Weiss introduced the concept of spaces of homogeneous type [6].

Let $X=(X,d,\mu)$ be a space of homogeneous type, i.e. X is a topological space endowed with a quasi-distance d and a positive measure μ such that

$$d(x,y)\geq 0 \text{ and } d(x,y)=0 \text{ if and only if } x=y,$$

$$d(x,y)=d(y,x),$$

$$d(x,y)\leq K_1(d(x,z)+d(z,y)), \tag{1}$$

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the balls $B(x,r)=\{y\in X : d(x,y)<r\}$, $r>0$, form a basis of neighborhoods of the point x, μ is defined on a σ-algebra of subsets of X which contains the balls, and

$$0<\mu(B(x,2r))\le K_2\,\mu(B(x,r))<\infty, \tag{2}$$

where $K_i\ge 1\,(i=1,2)$ are constants independent of $x,y,z\in X$ and $r>0$. As usual, the dilation of a ball $B=B(x,r)$ will be denoted by $\lambda B=B(x,\lambda r)$ for every $\lambda>0$.

In the sequel, we always assume that $\mu(X)=\infty$, the space of compactly supported continuous function is dense in $L^1(X,\mu)$ and that X is Q-homogeneous $(Q>0)$, i.e.

$$K_3^{-1}r^Q\le \mu(B(x,r))\le K_3 r^Q, \tag{3}$$

where $K_3\ge 1$ is a constant independent of x and r. The n-dimensional Euclidean space is n-homogeneous.

For $f\in L^1_{\mathrm{loc}}()$, the uncentered maximal operator M is defined by

$$Mf(x)=\sup_{B\ni x}\mu(B)^{-1}\int_B |f(y)-f_B|\,d\mu(y)$$

and the sharp maximal function of Fefferman and Stein $M^\sharp f$ is defined by

$$M^\sharp f(x)=\sup_{B\ni x}\mu(B)^{-1}\int_B |f(y)|\,d\mu(y)$$

where the supremum is taken over all balls $B\subset X$ containing $x\in X$, B is its complement and B denotes the μ measure of B. For a fixed $q\in(0,1)$, any suitable function h and $x\in X$, let $M_q^\sharp h(x)=\big(M^\sharp\big(|h|^q\big)(x)\big)^{1/q}$ and $M_q h(x)=\big(M\big(|h|^q\big)(x)\big)^{1/q}$.

The maximal commutator generated by the operator M and $b\in L^1_{\mathrm{loc}}()$ is defined by

$$M_b f(x)=\sup_{B\ni x}\mu(B)^{-1}\int_B |b(x)-b(y)||f(y)|\,d\mu(y).$$

The commutators generated by the operators M, $M^\sharp$ and a suitable function b are defined by

$$[b,M]f(x)=b(x)Mf(x)-M(bf)(x)$$

and

$$[b,M^\sharp]f(x)=b(x)M^\sharp f(x)-M^\sharp(bf)(x).$$

Obviously, the operators M_b and $[b,M]$ essentially differ from each other since M_b is positive and sublinear and $[b,M]$ is neither positive nor sublinear. The operators M, M_b, $[b,M]$ and $[b,M^\sharp]$ play an important role in real and harmonic analysis and applications (see, for instance [2, 8, 17, 18, 20, 22]).

The commutator estimates have many important applications, for example, in studying the regularity and boundedness of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and in characterizing certain function spaces (see, for instance [5, 9]). The boundedness of the Hardy-Littlewood maximal operator M on $L^p()$ is one of the most fundamental results in harmonic analysis. It has been extended to a range of other function spaces, and to many variations of the standard maximal operator. In particular, one can study commutators of M with BMO functions b. These turn out to be L^p bounded for $1 < p < \infty$ if and only if $b \in BMO$ and $b^- \equiv -\min\{b, 0\} \in L^\infty()$ [2]. This is useful, for instance, when studying the product of an H^1 function with a BMO function [4]. Note that, the boundedness of the operator M_b on L^p spaces was proved by Garcia-Cuerva et al. [8].

The commutator estimates play an important role in studying the regularity of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and their boundedness can be used to characterize certain function spaces (see, for instance [5, 13, 14, 15, 16, 19]).

In [1, 11] were obtained for the boundedness of the maximal commutator operator M_b and commutators of maximal operator $[b, M]$ on the Lorentz spaces $L^{p,q}()$, see also [12].

