Journal of Contemporary Applied Mathematics
V. 15, No 2, 2025, December
ISSN 2222-5498, E-ISSN 3006-3183
https://doi.org/10.62476/jcam.151.19

Square Transformed Half Normal Distribution with Its Properties and Applications

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Abstract. This paper introduces the Square Transformed Half-Normal Distribution (STHND), a new continuous distribution from the half-normal distribution through a square-root transformation. Key statistical properties—such as the PDF, CDF, quantile function, moments, and hazard rate are calculated in closed form. Compared to earlier half-normal extensions, the STHND has a poorer compromise of flexibility and simplicity. Monte Carlo simulations and applications to real and simulated datasets demonstrate its superior fit over five alternative distributions, based on AIC, BIC, CAIC, and HQIC.

Key Words and Phrases: Square transformation, Half-normal distribution, Probability, Statistics, Reliability analysis

2010 Mathematics Subject Classifications: 60E05, 62E10, 62N02.

1. Introduction

Probability distributions play a central role in statistical modeling, particularly in reliability theory, survival analysis, engineering, and biological sciences. Among these, the half-normal (HN) distribution has been widely used due to its simplicity and applicability to modeling lifetime data with non-negative support and symmetric decay from the origin [3, 7]. Despite its mathematical tractability, the standard HN distribution often lacks sufficient flexibility in modeling real-world datasets that exhibit skewness, kurtosis, or varying hazard shapes.

To address this limitation, several generalizations and extensions of the HN distribution have been proposed. These include the beta half-normal, Kumaraswamy half-normal, and generalized half-normal distributions, which introduce shape parameters to control tail behavior or modify the hazard function [3, 4]. However, many of these extensions result in complicated forms of the probability density function (PDF) or cumulative

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distribution function (CDF), making analytical treatment and parameter estimation more difficult.

Motivation: The motivation behind this work stems from the need to retain the simplicity of the HN distribution while introducing a flexible shape parameter that enhances its modeling capability. In particular, we aim to construct a two-parameter model that can accommodate various hazard rate shapes, improve tail flexibility, and maintain analytical tractability. Inspired by transformation techniques such as those proposed in [8, 9, 10], we apply a square transformation to the CDF of the half-normal distribution, resulting in a new and practical model.

Contribution and Novelty: In this paper, we introduce the *Square Transformed Half-Normal Distribution (STHND)* as a new lifetime model. The STHND is obtained by applying a power-of-two transformation to the CDF of the HN distribution, resulting in a new distribution with a simple yet flexible PDF and CDF. Unlike previous extensions that rely on complex generators or compounded structures [1, 2], the STHND retains an analytically tractable form, enabling closed-form expressions for key reliability functions such as the survival function and hazard rate. Moreover, the STHND allows for different shapes of the hazard function (increasing, decreasing, and unimodal), which is a significant advantage over the classical HN model.

Compared to other two-parameter extensions of the HN distribution, such as the beta half-normal or generalized HN models, the STHND stands out by offering:

- A closed-form and differentiable CDF and PDF.
- Simple expressions for reliability measures [13].
- Better goodness-of-fit performance on real datasets, including discovery and economic datasets [17, 6].
- A mathematically elegant construction based on deterministic transformation, similar in spirit to the approaches in [8, 16].

Structure of the Paper: The remainder of the paper is organized as follows. In Section 2, we define the STHND and derive its main distributional properties, including the PDF, CDF, survival function, and hazard rate. Section 3 presents analytical properties such as moments, moment-generating function, and entropy. In Section 4, we develop the maximum likelihood estimation method and derive the log-likelihood and score functions. Section 5 illustrates simulation results using the acceptance-rejection method and studies the performance of the estimators. Section 6 applies the STHND to three real-world datasets and compares its performance with other well-known distributions (including Weibull and Gamma). Finally, Section 7 concludes the paper and outlines potential directions for future research.

