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Renormalized Solution for a Noncoercive Elliptic Problem with L^1-data

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Abstract. This study investigates a non-coercive elliptic equation of the form $-\mathrm{div}(k(u)\nabla u) + \mathrm{div}(\mathbf{v}\,u) + bu = f$ in a domain Ω, subject to homogeneous Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. We define a notion of renormalized solution and we prove the existence of a solution.

Key Words and Phrases: Elliptic problem, Renormalized solution, Dirichlet boundary conditions, L^1-data.

2010 Mathematics Subject Classifications: 35B30, 47B44.

1. Introduction

In this paper, we study the following noncoercive elliptic problem whose prototype is

$$\begin{cases} -\mathrm{div}(k(u)\nabla u) + \mathrm{div}(\mathbf{v}u) + bu = f \text{ in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{1}$$

Here, $\Omega \subset \mathbb{R}^d$ is a bounded domain with $d \geq 3$, and $k(\cdot)$ is a continuous diffusion coefficient constrained by $0 < k_0 \leq k(u)$ with k_0 a positive real number. The vector field $\mathbf{v} \in (L^p(\Omega))^d$ with $p = d$ if $d \geq 3$, and $b \in L^2(\Omega)$ is a non-negative function. We are interested in proving the existence and uniqueness result for (1). The difficulties connected to this problem are due to the L^1-data and to the presence of the term $\mathrm{div}(\mathbf{v}u)$, which induces a lack of coercivity ; thus, in general, the operator $A(u) = -\mathrm{div}(k(u)\nabla u) + \mathrm{div}(\mathbf{v}u) + bu$ is not coercive unless $\|\mathbf{v}\|_{L^p(\Omega)^d}$ is small enough. But if $f \in L^1(\Omega)$, and is no more an element of the dual space of $H_0^1(\Omega)$, we need to give some meaning to the notion of solution.

Thus, DiPerna and Lions in [2] introduce the notion of renormalized solution for ordinary differential equations, which has been extended to the elliptic case in [5] (see also [6]).

In the present paper, we prove the existence result for the renormalized solution to (1). Similar problems to (1) have already been studied in the literature. A noncoercive linear case has been studied in [7]. In [10], the authors gave a definition of a renormalized solution for elliptic problems with measure data and proved the existence of (1) with $k \equiv Id$. Moreover, for $\mathbf{v} \in (L^p(\Omega))^d$ with $p = d$ if $d \geq 3$ or $p > 2$ if $d = 2$, and $f \in L^1(\Omega)$, Ouédraogo, A., and Yaméogo, W. B. in [9] study the following problem :

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) + \mathrm{div}(\mathbf{v}u) + bu = f \text{ in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{2}$$

where $A : \Omega \to \mathbb{R}^{N\times N}$ is a measurable matrix field such that for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$ one has $\alpha\xi\cdot\xi \leq A(x)\xi\cdot\xi \leq \beta\xi\cdot\xi$ for some $\alpha, \beta \in \mathbb{R}_+^*$. They prove that the approximate solution, by the finite volume method, converges to the renormalized solution of (2). In this paper, thanks to the convection term $\mathrm{div}(\mathbf{v}u)$, it is not possible to use minimization techniques to get the existence of a solution. Therefore, thanks to the pseudo-monotone operator theory, we show that there exist weak energy solutions u_ε to the approximate problem of (1). Another difficulty that arises is obtaining uniform (with respect to ε) a priori estimates on the solutions u_ε of approximating problems. Our strategy is to use the weak maximum principle to establish the uniform boundedness of u_ε and use the method of passing the limit to get the existence result for problem (1). One motivation for studying (1) is that it plays a role in modeling various physical phenomena, such as Thomas-Fermi models in atomic physics or the modeling of porous flow in reservoirs [4].