In this paper we obtain necessary and sufficient conditions for the boundedness of the maximal commutator operator M_b, the commutators of the maximal operator $[b, M]$ and the commutators of the sharp maximal operator $[b, M^\sharp]$ on the Lorentz spaces $L^{p,q}(X)$. We give some new characterizations for certain subclasses of $BMO(X)$.

The structure of the paper is as follows. In Section 2 we give some definitions and auxiliary results. In Section 3 we obtain necessary and sufficient conditions for the boundedness of the maximal commutator M_b on $L^{p,q}(X)$ Lorentz spaces. In Section 4 we give necessary and sufficient conditions for the boundedness of the commutators of the maximal operator $[b, M]$ and the commutators of the sharp maximal operator $[b, M^\sharp]$ on $L^{p,q}(X)$ spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Definition and some basic properties

We start with the definition of Lorentz spaces. Lorentz spaces are introduced by Lorentz in the 1950. These spaces are Banach spaces and generalizations of the more familiar L^p spaces, also they appear to be useful in the general interpolation theory.

Suppose that f is a measurable function on X, then we define

$$f^*(t) = \inf\{s > 0 : d_f(s) \le t\},$$

where

$$d_f(s) := \mu(\{x \in X : |f(x)| > s\}), \quad \forall s > 0.$$

Definition 1. *[3] The Lorentz space $L^{p,q} \equiv L^{p,q}(X)$, $0 < p,q \leq \infty$ is the collection of all measurable functions f on X such the quantity*

$$\|f\|_{L^{p,q}(X)} := \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L^q(0,\infty)} \tag{4}$$

is finite. Clearly $L^{p,p}(X) \equiv L^p(X)$ and $L^{p,1}(X) \equiv WL^p(X)$. The functional $\|\cdot\|_{L^{p,q}(X)}$ is a norm if and only if either $1 \leq q \leq p$ or $p = q = \infty$.

Lemma 1. $[7, Proposition\ 2.11]$ *Let $0 < q_1, q_2 < \infty$, and $0 < q_1, q_2 < \infty$. Suppose that $f \in L^{q_1,r_1}(X)$ and $g \in L^{q_2,r_2}(X)$. Then*

$$\|fg\|_{L^{q,r}(X)} \leq 2\|f\|_{L^{q_1,r_1}(X)} \|g\|_{L^{q_2,r_2}(X)}$$

where $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$.

The following result completely characterizes the boundedness of M on Lorentz spaces.

Lemma 2. $[7, Theorem\ 3.1]$ *Let $1 \leq p,q \leq \infty$.*

(i) *If $1 < p \leq \infty$, then the operator M is bounded on the Lorentz spaces $L^{p,q}(X)$.*

(ii) *If $p = 1$, then the operator M is bounded on the Lorentz spaces $L^{1,q}(X)$ to $WL^1(X)$.*

3. $L^{p,q}$-boundedness of the maximal commutator operator M_b

In this section we find necessary and sufficient conditions for the boundedness of the maximal commutator M_b on $L^{p,q}(X)$ Lorentz spaces.

Definition 2. *We define the space $BMO(X)$ as the set of all locally integrable functions f with finite norm*

$$\|f\|_* = \sup_{x \in X, t > 0} \mu(B(x,t))^{-1} \int_{B(x,t)} |f(y) - f_{B(x,t)}| \, d\mu(y) < \infty,$$

where $f_{B(x,t)} = \mu(B(x,r))^{-1} \int_{B(x,t)} f(y)\, d\mu(y)$.

Lemma 3. $[17, Lemma\ 1]$ *If $b \in BMO(X)$, then for any $q \in (0,1)$, there exists a positive constant C such that*

$$M_q^\sharp\big(M_b f\big)(x) \leq C\|b\|_* M\big(Mf\big)(x) \tag{5}$$

for every $x \in X$ and for all $f \in L^1_{\mathrm{loc}}(X)$.

Theorem 1. *Let $p,q \in (1,\infty)$. The following assertions are equivalent:*

(i) $b \in BMO(X)$.

