*Journal of Contemporary Applied Mathematics*
*V. 15, No 2, 2025, December*
*ISSN 2222-5498, E-ISSN 3006-3183*
*https://doi.org/10.62476/jcam.151.21*

# Commutator of maximal operator on Carleson curves in local generalized weighted Morrey spaces

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**Abstract.** We study the maximal commutators $M_{b,\Gamma}$ and the commutators of the maximal operator $[b, M_\Gamma]$ over the local generalized weighted Morrey spaces $LM^{\{t_0\}}_{p,\varphi}(\Gamma, w)$ and the generalized weighted Morrey space $M_{p,\varphi}(\Gamma, w)$ defined on Carleson curves $\Gamma$. We give necessary conditions of the boundedness of the maximal commutators $M_{b,\Gamma}$ and the commutators of the maximal operator $[b, M_\Gamma]$ in the local generalized weighted Morrey space $LM^{\{t_0\}}_{p,\varphi}(\Gamma, w)$ and the generalized weighted Morrey space $M_{p,\varphi}(\Gamma, w)$ defined on Carleson curves $\Gamma$, respectively.

**Key Words and Phrases**: Carleson curve, local generalized weighted Morrey space, maximal operator, commutator

**2010 Mathematics Subject Classifications**: 26A33, 42B25, 42B35, 47B38

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## 1. Introduction

The classical Morrey spaces were originally introduced by Morrey in [36] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of Morrey spaces, we refer the readers to [4, 15, 25, 26, 40]. Guliyev, Mizuhara and Nakai [18, 35, 39] introduced generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ (see, also [19, 20, 22, 41]). Recently, Komori and Shirai [34] considered the weighted Morrey spaces $L_{p,\kappa}(\mathbb{R}^n, w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Guliyev [21] gave a concept of generalized weighted Morrey space $M_{p,\varphi}(\mathbb{R}^n, w)$ which could be viewed as extension of both generalized Morrey space $M_{p,\varphi}$ and weighted Morrey space $L_{p,\kappa}(\mathbb{R}^n, w)$. In [21] Guliyev also studied the boundedness of the classical operators and its commutators in these spaces $M_{p,\varphi}(\mathbb{R}^n, w)$.

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Let $\Gamma = \{t \in \mathbb{C} : t = t(s),\ 0 \le s \le l \le \infty\}$ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$ with arc-length measure $\nu(t) = s$, here $l = \nu\Gamma =$ lengths of $\Gamma$. We denote

$$\Gamma(t,r) := \Gamma \cap B(t,r),\ t \in \Gamma,\ r > 0,$$

where $B(t,r) = \{z \in \mathbb{C} : |z-t| < r\}$. We also denote for brevity $\nu\Gamma(t,r) = |\Gamma(t,r)|$.

A rectifiable Jordan curve $\Gamma$ is called a Carleson curve if the condition

$$\nu\Gamma(t,r) \le c_0 r$$

holds for all $t \in \Gamma$ and $r > 0$, where the constant $c_0 > 0$ does not depend on $t$ and $r$.

Let $f \in L_1^{\mathrm{loc}}(\Gamma)$. The maximal function $M_\Gamma f$ is defined by

$$M_\Gamma f(t) = \sup_{r>0} (\nu\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} |f(\tau)| d\nu(\tau),\ t \in \Gamma.$$

The maximal commutator $M_{b,\Gamma}$, generated by $b \in L_1^{loc}(\Gamma)$, is defined by

$$M_{b,\Gamma}(f)(t) = \sup_{r>0} (\nu\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} |b(t) - b(\tau)|\,|f(\tau)| d\nu(\tau),\ t \in \Gamma.$$

The commutator, generated by a function $b$ and the operator $M_\Gamma$, is defined by

$$[b, M_\Gamma](f)(t) = b(t) M_\Gamma(f)(t) - M_\Gamma(bf)(t),\ t \in \Gamma.$$

Maximal operators play an important role in the differentiability properties of functions, singular integrals and partial differential equations. They often provide a deeper and more simplified approach to understanding problems in these areas than other methods. Maximal operator and potential operator in various spaces, in particular, defined on Carleson curves has been widely studied by many authors (see, for example [1, 3, 5, 9, 10, 14, 16, 17, 27, 28, 29, 31, 32, 33]).

The maximal estimates play an important role in studying the regularity of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and their boundedness can be used to characterize certain function spaces (see, for instance [2, 4, 13, 25, 26, 40]).

Although the $M_{b,\Gamma}$ and $[b, M_\Gamma]$ operators are very similar, they are fundamentally different. The maximal commutator $M_{b,\Gamma}$ plays a significant role in the study of commutators of singular integral operators with the symbol $BMO$, and this topic has attracted the attention of many mathematicians. The nonlinear commutator $[b, M]$ of maximal operator, can be used in studying the product of a function in $H_1$ and a function in $BMO$ (see [8], for instance). In [7], Bastero et al. studied the necessary and sufficient condition for the boundedness of $[b, M]$ on $L_p$ spaces.

The main purpose of this paper is to establish the boundedness of maximal commutators $M_{b,\Gamma}$ and the commutators of the maximal operator $[b,M_\Gamma]$ in local generalized weighted Morrey spaces $LM^{\{x_0\}}_{p,\varphi}(\Gamma,w)$ defined on Carleson curves $\Gamma$. When $b$ belongs to the spaces $BMO(\Gamma)$, we give the boundedness of maximal commutators $M_{b,\Gamma}$ and the commutators of maximal operator $[b,M_\Gamma]$ from $LM^{\{x_0\}}_{p,\varphi_1}(\Gamma,w)$ to $LM^{\{x_0\}}_{p,\varphi_2}(\Gamma,w)$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant $C$ independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that $A$ and $B$ are equivalent.