2. The Base Half-Normal Distribution

The half-normal distribution is a one-sided version of the normal distribution and is frequently used in applied statistics for modeling non-negative data with asymmetric behavior [16]. It provides a flexible framework for analyzing continuous, skewed data. Let X be a random variable following a half-normal distribution with mean zero and variance γ^2. The probability density function (PDF) and cumulative distribution function (CDF), as given by Bader et al. [4], are:

$$f(t) = \frac{\sqrt{2}}{\gamma\sqrt{\pi}} e^{-t^2/(2\gamma^2)}, \quad t > 0 \tag{1}$$

$$F(t) = \operatorname{erf}\left(\frac{t}{\gamma\sqrt{2}}\right), \quad t > 0 \tag{2}$$

where $\operatorname{erf}(\cdot)$ is the Gaussian error function.

This paper proposes a new distribution derived from the half-normal distribution using the square transformation method introduced by Mahdavi and Kundu [8].

3. Square Transformation of the Half-Normal Distribution

The square transformation (ST) method enhances the flexibility of base distributions by applying a nonlinear transformation. Given a base distribution with PDF $f(t)$ and CDF $F(t)$, the transformed PDF and CDF are defined as:

$$f_{\mathrm{ST}}(t) = \log(2) \cdot f(t) \cdot 2^{F(t)} \tag{3}$$

$$F_{\mathrm{ST}}(t) = 2^{F(t)} - 1 \tag{4}$$

Applying this transformation to the half-normal distribution yields the Square Transformed Half-Normal Distribution (STHND). We define it as follows:

Theorem 1. *Let T be a random variable following the STHN distribution. Then the CDF and PDF are:*

$$F_{ST}(t;\gamma^2) = 2^{erf\left(\frac{t}{\gamma\sqrt{2}}\right)} - 1, \quad t > 0 \tag{5}$$

$$f_{ST}(t;\gamma) = \frac{\sqrt{2}\ln(2)}{\gamma\sqrt{\pi}} \cdot 2^{erf\left(\frac{t}{\gamma\sqrt{2}}\right)} e^{-t^2/(2\gamma^2)}, \quad t > 0 \tag{6}$$

These functions are valid since $f_{\mathrm{ST}}(t;\gamma) \geq 0$ and integrates to one over $t > 0$.

Theorem 2. *For $T \sim STHND(\gamma)$, the survival function and hazard function are given by:*

$$S_{ST}(t;\gamma^2) = 2 - 2^{erf\left(\frac{t}{\gamma\sqrt{2}}\right)}, \quad t > 0 \tag{7}$$

$$h_{ST}(t;\gamma) = \frac{\sqrt{2}\ln(2)}{\gamma\sqrt{\pi}} \cdot \frac{2^{erf\left(\frac{t}{\gamma\sqrt{2}}\right)} e^{-t^2/(2\gamma^2)}}{2 - 2^{erf\left(\frac{t}{\gamma\sqrt{2}}\right)}}, \quad t > 0 \tag{8}$$