(ii) *The operator M_b is bounded on $L^{p,q}(X)$.*

(iii) *There exist a constant $C>0$ such that*

$$\sup_B \frac{\big\|\big(b(\cdot)-b_B\big)\chi_B\big\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}} \le C. \tag{6}$$

(iv) *There exist a constant $C>0$ such that*

$$\sup_B \frac{1}{\mu(B)} \big\|\big(b(\cdot)-b_B\big)\chi_B\big\|_{L^1(X)} \le C. \tag{7}$$

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(X)$. Combining Lemmas 2 and 3, we get

$$\begin{aligned}\|M_b f\|_{L^{p,q}(X)} &\lesssim \|M_q^\sharp\big(M_b f\big)\|_{L^{p,q}(X)} \lesssim \|b\|_* \|M\big(Mf\big)\|_{L^{p,q}(X)} \\ &\lesssim \|b\|_* \|Mf\|_{L^{p,q}(X)} \\ &\lesssim \|b\|_* \|f\|_{L^{p,q}(X)}.\end{aligned}$$

$(ii) \Rightarrow (i)$. Assume that M_b is bounded on $L^{p,q}(X)$. Let $B=B(x,r)$ be a fixed ball. We consider $f=\chi_B$. It is easy to compute that

$$\|\chi_B\|_{L^{p,q}(X)} \approx r^{\frac{Q}{p}}. \tag{8}$$

On the other hand, for all $x \in B$ we have

$$\begin{aligned}\big|b(x)-b_B\big| &\le \frac{1}{\mu(B)} \int_B |b(x)-b(y)|\, d\mu(y) \\ &= \frac{1}{\mu(B)} \int_B |b(x)-b(y)|\, \chi_B(y)\, d\mu(y) \\ &\le M_b(\chi_B)(x).\end{aligned}$$

Since M_b is bounded on $L^{p,q}(X)$, then by (8) we obtain

$$\frac{\|\big(b-b_B\big)\chi_B\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}} \le \frac{\|M_b(\chi_B)\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}} \lesssim \frac{\|\chi_B\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}} = 1, \tag{9}$$

which implies that (6) holds since the ball $B \subset X$ is arbitrary.

$(iii) \Rightarrow (iv)$. Assume that (6) holds, we will prove (7). For any fixed ball B, by Lemma 1, inequalities (6) and (8), it is easy to see

$$\frac{1}{\mu(B)} \int_B |b(x)-b(y)|\, d\mu(y) \lesssim \frac{1}{\mu(B)} \|\big(b-b_B\big)\chi_B\|_{L^{p,q}(X)} \, \|\chi_B\|_{L^{p',r'}(X)}$$

$$
\begin{aligned}
&\lesssim \frac{\|(b-b_B)\chi_B\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}} \\
&\lesssim 1.
\end{aligned}
$$

$(iv) \Rightarrow (i)$. For any fixed ball B, we have

$$
\begin{aligned}
\frac{1}{\mu(B)}\int_B |b(x)-b_B|\,d\mu(y) &= \frac{\|(b-b_B)\chi_B\|_{L^1(X)}}{\mu(B)} \\
&\le \sup_B \frac{\|(b-b_B)\chi_B\|_{L^1(X)}}{\mu(B)} \\
&\lesssim 1,
\end{aligned}
$$

which implies that $b \in BMO(X)$. Thus the proof of the theorem is completed.

4. $L^{p,q}$-boundedness of the commutator of maximal operator $[b,M]$

In this section we obtain necessary and sufficient conditions for the boundedness of the commutator of maximal operator $[b,M]$ on $L^{p,q}(X)$ Lorentz spaces.

For a function b defined on X, we denote

$$
b^-(x) := \begin{cases} 0\,, & \text{if } b(x) \ge 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}
$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b,M]$ and M_b are valid :

Let b be any non-negative locally integrable function. Then for all $f \in L^1_{\text{loc}}(X)$ and $x \in X$ the following inequality is valid

$$
\begin{aligned}
&\big|[b,M]f(x)\big| = \big|b(x)Mf(x) - M(bf)(x)\big| \\
&= \big|M(b(x)f)(x) - M(bf)(x)\big| \le M(|b(x)-b|f)(x) = M_b f(x).
\end{aligned}
$$

If b is any locally integrable function on X, then

$$
|[b,M]f(x)| \le M_b f(x) + 2b^-(x)Mf(x), \qquad x \in X \tag{10}
$$

holds for all $f \in L^1_{\text{loc}}(X)$ (see, for example [10, 22]).