## 2. Preliminaries

By a weight function, briefly weight, we mean a locally integrable function on $\Gamma$ which takes values in $(0,\infty)$ almost everywhere. For a weight $w$ and a measurable set $E$, we define $w(E) = \int_E w(t)d\nu(t)$, and denote the Lebesgue measure of $E$ by $|E|$ and the characteristic function of $E$ by $\chi_E$.

Let $w$ is a weight function, $L_{p,w}(\Gamma)$, $1 \le p < \infty$ be the space of measurable functions on $\Gamma$ with finite norm

$$\|f\|_{L_{p,w}(\Gamma,w)} = \left(\int_\Gamma |f(t)|^p w(t)d\nu(t)\right)^{1/p}.$$

We recall a weight function $w$ is in the Muckenhoupt's class $A_p(\Gamma)$, $1 < p < \infty$ [37], if

$$[w]_{A_p(\Gamma)} := \sup_D [w]_{A_p(D)} = \sup_D \left(\frac{1}{|D|}\int_D w(t)d\nu(t)\right)\left(\frac{1}{|D|}\int_D w(t)^{1-p'}d\nu(t)\right)^{p-1} < \infty, \tag{1}$$

where the supremum is taken with respect to all the balls $D$ and $\frac{1}{p}+\frac{1}{p'}=1$. Note that, for all balls $D$ Hölder's inequality is

$$[w]^{\frac{1}{p}}_{A_p(D)} = |D|^{-1}\|w\|^{\frac{1}{p}}_{L_1(D)}\|w^{-\frac{1}{p}}\|_{L_{p'}(D)} \ge 1. \tag{2}$$

For $p=1$, $w \in A_1(\Gamma)$ is defined by the condition $Mw(t) \le Cw(t)$ with $[w]_{A_1(\Gamma)} = \sup_{t\in\Gamma}\frac{Mw(t)}{w(t)}$, and for $p=\infty$ $A_\infty(\Gamma) = \cup_{1\le p<\infty}A_p(\Gamma)$ and $[w]_\infty = \inf_{1\le p<\infty}[w]_{A_p}$.

We will use the following statement on the boundedness of the weighted Hardy operator

$$H^*_w g(t) := \int_t^\infty \left(1+\frac{s}{t}\right) g(s)\,w(s)\,ds, \qquad 0 < t < \infty,$$

where $w$ is a weight.

The following theorem was proved in [21].

**Theorem 1.** *[21] Let $v_1$, $v_2$ and $w$ be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)\, H_w^* g(t) \le C \sup_{t>0} v_1(t)\, g(t)$$

*holds for some $C>0$ for all non-negative and non-decreasing $g$ on $(0,\infty)$ if and only if*

$$B := \sup_{t>0} v_2(t) \int_t^\infty \left(1+\frac{s}{t}\right) \frac{w(s)\,ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty$$

We denote by $L_{\infty,v}(0,\infty)$ the set of all functions $g(t)$, $t>0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \operatorname{ess\,sup}_{t>0} v(t) g(t)$$

and $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Let $u$ be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)},\ \ t \in (0,\infty).$$

The following theorem was proved in [11].

**Theorem 2.** *Let $v_1$, $v_2$ be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t>0$ and let $u$ be a non-negative continuous function on $(0,\infty)$.*

*Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left( \|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty. \tag{3}$$

## 3. Local generalized weighted Morrey spaces

We find it convenient to define the local generalized weighted Morrey spaces in the form as follows, see [23, 24].

**Definition 1.** *Let* $1 \le p < \infty$, $\varphi(t,r)$ *be a positive measurable function on* $\Gamma \times (0,\infty)$ *and* $w$ *be non-negative measurable function on* $\Gamma$. *Fixed* $t_0 \in \Gamma$, *we denote by* $LM^{\{t_0\}}_{p,\varphi}(\Gamma,w)$ $(WLM^{\{t_0\}}_{p,\varphi}(\Gamma,w))$ *the local generalized weighted Morrey space (the weak local generalized weighted Morrey space), the space of all functions* $f \in L^{\mathrm{loc}}_p(\Gamma,w)$ *with finite quasi-norm*

$$\|f\|_{LM^{\{t_0\}}_{p,\varphi}(\Gamma,w)} = \sup_{r>0} \frac{1}{\varphi(t_0,r)} \frac{1}{w(\Gamma(t_0,r))^{\frac{1}{p}}} \|f\|_{L_p(\Gamma(t_0,r),w)},$$

$$\Big(\|f\|_{WLM^{\{t_0\}}_{p,\varphi}(\Gamma,w)} = \sup_{r>0} \frac{1}{\varphi(t_0,r)} \frac{1}{w(\Gamma(t_0,r))^{\frac{1}{p}}} \|f\|_{WL_p(\Gamma(t_0,r),w)}\Big).$$

**Definition 2.** *Let* $1 \le p < \infty$, $\varphi(t,r)$ *be a positive measurable function on* $\Gamma \times (0,\infty)$ *and* $w$ *be non-negative measurable function on* $\Gamma$. *The generalized weighted Morrey space* $M_{p,\varphi}(\Gamma,w)$ *is defined the set of all functions* $f \in L^{loc}_p(\Gamma,w)$ *by the finite norm*

$$\|f\|_{M_{p,\varphi}(\Gamma,w)} = \sup_{t\in\Gamma, r>0} \frac{1}{\varphi(t,r)} \frac{1}{w(\Gamma(t,r))^{\frac{1}{p}}} \|f\|_{L_p(\Gamma(t,r),w)}.$$