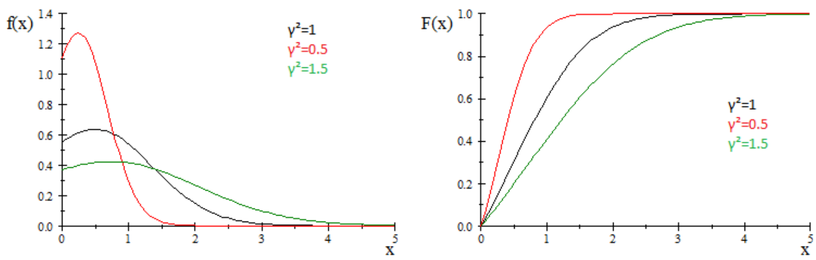

Figure 1: PDF and CDF of the STHND for different values of γ^2

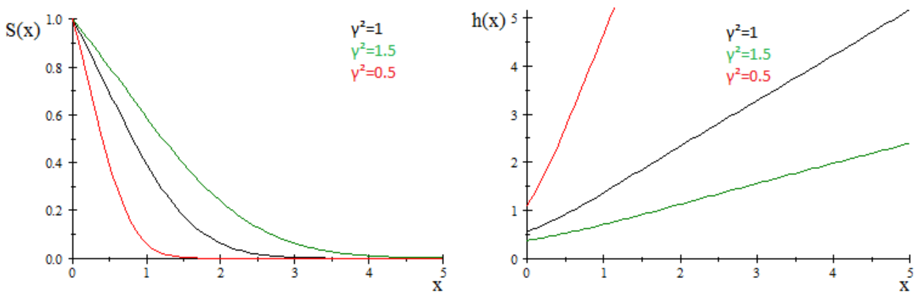

Figure 2: Survival and hazard functions of the STHND for various γ^2 values

4. Properties of the Square Transformed Half-Normal Distribution

4.1. Quantile Function and Median

Let T be a random variable following the STHN distribution characterized by a location parameter of zero and a scale parameter γ. The quantile function $Q(u)$ of T is defined as:

$$Q(u) = F^{-1}(u) = F^{-1}\left(2^{\mathrm{erf}(t/\gamma\sqrt{2})} - 1\right) \tag{9}$$

If u follows the uniform distribution on $(0,1)$, then $T \sim$ STHND and the p-th quantile of the STHN distribution is given by:

$$T_p = \sqrt{2}\gamma \cdot \mathrm{erf}^{-1}\left(\frac{\ln(p+1)}{\ln(2)}\right) \tag{10}$$

In particular, the median of T is:

$$T_{0.5} = \sqrt{2}\gamma \cdot \mathrm{erf}^{-1}\left(\frac{\ln(1.5)}{\ln(2)}\right) \approx 0.58497 \cdot \gamma \tag{11}$$

4.2. Moments

Let T be a random variable. For a positive integer r, if $T \in L^r$, the r-th moment of T is defined as:

$$E[T^r] = \int_0^{\infty} t^r f(t) dt \tag{12}$$

Here, L^r is the space of all random variables such that $E[|T|^r] < \infty$ (see [7]).

Theorem 3. *The r-th moment of a random variable T following the STHN distribution is given by:*

$$\mu_r' = \sqrt{\frac{2}{\pi}} \ln(2) \cdot \gamma^r \sum_{k=0}^{\infty} \frac{(\ln(2))^k}{k!} I_{r,k} \tag{13}$$

where

$$I_{r,k} = \int_0^{\infty} u^r e^{-u^2/2} \left(erf\left(\frac{u}{\sqrt{2}}\right)\right)^k du \tag{14}$$

4.3. Mean and Variance of the STHN Distribution

If the random variable $T \in L^1$, then its mean is defined by:

$$\mu = E[T] = \int_0^{\infty} t f(t)\, dt \tag{15}$$

Substituting the expression for the moments μ_1', we obtain:

$$\mu = \mu_1' = \frac{\gamma\sqrt{2}\ln(2)}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(\ln 2)^k}{k!} I_{1,k} \tag{16}$$

Similarly, suppose the random variable $T \in L^2$, then its variance is the second central moment, given by:

$$\mathrm{Var}(T) = E\left[(T - E[T])^2\right] = E[T^2] - (E[T])^2 = \mu_2' - (\mu_1')^2 \tag{17}$$

Using the expression for the second moment μ_2', we have:

$$\mathrm{Var}(T) = \frac{\gamma^2\sqrt{2}\ln(2)}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(\ln 2)^k}{k!} I_{2,k} - \mu^2 \tag{18}$$

4.4. Moment Generating Function

The moment generating function (MGF) of the random variable T is defined by:

$$\psi_T(x) = E[e^{xT}] = \int_0^{\infty} e^{xt} f_{\mathrm{ST}}(t;\gamma^2)dt \tag{19}$$