Denote by $M_b f$ the local maximal function of f:

$$
M_B f(x) := \sup_{B' \ni x:\, B' \subset B} \frac{1}{\mu(B')}\int_{B'} |f(y)|\,d\mu(y),\; x \in X.
$$

Applying Theorem 1, we obtain the following result.

Theorem 2. *Let $p,q \in (1,\infty)$. The following assertions are equivalent:*

(i) *$b \in BMO(X)$ and $b^- \in L^\infty(X)$.*

(ii) *The operator $[b,M]$ is bounded on $L^{p,q}(X)$.*

(iii) *There exist a constant $C>0$ such that*

$$\sup_B \frac{\big\|\big(b(\cdot)-M_B(b)(\cdot)\big)\chi_B\big\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}} \le C. \tag{11}$$

(iv) *There exist a constant $C>0$ such that*

$$\sup_B \frac{1}{\mu(B)} \big\|\big(b(\cdot)-M_B(b)(\cdot)\big)\chi_B\big\|_{L^1(X)} \le C. \tag{12}$$

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(X)$ and $b^- \in L^\infty(X)$. Combining Lemma 2 and Theorem 1, and inequality (10), we get

$$\begin{aligned}\|[b,M]f\|_{L^{p,q}(X)} &\le \|M_b f + 2b^- Mf\|_{L^{p,q}(X)}\\ &\le \|M_b f\|_{L^{p,q}(X)} + \|b^-\|_{L^\infty}\|Mf\|_{L^{p,q}(X)}\\ &\lesssim \big(\|b\|_* + \|b^-\|_{L^\infty}\big)\|f\|_{L^{p,q}(X)}.\end{aligned}$$

Thus, we obtain that $[b,M]$ is bounded on $L^{p,q}(X)$.

$(ii) \Rightarrow (iii)$. Assume that $[b,M]$ is bounded on $L^{p,q}(X)$. Let $B=B(x,r)$ be a fixed ball. Since

$$M(b\chi_B)\chi_B = M_B(b) \quad \text{and} \quad M(\chi_B)\chi_B = \chi_B,$$

we have

$$\begin{aligned}|M_B(b)-b\chi_B| &= |M(b\chi_B)\chi_B - bM(\chi_B)\chi_B|\\ &\le |M(b\chi_B)-bM(\chi_B)| = |[b,M]\chi_B|.\end{aligned}$$

Hence

$$\|M_B(b)-b\chi_B\|_{L^{p,q}(X)} \le \|[b,M]\chi_B\|_{L^{p,q}(X)}.$$

Thus we get

$$\frac{\|\big(b-M_B(b)\big)\chi_B\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}} \le \frac{\|[b,M](\chi_B)\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}} \lesssim \frac{\|\chi_B\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}} = 1,$$

which deduces that (iii).

$(iii) \Rightarrow (iv)$. Assume that (11) holds, then for any fixed ball B, by Lemma 1, we conclude that

$$\frac{1}{\mu(B)}\int_B |b(x)-M_B(b)(x)|\,d\mu(x) \lesssim \frac{1}{\mu(B)}\,\|\big(b-M_B(b)\big)\chi_B\|_{L^{p,q}(X)}\,\|\chi_B\|_{L^{p',q'}(X)}$$

$$\lesssim \frac{\|\big(b - M_B(b)\big)\chi_B\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}}$$
$$\lesssim 1.$$

$(iv) \Rightarrow (i)$. Assume that (12) holds, we will prove $b \in BMO(X)$ and $b^- \in L^\infty(X)$.
Denote by

$$E := \{x \in B : b(x) \leq b_B\}, \quad F := \{x \in B : b(x) > b_B\}.$$

Since

$$\int_E |b(t) - b_B|\, d\mu(y) = \int_F |b(y) - b_B|\, d\mu(y),$$

in view of the inequality $b(x) \leq b_B \leq M_B(b)$, $x \in E$, we get

$$\begin{aligned} \frac{1}{\mu(B)} \int_B |b - b_B| &= \frac{2}{\mu(B)} \int_E |b - b_B| \\ &\leq \frac{2}{\mu(B)} \int_E |b - M_B(b)| \\ &\leq \frac{2}{\mu(B)} \int_B |b - M_B(b)| \lesssim c. \end{aligned}$$

Consequently, $b \in BMO(X)$. In order to show that $b^- \in L^\infty(X)$, note that $M_B(b) \geq |b|$.
Hence

$$0 \leq b^- = |b| - b^+ \leq M_B(b) - b^+ + b^- = M_B(b) - b.$$

Thus

$$(b^-)_B \leq c,$$

and by the Lebesgue Differentiation theorem we get that

$$0 \leq b^-(x) = \lim_{\mu(B)\to 0} \frac{1}{\mu(B)} \int_B b^-(y)\, d\mu(y) \leq c \quad \text{for a.e. } x \in X.$$

Thus the proof of the theorem is completed.