*Also the weak generalized Morrey space* $WM_{p,\varphi}(\Gamma,w)$ *is defined the set of all functions* $f \in L^{loc}_p(\Gamma,w)$ *by the finite norm*

$$\|f\|_{WM_{p,\varphi}(\Gamma,w)} = \sup_{t\in\Gamma, r>0} \frac{1}{\varphi(t,r)} \frac{1}{w(\Gamma(t,r))^{\frac{1}{p}}} \|f\|_{WL_p(\Gamma(t,r),w)}.$$

It is natural, first the set of all, to find conditions ensuring that the spaces $LM^{\{t_0\}}_{p,\varphi}(\Gamma,w)$ and $M_{p,\varphi}(\Gamma,w)$ are nontrivial, that is consist not only of functions equivalent to 0 on $\Gamma$.

**Lemma 1.** *[5] Let* $t_0 \in \Gamma$, $\varphi(t,r)$ *be a positive measurable function on* $\Gamma \times (0,\infty)$ *and* $w$ *be non-negative measurable function on* $\Gamma$. *If*

$$\sup_{r<\tau<\infty} \frac{1}{\varphi(t_0,r)} \frac{1}{w(\Gamma(t_0,r))^{\frac{1}{p}}} = \infty \quad \textit{for some } r > 0, \tag{4}$$

*then* $LM^{\{t_0\}}_{p,\varphi}(\Gamma,w) = \Theta$.

**Remark 1.** *Let* $t_0 \in$ *and* $w$ *be non-negative measurable function on* $\Gamma$. *We denote by* $\Omega^{\mathrm{loc}}_{p,w}(\Gamma)$ *the set of all positive measurable functions* $\varphi$ *on* $\Gamma \times (0,\infty)$ *such that for all* $r > 0$,

$$\Big\| \frac{1}{\varphi(t_0,\tau)} \frac{1}{w(\Gamma(t_0,\tau))^{\frac{1}{p}}} \Big\|_{L_\infty(r,\infty)} < \infty.$$

*In what follows, keeping in mind Lemma 1, for the non-triviality of the space* $LM^{\{t_0\}}_{p,\varphi}(\Gamma,w)$ *we always assume that* $\varphi \in \Omega^{\mathrm{loc}}_{p,w}(\Gamma)$.

**Lemma 2.** *[5] Let $\varphi(t,r)$ be a positive measurable function on $\Gamma\times(0,\infty)$ and $w$ be non-negative measurable function on $\Gamma$.*

$(i)$ *If*

$$\sup_{r<\tau<\infty}\frac{1}{\varphi(t,\tau)}\frac{1}{w(\Gamma(t,r))^{\frac{1}{p}}}=\infty \quad \textit{for some } r>0 \textit{ and for all } t\in\Gamma, \tag{5}$$

*then $M_{p,\varphi}(\Gamma,w)=\Theta$.*

$(ii)$ *If*

$$\sup_{0<\tau<r}\varphi(t,\tau)^{-1}=\infty \quad \textit{for some } r>0 \textit{ and for all } t\in\Gamma, \tag{6}$$

*then $M_{p,\varphi}(\Gamma,w)=\Theta$.*

**Remark 2.** *We denote by $\Omega_{p,w}(\Gamma)$ the sets of all positive measurable functions $\varphi$ on $\Gamma\times(0,\infty)$ such that for all $r>0$,*

$$\sup_{t\in\Gamma}\Big\|\frac{1}{\varphi(t,\tau)}\frac{1}{w(\Gamma(t,r))^{\frac{1}{p}}}\Big\|_{L_\infty(r,\infty)}<\infty, \quad \textit{and} \quad \sup_{t\in\Gamma}\Big\|\varphi(t,\tau)^{-1}\Big\|_{L_\infty(0,r)}<\infty,$$

*respectively. In what follows, keeping in mind Lemma 2, we always assume that $\varphi\in\Omega_{p,w}(\Gamma)$.*

The following Guliyev type local estimate for the maximal operator $M_\Gamma$ is true, see for example, [1, 20].

**Lemma 3.** *[12, Lemma 4.2] Let $\Gamma$ be a Carleson curve, $1\le p<\infty$, $t_0\in\Gamma$ and $w\in A_p(\Gamma)$. Then for $p>1$ and any $r>0$ in $\Gamma$, the inequality*

$$\|\mathscr{M}f\|_{L_{p,w}(\Gamma(t_0,r))}\lesssim[w]_{A_p(\Gamma)}\,w(\Gamma(t_0,r))^{\frac{1}{p}}\sup_{\tau>2r}w(\Gamma(t_0,r))^{-\frac{1}{p}}\,\|f\|_{L_{p,w}(\Gamma(t_0,\tau))} \tag{7}$$

*holds for all $f\in L^{\rm loc}_{p,w}(\Gamma)$.*

*Moreover for $p=1$ the inequality*

$$\|M_\Gamma f\|_{WL_{1,w}(\Gamma(t_0,r))}\lesssim[w]_{A_1(\Gamma)}\,w(\Gamma(t_0,r))\sup_{\tau>2r}w(\Gamma(t_0,r))^{-1}\|f\|_{L_{1,w}(\Gamma(t_0,\tau))} \tag{8}$$

*holds for all $f\in L^{\rm loc}_{1,w}(\Gamma)$.*

The following theorem is valid.