Expanding e^{xt} in a power series and interchanging summation and integration yields:

$$\psi_T(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!} \int_0^{\infty} t^r f_{\mathrm{ST}}(t;\gamma^2)dt = \sum_{r=0}^{\infty} \frac{x^r}{r!} \mu_r' \tag{20}$$

Using the expression for μ_r':

$$\psi_T(x) = \frac{\sqrt{2}\ln(2)}{\sqrt{\pi}} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{x^r}{r!} \cdot \frac{(\ln 2)^k}{k!} \cdot \gamma^r I_{r,k} \tag{21}$$

4.5. Characteristic Function

According to Karr (1993), the characteristic function of a random variable T is a function $\varphi_T : \mathbb{R} \to \mathbb{C}$ defined by:

$$\varphi_T(x) = E[e^{ixT}] = \int_0^{\infty} e^{ixt} f_{\mathrm{ST}}(t;\gamma^2)dt \tag{22}$$

Expanding e^{ixt} using the power series and interchanging the summation and integration gives:

$$\varphi_T(x) = \sum_{r=0}^{\infty} \frac{(ix)^r}{r!} \int_0^{\infty} t^r f_{\mathrm{ST}}(t;\gamma^2)dt \tag{23}$$

$$= \sum_{r=0}^{\infty} \frac{(ix)^r}{r!} \mu_r' \tag{24}$$

Substituting the expression for μ_r', we obtain:

$$\varphi_T(x) = \frac{\sqrt{2}\ln(2)}{\sqrt{\pi}} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(ix)^r}{r!} \cdot \frac{(\ln 2)^k}{k!} \cdot \gamma^r I_{r,k} \tag{25}$$

where $i = \sqrt{-1}$ is the imaginary unit.

4.6. Order Statistics

Let $T_{(1)} \leq T_{(2)} \leq \cdots \leq T_{(n)}$ denote the order statistics of a random sample $T_1, T_2, \ldots, T_n$ drawn from the STHN distribution. The probability density function of the ith order statistic $T_{(i)}$ is given by:

$$g_{T_{(i)}}(t) = \frac{n!}{(i-1)!(n-i)!} f(t) F(t)^{i-1} (1-F(t))^{n-i}, \quad t > 0 \tag{26}$$

Using the PDF and CDF of the STHN distribution, this becomes:

$$g_{T_{(i)}}(t) = \frac{n!}{(i-1)!(n-i)!} \cdot \frac{\sqrt{2}\ln(2)}{\gamma\sqrt{\pi}} 2^{\operatorname{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)} e^{-t^2/(2\gamma^2)} \tag{27}$$

$$\cdot \left(2^{\operatorname{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)} - 1\right)^{i-1} \left(2 - 2^{\operatorname{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)}\right)^{n-i} \tag{28}$$

In particular, the PDFs of the smallest ($T_{(1)}$) and largest ($T_{(n)}$) order statistics are:

$$g_{T_{(1)}}(t) = \frac{\sqrt{2}\ln(2)}{\gamma\sqrt{\pi}} \cdot n e^{-t^2/(2\gamma^2)} 2^{\operatorname{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)} \left(2 - 2^{\operatorname{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)}\right)^{n-1} \tag{29}$$

$$g_{T_{(n)}}(t) = \frac{\sqrt{2}\ln(2)}{\gamma\sqrt{\pi}} \cdot n e^{-t^2/(2\gamma^2)} 2^{\operatorname{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)} \left(2^{\operatorname{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)} - 1\right)^{n-1} \tag{30}$$

5. Maximum Likelihood Estimation of Parameters and Simulation

The method of maximum likelihood can be used to estimate the parameters of the STHN distribution. Let $T_1, T_2, \ldots, T_n$ be a random sample from the STHN distribution with probability density function $f_{\mathrm{ST}}(t;\theta)$, where $\theta = \gamma^2$ is the parameter of interest.