Theorem 3. *Let $p, q \in (1, \infty)$. The following assertions are equivalent:*
(i) *$b \in BMO(X)$ and $b^- \in L^\infty(X)$.*
(ii) *The operator $[b, M^\sharp]$ is bounded on $L^{p,q}(X)$.*
(iii) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{\big\|\big(b(\cdot) - 2M^\sharp(b\chi_B)\big)\chi_B\big\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}} \leq C. \tag{13}$$

(iv) *There exist a constant $C > 0$ such that*

$$\sup_B \frac{1}{\mu(B)} \big\|\big(b(\cdot) - 2M^\sharp(b\chi_B)\big)\chi_B\big\|_{L^1(X)} \leq C. \tag{14}$$

Proof. $(i) \Rightarrow (ii)$. Since $b \in BMO(X)$ and $b^- \in L^\infty(X)$, then for any locally integrable function f and a.e. $x \in X$

$$\begin{aligned}
&\big|[b,M^\sharp]f(x)\big| = \Big|\sup_{B\ni x}\frac{b(x)}{\mu(B)}\int_B |f(y)-f_B|\,d\mu(y)\\
&-\sup_{B\ni x}\frac{1}{\mu(B)}\int_B |b(y)f(y)-(bf)_B|\,d\mu(y)\Big|\\
&\le \sup_{B\ni x}\frac{1}{\mu(B)}\int_B \big|(b(y)-b(x))f(y)+b(x)f_B-(bf)_B\big|\,d\mu(y)\\
&\le \sup_{B\ni x}\frac{1}{\mu(B)}\int_B \Big(|b(y)-b(x)|\,|f(y)|+\big|b(x)f_B-(bf)_B\big|\Big)\,d\mu(y)\\
&\lesssim M_b f(x)+\sup_{B\ni x}\Big|\frac{b(x)}{\mu(B)}\int_B f(z)\,d\mu(z)-\frac{1}{\mu(B)}\int_B b(z)f(z)\,d\mu(z)\Big|\\
&\lesssim M_b f(x)+\sup_{B\ni x}\frac{1}{\mu(B)}\int_B |b(x)-b(z)||f(z)|\,d\mu(z)\\
&\lesssim M_b f(x).
\end{aligned}$$

Then, it follows from Theorem 1 that $[b,M^\sharp]$ is bounded on $L^{p,q}(X)$.

$(ii) \Rightarrow (iii)$. Assume $[b,M^\sharp]$ is bounded on $L^{p,q}(X)$, we will prove (13). For any fixed ball B, we have (see [2, page 3333] or [22, page 1383] for details)

$$M^\sharp\big(\chi_B\big)(x)=\frac{1}{2}\quad\text{for all}\quad x\in B.$$

Then, for all $x \in B$,

$$\begin{aligned}
b(x)-2M^\sharp\big(b\chi_B\big)(x) &= 2\Big(\frac{b(x)}{2}-M^\sharp\big(b\chi_B\big)(x)\Big)\\
&= 2\Big(b(x)M^\sharp\big(\chi_B\big)(x)-M^\sharp\big(b\chi_B\big)(x)\Big)\\
&= [b,M^\sharp]\big(\chi_B\big)(x).
\end{aligned}$$

Since $[b,M^\sharp]$ is bounded on $L^{p,q}(X)$, then by applying (8), we have

$$\begin{aligned}
\frac{\big\|\big(b(\cdot)-2M^\sharp(b\chi_B)\big)\chi_B\big\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}} &= 2\frac{\big\|[b,M^\sharp]\big(\chi_B\big)\big\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}}\\
&\lesssim \frac{\|\chi_B\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}}\lesssim 1.
\end{aligned}$$

which implies (13).