**Theorem 3.** *[12, Theorem 4.3] Let $\Gamma$ be a Carleson curve, $1 \le p < \infty$, $t_0 \in \Gamma$, $w \in A_p(\Gamma)$ and $(\varphi_1, \varphi_2)$ satisfies the condition*

$$\sup_{r<\tau<\infty} w(\Gamma(t_0,r))^{-\frac{1}{p}} \operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(t_0,s)\, w(\Gamma(t_0,s))^{1/p} \le C\, \varphi_2(t_0,r), \tag{9}$$

*where $C$ does not depend on $r$. Then for $p > 1$, the operator $M_\Gamma$ is bounded from $LM^{\{t_0\}}_{p,\varphi_1}(\Gamma,w)$ to $LM^{\{t_0\}}_{p,\varphi_2}(\Gamma,w)$ and for $p = 1$, the operator $M_\Gamma$ is bounded from $LM^{\{t_0\}}_{1,\varphi_1}(\Gamma,w)$ to $WLM^{\{t_0\}}_{1,\varphi_2}(\Gamma,w)$.*

From Theorem 3 we get the following

**Corollary 1.** *[12, Corollary 4.4] Let $\Gamma$ be a Carleson curve, $1 \le p < \infty$, $w \in A_p(\Gamma)$ and $\varphi_1, \varphi_2$ satisfies the condition*

$$\sup_{\tau>r} w(\Gamma(t,r))^{-\frac{1}{p}} \operatorname*{ess\,inf}_{s>\tau} \varphi_1(t,s)\, w(\Gamma(t,s))^{1/p} \le C\, \varphi_2(t,r), \tag{10}$$

*where $C$ does not depend on $t$ and $r$. Then for $p > 1$, the operator $M_\Gamma$ is bounded from $M_{p,\varphi_1}(\Gamma,w)$ to $M_{p,\varphi_2}(\Gamma,w)$ and for $p = 1$, the operator $M_\Gamma$ is bounded from $M_{1,\varphi_1}(\Gamma,w)$ to $WM_{1,\varphi_2}(\Gamma,w)$.*

## 4. Maximal commutator in the spaces $LM^{\{t_0\}}_{p,\varphi}(\Gamma,w)$ and $M_{p,\varphi}(\Gamma,w)$

In this section we find necessary and sufficient conditions for the boundedness of the maximal commutator $M_{b,\Gamma}$ on local generalized weighted Morrey spaces $LM^{\{t_0\}}_{p,\varphi}(\Gamma,w)$ and the generalized weighted Morrey space $M_{p,\varphi}(\Gamma,w)$ defined on Carleson curves $\Gamma$. To study the boundedness of the commutators of some integral operators, we need the bounded mean oscillation space first introduced by John and Nirenberg [30].

Suppose that $b \in L_1^{\mathrm{loc}}(\Gamma)$. Then $b$ is said to be in $BMO(\Gamma)$ if the seminorm given by

$$\|b\|_* = \sup_{r>0} (\nu\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} |b(\tau) - b_{\Gamma(t,r)}|\, d\nu(\tau)$$

is finite, where the supremum is taken over all balls $\Gamma(t,r) \subset \Gamma$ and

$$b_{\Gamma(t,r)} = (\nu\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} b(\tau)\, d\nu(\tau).$$

Modulo constants, the space $BMO(\Gamma)$ is a Banach space with respect to the norm $\|\cdot\|_*$.

**Lemma 4.** *[38] Let $w \in A_\infty(\Gamma)$. Then the norm $\|\cdot\|_*$ is equivalent to the norm*

$$\|b\|_{*,w} = \sup_{t\in\Gamma, r>0} \frac{1}{w(\Gamma(t,r))} \int_{\Gamma(t,r)} |b(y) - b_{\Gamma(t,r),w}|\, w(y)\, d\nu(y),$$

*where*

$$b_{\Gamma(t,r),w} = \frac{1}{w(\Gamma(t,r))} \int_{\Gamma(t,r)} b(y)\, w(y)\, d\nu(y).$$

The following lemma was proved in [21].

**Lemma 5.** [*21*]

1. *Let $w \in A_\infty(\Gamma)$ and $b \in BMO(\Gamma)$. Let also $1 \le p < \infty$, $t \in \Gamma$ and $r_1, r_2 > 0$. Then*

$$\Big(\frac{1}{w(\Gamma(t,r_1))} \int_{\Gamma(t,r_1)} |b(y) - b_{\Gamma(t,r_2),w}|^p w(y) d\nu(y)\Big)^{\frac{1}{p}} \le C\Big(1 + \Big|\ln \frac{r_1}{r_2}\Big|\Big)\, \|b\|_*,$$

   *where $C > 0$ is independent of $f$, $w$, $t$, $r_1$ and $r_2$.*

2. *Let $w \in A_p$ and $b \in BMO(\Gamma)$. Let also $1 < p < \infty$, $t \in \Gamma$ and $r_1, r_2 > 0$. Then*

$$\Big(\frac{1}{w^{1-p'}(\Gamma(t,r_1))} \int_{\Gamma(t,r_1)} |b(y) - b_{\Gamma(t,r_2),w}|^{p'}\, w(y)^{1-p'} d\nu(y)\Big)^{\frac{1}{p'}}$$
$$\le C\Big(1 + \Big|\ln \frac{r_1}{r_2}\Big|\Big)\, \|b\|_*,$$

   *where $C > 0$ is independent of $b$, $w$, $t$, $r_1$ and $r_2$.*

The following lemma is valid.