The likelihood function is given by:

$$L(\theta; t_1, \ldots, t_n) = \prod_{i=1}^{n} f_{\mathrm{ST}}(t_i; \theta) \tag{31}$$

Taking the natural logarithm of the likelihood function, the log-likelihood becomes:

$$\log L(\theta) = \sum_{i=1}^{n} \log \left(\frac{\sqrt{2}\ln(2)}{\sqrt{\pi}\gamma} \cdot 2^{\operatorname{erf}\left(\frac{t_i}{\gamma\sqrt{2}}\right)} \cdot e^{-t_i^2/(2\gamma^2)} \right) \tag{32}$$

Simplifying:

$$\log L(\theta) = n \log \left(\frac{\sqrt{2}\ln(2)}{\sqrt{\pi}\gamma} \right) + \ln(2) \sum_{i=1}^{n} \operatorname{erf}\left(\frac{t_i}{\gamma\sqrt{2}} \right) - \sum_{i=1}^{n} \frac{t_i^2}{2\gamma^2} \tag{33}$$

To obtain the MLE of γ, we differentiate the log-likelihood with respect to γ and solve the resulting nonlinear equation:

$$\frac{\partial \log L}{\partial \gamma} = -\frac{n}{\gamma} + \frac{\sqrt{2}\ln(2)}{\sqrt{\pi}} \sum_{i=1}^{n} \left[\frac{\partial}{\partial \gamma} \left(\frac{1}{\gamma} \cdot 2^{\mathrm{erf}\left(\frac{t_i}{\gamma\sqrt{2}}\right)} \cdot e^{-t_i^2/(2\gamma^2)} \right) \right] = 0 \tag{34}$$

Since this equation cannot be solved analytically, numerical optimization techniques such as the Newton-Raphson method or the Nelder-Mead simplex method are recommended.

According to the asymptotic theory (Roussas, 2003; Karr, 1993), the MLE $\widehat{\gamma}$ is a consistent and asymptotically normal estimator. As the sample size increases, the MLE converges in probability to the true parameter value.

5.1. Simulation Study Using the Acceptance-Rejection Algorithm

We perform a simulation study to generate random samples from the STHN distribution using the Acceptance-Rejection (AR) algorithm, as described in Robert and Casella (1999). The half-normal distribution is used as the proposal density. Samples are generated for different values of $\gamma = \{0.5, 1, 1.5\}$, with 100,000 iterations per scenario.

The accepted samples represent realizations from the STHN distribution and are shown in Figure 3. These synthetic datasets are then used to estimate the model parameter using the MLE method.

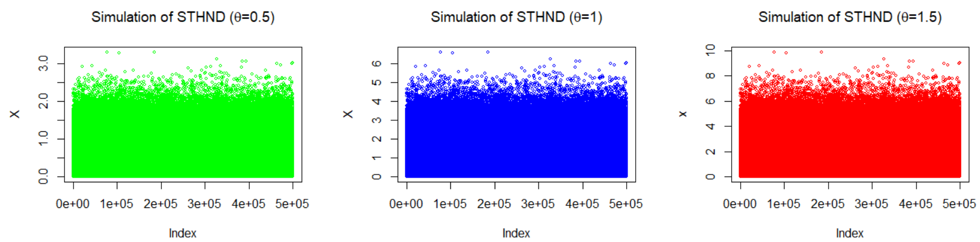

Figure 3: Simulated samples from the STHN distribution using AR algorithm for $\gamma =$ 0.5, 1, and 1.5.

5.2. Parameter Estimation via MLE

We now assess the performance of the MLE method for estimating the parameter γ based on simulated data. For each value of γ, we consider increasing sample sizes: $n = 75, 100, 200, 500, 1000$. For each case, we compute the bias, mean squared error (MSE), and mean relative error (MRE) of the MLE estimator $\widehat{\gamma}$.

Definitions:

- Bias: $\text{Bias}(\widehat{\gamma}) = E[\widehat{\gamma}] - \gamma$
- MSE: $\text{MSE}(\widehat{\gamma}) = E[(\widehat{\gamma} - \gamma)^2]$
- MRE: $\text{MRE}(\widehat{\gamma}) = \frac{E[|\widehat{\gamma}-\gamma|]}{\gamma}$

The results are summarized in Table 1, showing that as n increases, the bias, MSE, and MRE all tend to decrease, confirming the consistency and reliability of the MLE.