This paper is organized as follows : in Section 2, we introduce the main result of this study. We define the notion of a renormalized solution of (1), followed by the statement of Theorem 1, which constitutes the central result of the paper. Sect. 3 focuses on the detailed proof of the main result. We outline the essential steps, establish intermediate results, and provide a rigorous proof of the result. Sect. 4 summarizes the main contribution of this work, discusses its implications, and highlights potential directions for future research.

2. Main result

Throughout this paper, we denote $meas(E)$, or $|E|$ the Lebesgue measure of E, χ_E represents the characteristic function of E. For all $\kappa > 0$, we denote by T_κ

the truncation function at level κ defined by $T_\kappa(r) = \max(-\kappa, \min(\kappa, r))$ for all $r \in \mathbb{R}$ and we define the continuous function S_m by

$$S_m(r) = 1 - |T_{m+1}(r) - T_m(r)|. \tag{3}$$

Now, we recall the gradient of measurable functions whose truncated versions have finite energy.

Lemma 1 (see [1]). *Let $u : \Omega \to \mathbb{R}$ be a measurable function such that $T_\kappa(u) \in H_0^1(\Omega)$ for every $\kappa > 0$. Then there exists a unique measurable function $w : \Omega \to \mathbb{R}^d$ such that*

$$\nabla T_\kappa(u) = w\chi_{\{|u|<\kappa\}} \text{ a.e. in } \Omega.$$

We will define the gradient of u as the function w, and we will denote it by $w \equiv \nabla u$.

The definition of a renormalized solution to problem (1) is the following.

Definition 1. *A measurable function $u : \Omega \to \mathbb{R}$ is a renormalized solution of (1) if it satisfies the following conditions:*

$$\forall\kappa > 0, \quad T_\kappa(u) \in H_0^1(\Omega); \tag{4}$$

for any function $S \in W^{1,\infty}(\mathbb{R})$ with supp(S) compact, u satisfies the equation

$$\begin{aligned}-\mathrm{div}(S(u)\, k(u)\nabla u) + S'(u)\, k(u)\, \nabla u \cdot \nabla u + \mathrm{div}(S(u)\, u\, \mathbf{v})\\ -S'(u)\, u\mathbf{v} \cdot \nabla u + bu\, S(u) = f S(u) \text{ in } \mathcal{D}'(\Omega),\end{aligned} \tag{5}$$

$$\lim_{m\to\infty} \int_{\{m<|T_\kappa(u)|<m+1\}} k(u)|\nabla T_\kappa(u)|^2\, dx = 0. \tag{6}$$

The following remarks concern a few comments on Definition 1.

Remark 1. *Condition (4) enables the definition of ∇u almost everywhere in Ω. In (5), which is derived by point-wise multiplying (1) by $S(u)$, all terms are well-defined. Specifically, because Supp(S) is compact, we have supp$(S) \subset [-\kappa, \kappa]$ for some sufficiently large $\kappa > 0$. This implies that $S(u)\, k(u)\nabla u = S(u)\, k(T_\kappa(u))\nabla T_\kappa(u)$ a.e. in Ω, and thus it belongs to $(L^2(\Omega))^d$. Similarly, $S'(u)\, k(u)\, \nabla u \cdot \nabla u$ can be identified with $S'(u)\, k(T_\kappa(u))\, \nabla T_\kappa(u) \cdot \nabla T_\kappa(u)$, which is in $L^1(\Omega)$. Condition (6) is a standard requirement in the theory of renormalized solutions and provides further information about ∇u for large values of $|u|$.*

Our purpose is to establish the following existence result for (1).

Theorem 1. *Let $f \in L^1(\Omega)$ and assuming that $\mathbf{v} \in (L^p(\Omega))^d$ with $p = d$ if $d \geq 3$. Then, the problem (1) has at least one renormalized solution $u \in H_0^1(\Omega)$.*

3. Proof of the Theorem 1

The proof proceeds in several steps. First, we introduce the approximate problem for (1) and prove the existence of a weak solution. Then, we establish some a priori estimates, extract sub-sequences, and analyze their convergence. This leads to a measurable function u, which is finite almost everywhere in Ω. Finally, using an appropriate test function, we pass to the limit in the approximate problem and prove that u is the desired renormalized solution to (1).