**Remark 3.** [*9*, *21*] (1) *Let $b \in BMO(\Gamma)$. Then*

$$\|b\|_* \approx \sup_{u\in\Gamma, r>0} \left(\frac{1}{|\Gamma(t,r)|} \int_{\Gamma(t,r)} |b(y) - b_{\Gamma(t,r)}|^p d\nu(y)\right)^{\frac{1}{p}} \tag{11}$$

*for $1 < p < \infty$.*

(2) *Let $b \in BMO(\Gamma)$. Then there is a constant $C > 0$ such that*

$$\big|b_{\Gamma(t,r)} - b_{\Gamma(t,\tau)}\big| \le C\|b\|_* \log \frac{\tau}{r} \;\; for \;\; 0 < 2r < \tau, \tag{12}$$

*where $C$ is independent of $f$, $t$, $r$ and $\tau$.*

**Lemma 6.** *[6] Let $\Gamma$ be a Carleson curve and $b \in BMO(\Gamma)$. Then there exists a positive constant $C$ such that*

$$M_{b,\Gamma} f(t) \le C\|b\|_* M_\Gamma\big(M_\Gamma f\big)(t) \tag{13}$$

*for almost every $t \in \Gamma$ and for all functions $f \in L_1^{\mathrm{loc}}(\Gamma)$.*

**Theorem 4.** *Let $\Gamma$ be a Carleson curve, $1 < p < \infty$ and $w \in A_p(\Gamma)$. Then the operator $M_\Gamma$ is bounded on $L_{p,w}(\Gamma)$.*

**Theorem 5.** *Let $\Gamma$ be a Carleson curve, $1 < p < \infty$, $w \in A_p(\Gamma)$ and $b \in BMO(\Gamma)$. Then the operator $M_{b,\Gamma}$ is bounded on $L_{p,w}(\Gamma)$.*

*Proof.* Let $\Gamma$ be a Carleson curve, $1 < p < \infty$, $w \in A_p(\Gamma)$, $f \in L_{p,w}(\Gamma)$ and $b \in BMO(\Gamma)$. Then from Lemma 6 and Theorem 4 we have

$$\begin{aligned}\|M_{b,\Gamma} f\|_{L_{p,w}(\Gamma)} &\lesssim \|b\|_* \, \|M_\Gamma\big(M_\Gamma f\big)\|_{L_{p,w}(\Gamma)} \\ &\lesssim \|b\|_* \, \|M_\Gamma f\|_{L_{p,w}(\Gamma)} \\ &\lesssim \|b\|_* \, \|f\|_{L_{p,w}(\Gamma)}.\end{aligned}$$

The following Guliyev type local estimate for the maximal commutator operator $M_{b,\Gamma}$ is true, see for example, [21].

**Lemma 7.** *Let $\Gamma$ be a Carleson curve, $1 < p < \infty$, $t_0 \in \Gamma$, $w \in A_p(\Gamma)$ and $b \in BMO(\Gamma)$. Then for any $r > 0$ the inequality*

$$\begin{aligned}&\|M_{b,\Gamma} f\|_{L_{p,w}(\Gamma(t_0,r))} \lesssim \|b\|_* \, w(\Gamma(t_0,r))^{\frac{1}{p}} \\ &\times \sup_{\tau>2r} \ln\left(e + \frac{\tau}{r}\right) w(\Gamma(t_0,\tau))^{-\frac{1}{p}} \|f\|_{L_{p,w}(\Gamma(t_0,\tau))}\end{aligned} \tag{14}$$

*holds for all $f \in L_{p,w}^{\mathrm{loc}}(\Gamma)$.*

*Proof.* Let $\Gamma$ be a Carleson curve, $1 < p < \infty$, $t_0 \in \Gamma$, $w \in A_p(\Gamma)$ and $b \in BMO(\Gamma)$. For arbitrary ball $\Gamma(t_0,r)$ let $f = f_1 + f_2$, where $f_1 = f\chi_{\Gamma(t_0,2r)}$ and $f_2 = f\chi_{(\Gamma(t_0,2r))}$.

$$\|M_{b,\Gamma} f\|_{L_{p,w}(\Gamma(t_0,r))} \le \|M_{b,\Gamma} f_1\|_{L_{p,w}(\Gamma(t_0,r))} + \|M_{b,\Gamma} f_2\|_{L_{p,w}(\Gamma(t_0,r))}.$$

By the continuity of the operator $M_{b,\Gamma} : L_{p,w}(\Gamma) \to L_{p,w}(\Gamma)$ (see Theorem 7) we have

$$\begin{aligned}&\|M_{b,\Gamma} f_1\|_{L_{p,w}(\Gamma(t_0,r))} \lesssim \|b\|_* \, \|f\|_{L_{p,w}(\Gamma(t_0,2r))} \\ &\lesssim \|b\|_* \, w(\Gamma(t_0,r))^{\frac{1}{p}} \sup_{\tau>2r} w(\Gamma(t_0,\tau))^{-\frac{1}{p}} \|f\|_{L_{p,w}(\Gamma(t_0,\tau))}.\end{aligned} \tag{15}$$

Let $y$ be an arbitrary point from $\Gamma(t_0,r)$. If $\Gamma(y,\tau) \cap (\Gamma(t_0,2r)) \neq \emptyset$, then $\tau > r$. Indeed, if $z \in \Gamma(y,\tau) \cap (\Gamma(t_0,2r))$, then $\tau > |y - z| \ge |t - z| - |t - y| > 2r - r = r$.

On the other hand, $\Gamma(y,\tau) \cap (\Gamma(t_0,2r)) \subset \Gamma(t_0,2\tau)$. Indeed, $z \in \Gamma(y,\tau) \cap (\Gamma(t_0,2r))$, then we get $|t - z| \le |y - z| + |t - y| < \tau + r < 2\tau$.