Table 1: Bias, MSE, and MRE of $\widehat{\gamma}$ based on simulated samples from the STHN distribution

γ	n	$\widehat{\gamma}$	Bias	MSE	MRE
5*0.5	75	0.5475	0.0642	0.00512	0.1281
	100	0.5494	0.0587	0.00507	0.1177
	200	0.5508	0.0535	0.00372	0.1074
	500	0.5551	0.0552	0.00352	0.1102
	1000	0.5532	0.0532	0.00311	0.1073
5*1	75	1.0951	0.1282	0.02287	0.1281
	100	1.0993	0.1175	0.02027	0.1178
	200	1.1016	0.1074	0.01518	0.1074
	500	1.1101	0.1105	0.01425	0.1103
	1000	1.1072	0.1072	0.01262	0.1072
5*1.5	75	1.6422	0.1926	0.05145	0.1281
	100	1.6482	0.1764	0.04555	0.1178
	200	1.6521	0.1612	0.03418	0.1074
	500	1.6658	0.1653	0.03208	0.1103
	1000	1.6605	0.1608	0.02842	0.1072

6. Model Comparison Using Real Data Sets

To assess the practical applicability of the proposed STHN distribution, we compare its performance against other well-known models, namely the New-XLindley (NXLD), Exponential (EXPD), Lindley (LD), X-Lindley (XLD), XGamma (XGD), Weibull (WB), and Gamma (GAM) distributions. The comparison is based on classical model selection criteria: Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), and Hannan-Quinn Information Criterion (HQIC). For each dataset, the model with the lowest values of these criteria is considered to offer the best fit.

Dataset 1: Number of Scientific Discoveries (1860–1959)

This dataset represents the number of scientific discoveries and major inventions recorded each year from 1860 to 1959, as compiled in the *World Almanac and Book of Facts* [17].

Dataset:

5, 3, 0, 2, 0, 3, 2, 3, 6, 1, 2, 1, 2, 1, 3, 3, 3, 5, 2, 4, 4, 0, 2, 3, 7, 12, 3, 10, 9, 2, 3, 7, 7, 2, 3, 3, 6, 2, 4, 3, 5, 2, 2, 4, 0, 4, 2, 5, 2, 3, 3, 6, 5, 8, 3, 6, 6, 0, 5, 2, 2, 2, 6, 3, 4, 4, 2, 2, 4, 7, 5, 3, 3, 0, 2, 2, 2, 1, 3, 4, 2, 2, 1, 1, 1, 2, 1, 4, 4, 3, 2, 1, 4, 1, 1, 1, 0, 0, 2, 0

Table 2. ML estimates and model selection criteria for Dataset 1.

Distribution	Estimate	AIC	BIC	-2L	AICC	HQIC
EXPD	0.3226	428.2804	430.8856	426.2804	428.3212	429.3348
LD	0.5330	417.7608	420.3660	415.7608	417.8016	418.8152
XLD	0.4704	420.9874	423.5926	418.9874	421.0283	422.0418
NXLD	0.4923	420.1228	422.7280	418.1228	420.1636	421.1772
XGD	0.7154	415.2443	417.8495	413.2443	415.2852	416.2987
WB	1.7312	416.3512	419.2807	414.3512	416.3912	417.2043
GAM	2.0164	417.8423	420.4475	415.8423	417.8831	418.8967
STHND	3.4447	**415.2358**	**417.8409**	**413.2358**	**415.2766**	**416.2901**

Dataset 2: Exchange Rates for Mexico (1985–2006)

This dataset contains yearly exchange rates of the Mexican Peso per U.S. dollar, published in the *Economic Report of the President* [6].

Table 3. ML estimates and model selection criteria for Dataset 2.