3.1. Approximate problems

In order to get the existence of weak solutions, we introduce the approximate problem to (1), namely

$$\begin{cases} -\mathrm{div}(k_\epsilon(T_{1/\epsilon}(u_\epsilon))\nabla u_\epsilon) + \mathrm{div}(\mathbf{v}\, u_\epsilon) + b\, u_\epsilon = f_\epsilon \text{ in } \Omega, \\ \\ u_\epsilon \in H_0^1(\Omega), \end{cases} \tag{7}$$

with $f_\epsilon = T_{\frac{1}{\epsilon}}(f)$ a sequence in $L^\infty(\Omega)$ such that $f_\epsilon \to f$ in $L^1(\Omega)$ and $|f_\epsilon| \leq |f|$. Let us prove the following result.

Lemma 2. *Let $f_\epsilon \in L^\infty(\Omega)$ and assuming that $\mathbf{v} \in (L^p(\Omega))^d$ with $p = d$ if $d \geq 3$. Then, there exists at least one weak solution u_ϵ for the problem* (7) *in the sense that $u_\epsilon \in H_0^1(\Omega)$ and for all $\phi \in H_0^1(\Omega)$,*

$$\int_\Omega k_\epsilon(T_{1/\epsilon}(u_\epsilon))\, \nabla u_\epsilon \cdot \nabla \phi dx + \int_\Omega b\, u_\epsilon\, \phi\, dx = \int_\Omega u_\epsilon\, \mathbf{v} \cdot \nabla \phi\, dx + \int_\Omega f_\epsilon\, \phi dx. \tag{8}$$

To prove Lemma 2, we will use an approach based on the following result.

Lemma 3. *Let X and Y be two Banach spaces, with X reflexive and $X \subset Y$ with compact embedding. If the operator $A : X \to X'$ is bounded, coercive and pseudomonotone and $h : Y \to Y'$ is a continuous and bounded map in the sense that*

$$\|h(u)\|_{X'} \leq C, \ \forall u \in Y, \ \textit{for } C > 0.$$

Then, the operational equation $Au = h(u),\ u \in X$ admits a solution.

Proof. (of Lemma 2). We introduce the operators

$$\langle Au, \phi \rangle = \int_\Omega k(u)\, \nabla u \cdot \nabla \phi dx$$

and

$$\langle h_\epsilon(u), \phi\rangle = \int_\Omega T_{\frac{1}{\epsilon}}(u)\,\mathbf{v}\cdot\nabla\phi\,dx - \int_\Omega b\,T_{\frac{1}{\epsilon}}(u)\,\phi\,dx + \int_\Omega T_{\frac{1}{\epsilon}}(f)\,\phi dx,$$

with $u, \phi \in H_0^1(\Omega)$.
Since $L^2(\Omega)$ is a Banach space and $H_0^1(\Omega)$ is a Hilbert space, thus a reflexive Banach space. Moreover, $H_0^1(\Omega)$ is compactly embedded into $L^2(\Omega)$. Therefore, we have $X = H_0^1(\Omega)$ and $Y = L^2(\Omega)$. For the remainder of the proof, we will prove that $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is bounded, coercive, and pseudomonotone. Additionally, the map $h_\epsilon : L^2(\Omega) \to H^{-1}(\Omega)$ is shown to be continuous and bounded. Consequently, by applying Lemma 3, we conclude that the problem (7) admits at least one solution $u_\epsilon \in H_0^1(\Omega)$.
To complete the proof of Lemma 2, we must establish an a priori $L^\infty(\Omega)$ estimate for the sequence $\{u_\epsilon\}_{\epsilon>0}$. We apply Stampacchia's method, which relies on the boundedness of $\log(1+|u_\epsilon|)$, by choosing the test function $\phi = \frac{(u_\epsilon-\ell)^+}{1+|u_\epsilon|}$ in the weak formulation (8). This function belongs to $H_0^1(\Omega)$), satisfies $\|\phi\|_\infty \le 1$, and its gradient is given by:

$$\nabla\phi = \nabla\left(\frac{(u_\epsilon-\ell)^+}{1+|u_\epsilon|}\right) = (1+\ell)\frac{\nabla u_\epsilon}{(1+|u_\epsilon|)^2}\chi_{|u_\epsilon|>\ell}.$$