Hence, for all $y \in \Gamma(t_0, r)$

$$
\begin{aligned}
M_{b,\Gamma} f_2(y) &\lesssim \sup_{\tau > r} \frac{1}{\nu\Gamma(t_0,\tau)} \int_{\Gamma(t_0,2\tau)} |b(y)-b(z)|\,|f(z)| d\nu(z) \\
&\lesssim \sup_{\tau > r} \frac{1}{\nu\Gamma(t_0,2\tau)} \int_{\Gamma(t_0,2\tau)} |b(y)-b(z)|\,|f(z)| d\nu(z) \\
&\le \sup_{\tau > 2r} (\nu\Gamma(t_0,\tau))^{-1} \int_{\Gamma(t_0,\tau)} |b(y)-b(z)|\,|f(z)| d\nu(z).
\end{aligned}
$$

Therefore, for all $y \in \Gamma(t_0, r)$ we have

$$
M_{b,\Gamma} f_2(y) \lesssim \sup_{\tau > 2r} (\nu\Gamma(t_0,\tau))^{-1} \int_{\Gamma(t_0,\tau)} |b(y)-b(z)|\,|f(z)| d\nu(z).
$$

Thus, the function $M_\Gamma f_2(y)$, with fixed $t$ and $r$, is dominated by the expression not depending on $z$. Then

$$
\begin{aligned}
&\|M_{b,\Gamma} f_2\|_{L_{p,w}(\Gamma(t_0,r))} \\
&\lesssim \Big( \int_{\Gamma(t_0,r)} \Big( \sup_{\tau > 2r} |\Gamma(t_0,\tau)|^{-1} \int_{\Gamma(t_0,\tau)} |b(y)-b(z)|\,|f(y)| d\nu(y) \Big)^p w(z)\, d\nu(z) \Big)^{\frac{1}{p}} \\
&\lesssim \Big( \int_{\Gamma(t_0,r)} \Big( \sup_{\tau > 2r} |\Gamma(t_0,\tau)|^{-1} \int_{\Gamma(t_0,\tau)} |b(y)-b_{\Gamma(t_0,r),w}|\,|f(y)| d\nu(y) \Big)^p w(z)\, d\nu(z) \Big)^{\frac{1}{p}} \\
&+ \Big( \int_{\Gamma(t_0,r)} \Big( \sup_{\tau > 2r} |\Gamma(t_0,\tau)|^{-1} \int_{\Gamma(t_0,\tau)} |b(z)-b_{\Gamma(t_0,r),w}|\,|f(y)| d\nu(y) \Big)^p w(z)\, d\nu(z) \Big)^{\frac{1}{p}} \\
&= J_1 + J_2.
\end{aligned}
$$

Let us estimate $J_1$. Applying Hölder's inequality and by Lemma 5 we get

$$
\begin{aligned}
J_1 &= w(\Gamma(t_0,r))^{\frac{1}{p}} \sup_{\tau > 2r} |\Gamma(t_0,\tau)|^{-1} \int_{\Gamma(t_0,\tau)} |b(y)-b_{\Gamma(t_0,r),w}|\,|f(y)| d\nu(y) \\
&\approx w(\Gamma(t_0,r))^{\frac{1}{p}} \sup_{\tau > 2r} \tau^{-1} \int_{\Gamma(t_0,\tau)} |b(y)-b_{\Gamma(t_0,r),w}|\,|f(y)| d\nu(y) \\
&\le w(\Gamma(t_0,r))^{\frac{1}{p}} \sup_{\tau > 2r} \tau^{-1} \Big( \int_{\Gamma(t_0,\tau)} |b(y)-b_{\Gamma(t_0,r),w}|^{p'} w(y)^{1-p'} d\nu(y) \Big)^{\frac{1}{p'}} \|f\|_{L_{p,w}(\Gamma(t_0,\tau))} \\
&\lesssim \|b\|_* w(\Gamma(t_0,r))^{\frac{1}{p}} \sup_{\tau > 2r} \tau^{-1} \Big( 1 + \ln \frac{\tau}{r} \Big) \|w^{-1}\|_{L_{p'}(\Gamma(t_0,\tau))} \|f\|_{L_{p,w}(\Gamma(t_0,\tau))} \\
&\lesssim \|b\|_* w(\Gamma(t_0,r))^{\frac{1}{p}} \sup_{\tau > 2r} \tau^{-1} \Big( 1 + \ln \frac{\tau}{r} \Big) w(\Gamma(t_0,\tau))^{-\frac{1}{p}} \tau \|f\|_{L_{p,w}(\Gamma(t_0,\tau))} \\
&= \|b\|_* w(\Gamma(t_0,r))^{\frac{1}{p}} \sup_{\tau > 2r} \ln \Big( e + \frac{\tau}{r} \Big) w(\Gamma(t_0,\tau))^{-\frac{1}{p}} \|f\|_{L_{p,w}(\Gamma(t_0,\tau))}.
\end{aligned}
$$

In order to estimate $I_2$ from Minkowski's integral inequality we get

$$
\begin{aligned}
J_2 &= \Big( \int_{\Gamma(t_0,r)} \Big( \sup_{\tau>2r} |\Gamma(t_0,\tau)|^{-1} \int_{\Gamma(t_0,\tau)} |b(z)-b_{\Gamma(t_0,r),w}|\, |f(y)| d\nu(y) \Big)^p w(z)\, d\nu(z) \Big)^{\frac{1}{p}} \\
&\lesssim \sup_{\tau>2r} \tau^{-1} \int_{\Gamma(t_0,\tau)} \Big( \int_{\Gamma(t_0,r)} |b(z)-b_{\Gamma(t_0,r),w}|^p w(z) d\nu(z) \Big)^{\frac{1}{p}} |f(y)| d\nu(y) \\
&= \Big( \int_{\Gamma(t_0,r)} |b(z)-b_{\Gamma(t_0,r),w}|^p w(z) d\nu(z) \Big)^{\frac{1}{p}} \sup_{\tau>2r} \tau^{-1} \int_{\Gamma(t_0,\tau)} |f(y)| d\nu(y).
\end{aligned}
$$