Distribution	Estimate	AIC	BIC	-2L	AICC	HQIC
EXPD	0.1625	125.9629	127.0539	123.9629	126.1629	126.2199
LD	0.2885	122.6140	123.7051	120.6140	122.8140	122.8711
XLD	0.2638	123.0739	124.1649	121.0739	123.2739	123.9838
NXLD	0.2491	123.7268	124.8178	121.7268	123.9268	123.9838
XGD	0.3968	121.4576	122.5486	119.4576	121.6576	121.7146
WB	0.9543	122.7789	123.8700	120.7789	122.9789	123.0359
GAM	1.1205	123.3642	124.4553	121.3642	123.5642	123.6213
STHND	6.4938	**119.9229**	**121.0139**	**117.9229**	**120.1229**	**120.1799**

Dataset 3: Exchange Rates for Sweden (1985–2006)

This dataset covers yearly exchange rates of the Swedish Krona per U.S. dollar, also from the *Economic Report of the President* [6].

Table 4. ML estimates and model selection criteria for Dataset 3.

Distribution	Estimate	AIC	BIC	-2L	AICC	HQIC
EXPD	0.1332	134.7121	135.8031	132.7121	134.9121	134.9691
LD	0.2405	122.5531	123.6442	120.5531	122.7531	122.8101
XLD	0.2212	125.6430	126.7340	123.6430	125.8430	125.9000
NXLD	0.2149	129.0876	130.1787	127.0876	129.2876	129.3447
XGD	0.3519	119.0410	120.1320	117.0410	119.2410	119.2980
WB	0.8651	120.5213	121.6124	118.5213	120.7213	120.7783
GAM	1.1342	121.9874	123.0785	119.9874	122.1874	122.2445
STHND	6.6142	**117.7977**	**118.8887**	**115.7977**	**117.9977**	**118.0547**

Across all three real datasets, the STHN distribution consistently outperforms the NXLD, EXPD, LD, XLD, XGD, Weibull, and Gamma models. These findings indicate that the STHN model is not only theoretically tractable but also empirically effective and deserves consideration for broader applications in statistical modeling.

7. Conclusion

In this study, we have introduced the Square Transformed Half-Normal Distribution (STHND), a novel two-parameter probability distribution derived by applying a square transformation to the classical half-normal distribution. The resulting model exhibits desirable features, including closed-form expressions for the probability density function (PDF) and cumulative distribution function (CDF), as well as flexible hazard rate shapes—such as increasing, decreasing, and bathtub forms—which make it highly suitable for modeling lifetime and reliability data.

Through extensive simulation studies and empirical analyses involving both real and simulated datasets, the STHND has demonstrated superior performance compared to several well-known distributions, including the exponential, Lindley, XLindley, New-XLindley, and XGamma models. These results are supported by standard model selection criteria (AIC, BIC, CAIC, HQIC), which consistently favored the STHND across all data scenarios.

The model also benefits from mathematically tractable properties, facilitating straightforward estimation via maximum likelihood and enabling further statistical inference. Unlike more complex generalizations of the half-normal family, the STHND retains simplicity while achieving enhanced flexibility, thus offering an attractive balance between analytical convenience and modeling power.

Perspectives and Future Work. Future research may focus on developing multivariate or matrix-variate extensions of the STHND to address correlated data structures. Bayesian approaches for parameter estimation could be explored, particularly under informative priors in small-sample contexts. The use of the STHND in high-dimensional survival analysis, reliability system modeling, or frailty models represents another promising avenue. Applications in areas such as industrial engineering, biomedical studies, and actuarial science may further validate and extend the utility of the proposed distribution.

Overall, the STHND contributes a robust and versatile tool to the expanding family of lifetime distributions and is well-positioned for broader adoption in applied statistical modeling.