Substituting this into (8), it follows that

$$\begin{aligned}(1+\ell)\int_\Omega k_\epsilon(T_{1/\epsilon}(u_\epsilon))\frac{|\nabla u_\epsilon|^2}{(1+|u_\epsilon|)^2}dx + \int_\Omega b\,u_\epsilon\,\phi\,dx \\ \le (1+\ell)\int_\Omega u_\epsilon\,\mathbf{v}\cdot\frac{\nabla u_\epsilon}{(1+|u_\epsilon|)^2}\,dx + \int_{\{u_\epsilon>\ell\}}|f|dx. \quad (9)\end{aligned}$$

Observe that $\frac{u_\epsilon}{1+|u_\epsilon|} \le 1$. Also, since $g_\ell(u_\epsilon)$ has the same sign as u_ϵ and $b \ge 0$, we can rewrite (9) as follows

$$\int_{\{|u_\epsilon|>\ell\}} k_0\frac{|\nabla u_\epsilon|^2}{(1+|u_\epsilon|)^2}dx \le \int_{\{|u_\epsilon|>\ell\}}\mathbf{v}\cdot\frac{\nabla u_\epsilon}{1+|u_\epsilon|}\,dx + \frac{1}{(1+\ell)}\int_{\{|u_\epsilon|>\ell\}}|f|dx. \quad (10)$$

The application of Young's inequality to the first term on the right-hand side of (10) results in

$$\frac{k_0}{2}\int_{\{|u_\epsilon|>\ell\}}\frac{|\nabla u_\epsilon|^2}{(1+|u_\epsilon|)^2}dx \le \frac{1}{2k_0}\int_{\{|u_\epsilon|>\ell\}}|\mathbf{v}|^2dx + \frac{1}{(1+\ell)}\int_{\{|u_\epsilon|>\ell\}}|f|dx. \quad (11)$$

We rewrite (11) as follows by taking $\ell = e^k - 1$

$$\int_{\{\log(1+|u_\epsilon|)|>k\}} |\nabla \log(1+|u_\epsilon|)|^2 dx \leq \frac{1}{k_0^2}\int_{\{\log(1+|u_\epsilon|)|>k\}} |\mathbf{v}|^2 dx + \frac{2}{k_0(1+\ell)}\int_{\{\log(1+|u_\epsilon|)|>k\}} |f| dx. \quad (12)$$

Next, it follows from Stampacchia's theorem (see [7]) that there exists a positive constant M such that

$$\|\log(1+|u_\epsilon|)\|_\infty \leq M,$$

so

$$\|u_\epsilon\|_\infty \leq e^M - 1. \quad (13)$$

3.2. A priori estimates

Lemma 4. *For $\kappa > 0$, the sequence $(T_\kappa(u_\epsilon))_{\epsilon>0}$ is bounded in $H_0^1(\Omega)$.*

Proof. Using $\phi = T_\kappa(u_\epsilon)$ as test function in (8), we get

$$\int_\Omega k_\epsilon(u_\epsilon)\nabla u_\epsilon \cdot \nabla T_\kappa(u_\epsilon)\, dx + \int_\Omega b\, u_\epsilon\, T_\kappa(u_\epsilon)\, dx = \int_\Omega u_\epsilon\, \mathbf{v} \cdot \nabla T_\kappa(u_\epsilon)\, dx + \int_\Omega f_\epsilon T_\kappa(u_\epsilon)\, dx. \quad (14)$$