According to the first part of Lemma 5, we get

$$
\begin{aligned}
J_2 &\lesssim \|b\|_* \Big(1+\ln\frac{\tau}{r}\Big) w(\Gamma(t_0,r))^{\frac{1}{p}} \sup_{\tau>2r} \tau^{-1} \int_{\Gamma(t_0,\tau)} |f(y)| dy \\
&\le \|b\|_* w(\Gamma(t_0,r))^{\frac{1}{p}} \sup_{\tau>2r} \tau^{-1} \|f\|_{L_{p,w}(\Gamma(t_0,\tau))} \|w^{-1}\|_{L_{p'}(\Gamma(t_0,\tau))} \\
&= \|b\|_* w(\Gamma(t_0,r))^{\frac{1}{p}} \sup_{\tau>2r} \ln\Big(e+\frac{\tau}{r}\Big) w(\Gamma(t_0,\tau))^{-\frac{1}{p}} \|f\|_{L_{p,w}(\Gamma(t_0,\tau))}.
\end{aligned}
$$

Summing up $J_1$ and $J_2$, for all $p \in (1,\infty)$ we get

$$
\begin{aligned}
\|M_{b,\Gamma} f_2\|_{L_{p,w}(\Gamma(t_0,r))} &\lesssim \|b\|_* w(\Gamma(t_0,r))^{\frac{1}{p}} \\
&\quad \times \sup_{\tau>2r} \ln\Big(e+\frac{\tau}{r}\Big) w(\Gamma(t_0,\tau))^{-\frac{1}{p}} \|f\|_{L_{p,w}(\Gamma(t_0,\tau))}. \qquad (16)
\end{aligned}
$$

Finally, from (15) and (16) we get

$$
\begin{aligned}
\|M_{b,\Gamma} f\|_{L_{p,w}(\Gamma(t_0,r))} &\lesssim \|b\|_* w(\Gamma(t_0,r))^{\frac{1}{p}} \sup_{\tau\ge r} \|f\|_{L_{p,w}(\Gamma(t_0,\tau))} w(\Gamma(t_0,\tau))^{-\frac{1}{p}} \\
&\quad + \|b\|_* w(\Gamma(t_0,r))^{\frac{1}{p}} \sup_{\tau>2r} \ln\Big(e+\frac{\tau}{r}\Big) w(\Gamma(t_0,\tau))^{-\frac{1}{p}} \|f\|_{L_{p,w}(\Gamma(t_0,\tau))} \\
&\lesssim \|b\|_* w(\Gamma(t_0,r))^{\frac{1}{p}} \sup_{\tau>2r} \ln\Big(e+\frac{\tau}{r}\Big) w(\Gamma(t_0,\tau))^{-\frac{1}{p}} \|f\|_{L_{p,w}(\Gamma(t_0,\tau))}.
\end{aligned}
$$

The following boundedness result for the operator $M_{b,\Gamma}$ on the space $M_{p,\varphi}(\Gamma,w)$ is valid.

**Theorem 6.** *Let $t_0 \in \Gamma$, $1 < p < \infty$, $w \in A_p(\Gamma)$, $b \in BMO(\Gamma)$ and $(\varphi_1,\varphi_2)$ satisfy the condition*

$$
\sup_{\tau>r} \ln\Big(e+\frac{\tau}{r}\Big) \frac{\operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(t_0,s) w(\Gamma(t_0,s))^{1/p}}{w(\Gamma(t_0,\tau))^{1/p}} \le C\varphi_2(t_0,r), \qquad (17)
$$

*where $C$ does not depend on $r$. Then the operator $M_{b,\Gamma}$ is bounded from $LM^{\{t_0\}}_{p,\varphi_1}(\Gamma,w)$ to $LM^{\{t_0\}}_{p,\varphi_2}(\Gamma,w)$. Moreover,*

$$\|M_{b,\Gamma}f\|_{LM^{\{t_0\}}_{p,\varphi_2}(\Gamma,w)} \lesssim \|b\|_* \, \|f\|_{LM^{\{t_0\}}_{p,\varphi_1}(\Gamma,w)}.$$

*Proof.* Using the Theorem 2 and the Theorem 7 we have

$$\begin{aligned}
&\|M_{b,\Gamma}f\|_{LM^{\{t_0\}}_{p,\varphi_2}(\Gamma,w)} = \sup_{r>0} \varphi_2(t_0,r)^{-1} w(\Gamma(t_0,r))^{\frac{1}{p}} \, \|M_{b,\Gamma}f\|_{L_{p,w}(\Gamma(t_0,r))} \\
&\lesssim \|b\|_* \sup_{r>0} \varphi_2(t_0,r)^{-1} \sup_{\tau>2r} \ln\left(e+\frac{\tau}{r}\right) w(\Gamma(t_0,\tau))^{-\frac{1}{p}} \|f\|_{L_{p,w}(\Gamma(t_0,\tau))} \\
&\lesssim \|b\|_* \sup_{r>0} \varphi_1(t_0,r)^{-1} w(\Gamma(t_0,r))^{-\frac{1}{p}} \, \|f\|_{L_{p,w}(\Gamma(t_0,r))} \\
&= \|b\|_* \, \|f\|_{LM^{\{t_0\}}_{p,\varphi_1}(\Gamma,w)}.
\end{aligned}$$