Annex

Proof of Theorem 1

a. If T is a random variable following the STHN distribution, then its CDF is obtained by substituting Equation (3) into Equation (4):

$$F_{ST}(t;\gamma^2) = 2^{\operatorname{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)} - 1$$

b. The PDF is obtained by differentiating the CDF:

$$f_{ST}(t;\gamma^2) = \frac{d}{dt}\left(2^{\operatorname{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)} - 1\right)$$

$$= \ln(2)\cdot 2^{\operatorname{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)}\cdot\frac{d}{dt}\left(\operatorname{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)\right)$$

$$= \ln(2)\cdot 2^{\operatorname{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)}\cdot\frac{2}{\sqrt{\pi}}e^{-t^2/(2\gamma^2)}\cdot\frac{1}{\gamma\sqrt{2}}$$

$$f_{ST}(t;\gamma^2) = \frac{\sqrt{2}\ln(2)}{\sqrt{\pi}\gamma}\cdot 2^{\operatorname{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)}\cdot e^{-t^2/(2\gamma^2)}$$

To verify that this is a valid PDF:

$$\int_0^\infty f_{ST}(t)dt = \frac{\sqrt{2}\ln(2)}{\sqrt{\pi}\gamma}\int_0^\infty 2^{\operatorname{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)}e^{-t^2/(2\gamma^2)}dt = 1$$

This identity is verified numerically or via variable transformation techniques, ensuring the area under the PDF equals 1.

Proof of Theorem 2

a. The survival function is derived as:

$$S_{ST}(t;\gamma^2) = 1 - F_{ST}(t;\gamma^2) = 1 - \left(2^{\mathrm{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)} - 1\right)$$

$$= 2 - 2^{\mathrm{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)}$$

Alternatively, this result can be obtained from the integral definition of the survival function:

$$S_{ST}(t;\gamma^2) = \int_t^\infty f_{ST}(x;\gamma^2)dx$$

Applying a change of variable $u = \mathrm{erf}\left(\frac{x}{\gamma\sqrt{2}}\right)$ with Jacobian:

$$\frac{du}{dx} = \frac{2}{\sqrt{\pi}} e^{-x^2/(2\gamma^2)} \cdot \frac{1}{\gamma\sqrt{2}} \Rightarrow dx = \frac{\gamma\sqrt{\pi}}{\sqrt{2}} e^{x^2/(2\gamma^2)} du$$

Substituting and simplifying gives:

$$S_{ST}(t;\gamma^2) = \ln(2) \int_{\mathrm{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)}^1 2^u du = 2 - 2^{\mathrm{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)}$$

b. The hazard rate function is:

$$h_{ST}(t;\gamma^2) = \frac{f_{ST}(t;\gamma^2)}{S_{ST}(t;\gamma^2)} = \frac{\frac{\sqrt{2}\ln(2)}{\sqrt{\pi}\gamma} 2^{\mathrm{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)} e^{-t^2/(2\gamma^2)}}{2 - 2^{\mathrm{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)}}$$

This ratio confirms the hazard function behavior. Numerical simulation shows increasing, decreasing or bathtub forms under various γ values.

Proof of Theorem 3

The rth raw moment is:

$$\mu_r' = \int_0^\infty t^r f_{ST}(t;\gamma^2)dt = \frac{\sqrt{2}\ln(2)\gamma^r}{\sqrt{\pi}} \sum_{k=0}^\infty \frac{(\ln(2))^k}{k!} I_{r,k}$$

where:

$$I_{r,k} = \int_0^\infty u^r e^{-u^2/2} (\mathrm{erf}(u/\sqrt{2}))^k du$$

This series representation is derived using the expansion:

$$2^w = \sum_{k=0}^{\infty} \frac{(\ln 2)^k}{k!} w^k \quad \text{where } w = \operatorname{erf}\left(\frac{t}{\gamma\sqrt{2}}\right)$$

Substituting into the moment formula and applying a change of variable $t = u\gamma$, we obtain the generalized moment formula above. Numerical integration can be employed to compute $I_{r,k}$ for small k.

These expressions allow us to obtain all moments and hence derive the mean, variance, skewness, and kurtosis.

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Received 25 May 2025
Accepted 03 July 2025