From the right-hand side of (14), we use Hölder's inequality to get

$$\int_\Omega f_\epsilon T_\kappa(u_\epsilon)\, dx \leq \kappa \|f_\epsilon\|_1 \leq \kappa \|f\|_1. \quad (15)$$

For the first term of the left-hand side of (14), we have

$$\int_\Omega k_\epsilon(u_\epsilon)\nabla u_\epsilon \cdot \nabla T_\kappa(u_\epsilon)\, dx = \int_{\{|u_\epsilon|\leq\kappa\}} k_\epsilon(u_\epsilon)\nabla u_\epsilon \cdot \nabla u_\epsilon\, dx \geq k_0 \int_{\{|u_\epsilon|\leq\kappa\}} |\nabla u_\epsilon|^2 dx = k_0 \int_\Omega |\nabla T_\kappa(u_\epsilon)|^2 dx. \quad (16)$$

Observe that $|T_\kappa(u_\epsilon)\nabla T_\kappa(u_\epsilon)| \leq \kappa|\nabla T_\kappa(u_\epsilon)|$ and as $p > 2$, then $L^p(\Omega)^d \hookrightarrow L^2(\Omega)^d$. Next, by using Young inequality, we can estimate the first terms on the right-hand side of (14) as follows:

$$\left|\int_\Omega T_\kappa(u_\epsilon)\, \mathbf{v} \cdot \nabla T_\kappa(u_\epsilon) dx\right| \leq \kappa \int_\Omega |\mathbf{v}|\, |\nabla T_\kappa(u_\epsilon)| dx$$

$$\leq \frac{\kappa^2}{2k_0}\int_\Omega |\mathbf{v}|^2 dx + \frac{k_0}{2}\int_\Omega |\nabla T_\kappa(u_\epsilon)|^2 dx$$
$$\leq \frac{S(\Omega,p)\kappa^2}{2k_0}\int_\Omega |\mathbf{v}|^p dx + \frac{k_0}{2}\int_\Omega |\nabla T_\kappa(u_\epsilon)|^2 dx, \quad (17)$$

where $S(\Omega,p)$ is the best constant of Sobolev.
Combining (14)-(17) and using the fact that $bu_\epsilon T_\kappa(u_\epsilon) \geq 0$ (since b is non-negative and T_κ has the same sign as u_ϵ), we deduce that

$$\frac{k_0}{2}\int_\Omega |\nabla T_\kappa(u_\epsilon)|^2\, dx \leq \frac{S(\Omega,p)\kappa^2}{2k_0}\int_\Omega |\mathbf{v}|^p dx + \kappa\|f\|_1. \quad (18)$$

Thus, (18) leads to

$$\int_\Omega |\nabla T_\kappa(u_\epsilon)|^2\, dx \leq C \quad (19)$$

with $C = \frac{S(\Omega,p)\kappa^2}{k_0^2}\int_\Omega |\mathbf{v}|^p dx + \frac{2\kappa}{k_0}\|f\|$.
Using the Poincaré inequality, (19) yields

$$\|T_\kappa(u_\epsilon)\|^2_{H_0^1(\Omega)} \leq C_0\|\nabla T_\kappa(u_\epsilon)\|^2_{(L^2(\Omega))^d} \leq C_0 C,$$

where $C_0 > 0$ is a constant independent of ϵ. Hence, the sequence $\{T_\kappa(u_\epsilon)\}_\epsilon > 0$ is bounded in $H_0^1(\Omega)$, for every $\kappa > 0$.

Now, we show that u is finite a.e. in Ω through a "log–type" estimate on u_ϵ.