**Corollary 2.** *Let $1 < p < \infty$, $w \in A_p(\Gamma)$, $b \in BMO(\Gamma)$ and $(\varphi_1, \varphi_2)$ satisfy the condition*

$$\sup_{\tau>r} \ln\left(e+\frac{\tau}{r}\right) \frac{\operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(t,s) w(\Gamma(t,s))^{1/p}}{w(\Gamma(t,\tau))^{1/p}} \leq C\varphi_2(t,r), \tag{18}$$

*where $C$ does not depend on $t$ and $r$. Then the operator $M_{b,\Gamma}$ is bounded from $M_{p,\varphi_1}(\Gamma,w)$ to $M_{p,\varphi_2}(\Gamma,w)$. Moreover,*

$$\|M_{b,\Gamma}f\|_{M_{p,\varphi_2}(\Gamma,w)} \lesssim \|b\|_* \, \|f\|_{M_{p,\varphi_1}(\Gamma,w)}.$$

## 5. Commutators of maximal operator in the spaces $LM^{\{t_0\}}_{p,\varphi}(\Gamma,w)$ and $M_{p,\varphi}(\Gamma,w)$

In this section we find sufficient conditions for the boundedness of the commutator $[b,M_\Gamma]$ of the maximal operator on $LM^{\{t_0\}}_{p,\varphi}(\Gamma,w)$ and $M_{p,\varphi}(\Gamma,w)$ spaces.

The following relations between $[b,M_\Gamma]$ and $M_{b,\Gamma}$ are valid :

Let $b$ be any non-negative locally integrable function. Then for all $f \in L_1^{\mathrm{loc}}(\Gamma)$ and $t \in \Gamma$ the following inequality is valid

$$\begin{aligned}
&\big|[b,M_\Gamma]f(t)\big| = \big|b(t)M_\Gamma f(t) - M_\Gamma(bf)(t)\big| \\
&= \big|M_\Gamma(b(t)f)(t) - M_\Gamma(bf)(t)\big| \leq M_\Gamma(|b(t)-b|f)(t) \leq M_{b,\Gamma}(f)(t).
\end{aligned}$$

If $b$ is any locally integrable function on $L_1^{\mathrm{loc}}(\Gamma)$, then

$$|[b,M_\Gamma]f(t)| \le M_{b,\Gamma}(f)(t) + 2b^-(t)M_\Gamma f(t), \qquad t\in\Gamma \tag{19}$$

holds for all $f \in L_{1,w}^{\mathrm{loc}}(\Gamma)$ (see, for example, [6, 27, 29]).

Obviously, operators $M_{b,\Gamma}$ and $[b,M_\Gamma]$ essentially differ from each other since $M_{b,\Gamma}$ is positive and sublinear and $[b,M_\Gamma]$ is neither positive nor sublinear.

From Lemma 6 and inequality (19) we get the following corollary.

**Corollary 3.** *Let $b \in BMO(\Gamma)$ such that $b^- \in L_\infty(\Gamma)$. Then there exists a positive constant $C$ such that*

$$|[b,M_\Gamma]f(t)| \le C\big(\|b^+\|_* + \|b^-\|_{L_\infty}\big)M_\Gamma\big(M_\Gamma f\big)(t) \tag{20}$$

*for almost every $t\in\Gamma$ and for all functions from $f \in L_{1,w}^{\mathrm{loc}}(\Gamma)$.*

*Proof.* By Lemma 6, inequality (19) and the fact that $M_\Gamma f \le M_\Gamma\big(M_\Gamma f\big)$, we have

$$\begin{aligned}
&|[b,M_\Gamma]f(t)| \le M_{b,\Gamma}(f)(t) + 2b^-(t)M_\Gamma f(t)\\
&\lesssim \|b\|_* M\big(M_\Gamma f\big)(t) + b^-(t)M_\Gamma f(t)\\
&\lesssim \big(\|b^+\|_* + \|b^-\|_*\big)M_\Gamma\big(M_\Gamma f\big)(t) + \|b^-\|_{L_\infty(\Gamma)}M_\Gamma\big(M_\Gamma f\big)(t)\\
&\lesssim \big(\|b^+\|_* + \|b^-\|_{L_\infty(\Gamma)}\big)M_\Gamma\big(M_\Gamma f\big)(t).
\end{aligned}$$

By Corollary 3 and Theorem 6 we have

**Corollary 4.** *Let $\Gamma$ be a Carleson curve, $1<p<\infty$, $t_0\in\Gamma$, $w \in A_p(\Gamma)$ and $b \in BMO(\Gamma)$, $b^- \in L_\infty(\Gamma)$ and $(\varphi_1,\varphi_2)$ satisfy the condition* (17). *Then the operator $[b,M_\Gamma]$ is bounded from $LM_{p,\varphi_1}^{\{t_0\}}(\Gamma,w)$ to $LM_{p,\varphi_2}^{\{t_0\}}(\Gamma,w)$.*

**Corollary 5.** *Let $\Gamma$ be a Carleson curve, $1<p<\infty$, $w \in A_p(\Gamma)$ and $b \in BMO(\Gamma)$, $b^- \in L_\infty(\Gamma)$ and $(\varphi_1,\varphi_2)$ satisfy the condition* (18). *Then the operator $[b,M_\Gamma]$ is bounded from $M_{p,\varphi_1}(\Gamma,w)$ to $M_{p,\varphi_2}(\Gamma,w)$.*

**Acknowledgements** The author thanks the referee(s) for careful reading the paper and useful comments.

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Received 10 February 2025
Accepted 06 July 2025