$(iii) \Rightarrow (iv)$. Assume (13) holds, we will prove (14). For any fixed ball B, combining Lemma 1 with (13) deduces that

$$\begin{aligned}
&\frac{1}{\mu(B)} \left\| \left(b(\cdot) - 2M^{\sharp}(b\chi_{B})\right)\chi_{B} \right\|_{L^{1}(X)} \\
&\leq \frac{\left\| \left(b(\cdot) - 2M^{\sharp}(b\chi_{B})\right)\chi_{B} \right\|_{L^{p,q}(X)}}{\|\chi_{B}\|_{L^{p,q}(X)}} \leq C,
\end{aligned}$$

which implies (14) holds since the constant C is independent of B.

$(iv) \Rightarrow (i)$. We first prove $b \in BMO(X)$. For any fixed ball B, we have (see (2) in [2] for details)

$$\left|b_B\right| \leq 2M^{\sharp}\left(b\chi_{B}\right)(x), \quad \text{for any} \quad x \in B. \tag{15}$$

For any ball B, let $E = \{y \in B : b(y) \leq b_B\}$ and $F = \{y \in B : b(y) > b_B\}$. The following equality is true (see [2, page 3331]):

$$\int_{E} |b(y) - b_B|\, d\mu(y) = \int_{F} |b(y) - b_B|\, d\mu(y). \tag{16}$$

Since $b(y) \leq b_B \leq |b_B| \leq 2M^{\sharp}\left(b\chi_{B}\right)(y)$ for any $y \in E$, we obtain

$$|b(y) - b_B| \leq \left|b(y) - 2M^{\sharp}\left(b\chi_{B}\right)(y)\right|, \; y \in E. \tag{17}$$

Then from (16) and (17) we have

$$\begin{aligned}
&\frac{1}{\mu(B)} \int_{B} |b(y) - b_B|\, d\mu(y) = \frac{2}{\mu(B)} \int_{E} |b(y) - b_B|\, d\mu(y) \\
&\leq \frac{2}{\mu(B)} \int_{E} \left|b(y) - 2M^{\sharp}\left(b\chi_{B}\right)(y)\right| d\mu(y) \\
&\leq \frac{2}{\mu(B)} \int_{B} \left|b(y) - 2M^{\sharp}\left(b\chi_{B}\right)(y)\right| d\mu(y).
\end{aligned}$$

Applying from (14) we get $b \in BMO(X)$.

In order to show that $b^{-} \in L^{\infty}(X)$, note that by (15) for $x \in B$ $\left|b_B\right| \leq 2M^{\sharp}\left(b\chi_{B}\right)$. Hence

$$\left|b_B\right| - b^{+} + b^{-} = \left|b_B\right| - b(x) = 2M^{\sharp}\left(b\chi_{B}\right)(x) - b(x).$$

Therefore

$$\begin{aligned}
&\left|b_B\right| - \frac{1}{\mu(B)} \int_{B} b^{+}(x)dx + \frac{1}{\mu(B)} \int_{B} b^{-}(x)dx \\
&= \frac{1}{\mu(B)} \int_{B} \left(\left|b_B\right| - b^{+}(x) + b^{-}(x)\right) dx
\end{aligned}$$

$$\leq \frac{1}{\mu(B)} \int_B \big(2M^\sharp\big(b\chi_B\big)(x) - b(x)\big)dx$$
$$\leq \frac{1}{\mu(B)} \int_B \big|b(x) - 2M^\sharp\big(b\chi_B\big)(x)\big|dx. \tag{18}$$

Then from Lemma 1 and (8) we get

$$\frac{1}{\mu(B)} \int_B \big|b(x) - 2M^\sharp\big(b\chi_B\big)(x)\big|dx$$
$$\frac{1}{\mu(B)} \big\|\big(b(\cdot) - 2M^\sharp(b\chi_B)\big)\chi_B\big\|_{L^{p,q}(X)} \, \|\chi_B\|_{L^{p',q'}(X)}$$
$$\leq \frac{\big\|\big(b(\cdot) - 2M^\sharp(b\chi_B)\big)\chi_B\big\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}} \leq C, \tag{19}$$

From (18) and (19) we obtain

$$\big|b_B\big| - \frac{1}{\mu(B)} \int_B b^+(x)dx + \frac{1}{\mu(B)} \int_B b^-(x)dx \leq C.$$

By the Lebesgue differentiation theorem we get that

$$2|b^-(x)| = 2b^-(x) = |b(x)| - b^+(x) - b^-(x) \leq C.$$

This implies that $b^- \in L^\infty()$.

Thus the proof of the Theorem 3 is completed.

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