Lemma 5. *Let $u_\epsilon \in H_0^1(\Omega)$ be a weak solution of (8). Then there exists $C > 0$, such that*

$$\|\log(1+|u_\epsilon|)\|^2_{H_0^1(\Omega)} \leq C, \quad (20)$$

$$meas\{|u_\epsilon| \geq \kappa\} \leq \frac{C}{(\log(1+\kappa))^2},\ \textit{for all } \kappa \textit{ large enough} \quad (21)$$

and

$$meas\{|\nabla u_\epsilon| \geq \kappa\} \leq \frac{C}{\kappa} + \frac{C}{(\log(1+\kappa))^2},\ \textit{for all } \kappa \textit{ large enough.} \quad (22)$$

Let's start by proving (20). For a Lipschitz function ϕ defined by $\phi(u_\epsilon) = \int_0^{u_\epsilon} \dfrac{1}{(1+|t|)^2}dt$ with $\phi(0) = 0$, from Stampacchia Lemma we have $\phi(u_\epsilon) \in H_0^1(\Omega)$ given $u_\epsilon \in H_0^1(\Omega)$. Note that $\nabla\phi(u) = (\nabla u)/(1+|u|)^2$ and $|\phi(u)| \leq 1$, therefore taking $\phi(u_\epsilon)$ as a test function in (8) leads to

$$\int_\Omega k_\epsilon(u_\epsilon)\frac{|\nabla u_\epsilon|^2}{(1+|u_\epsilon|)^2}\, dx + \int_\Omega bu_\epsilon\phi(u_\epsilon)\, dx \leq \|f\|_1 + \int_\Omega |u_\epsilon||\mathbf{v}|\frac{|\nabla u_\epsilon|}{(1+|u_\epsilon|)^2}\, dx. \quad (23)$$

Since b is a non-negative function and moreover $\phi(u_\epsilon)$ and u_ϵ have the same sign, then the second term in the left-hand side of (23) is non-negative. This leads to

$$\int_\Omega k_\epsilon(u_\epsilon)\frac{|\nabla u_\epsilon|^2}{(1+|u_\epsilon|)^2}\,dx \le \|f\|_1 + \int_\Omega |u_\epsilon||\mathbf{v}|\frac{|\nabla u_\epsilon|}{(1+|u_\epsilon|)^2}\,dx. \tag{24}$$

Furthermore, we have $|u_\epsilon| \le 1+|u_\epsilon|$. Then, using Young's inequality and Sobolev embedding, the second term on the right-hand side of (24) can be estimated as follow

$$\begin{aligned}\left|\int_\Omega u_\epsilon \mathbf{v}\frac{\nabla u_\epsilon}{(1+|u_\epsilon|)^2}\,dx\right| &\le \int_\Omega \frac{|\mathbf{v}||\nabla u_\epsilon|}{1+|u_\epsilon|}dx \\ &\le \frac{S(\Omega,p)}{2k_0}\|\mathbf{v}\|^p_{L^p(\Omega)^d} + \frac{k_0}{2}\|\nabla \log(1+|u_\epsilon|)\|^2_{L^2(\Omega)}.\end{aligned} \tag{25}$$

For the first term of (23), we have

$$k_0\|\nabla \log(1+|u_\epsilon|)\|^2_{L^2(\Omega)} \le \int_\Omega k_0\Big(\frac{\nabla|u_\epsilon|}{1+|u_\epsilon|}\Big)^2 dx \le \int_\Omega k_\epsilon(u_\epsilon)\Big(\frac{\nabla|u_\epsilon|}{1+|u_\epsilon|}\Big)^2 dx. \tag{26}$$

Combining (24)-(25) and (26) yields

$$\|\nabla \log(1+|u_\epsilon|)\|^2_{L^2(\Omega)} \le \frac{2\|f\|_1}{k_0} + \frac{S(\Omega,p)}{k_0^2}\|\mathbf{v}\|^p_{L^p(\Omega)^d}.$$

Then, applying Poincaré's inequality, we find

$$\|\log(1+|u_\epsilon|)\|^2_{H^1_0(\Omega)} \le C,$$

where $C = \frac{2\|f\|_1}{k_0} + \frac{S(\Omega,p)}{k_0^2}\|\mathbf{v}\|^p_{L^p(\Omega)^d}$ is a positive constant, thereby proving (20).

From inequality (21), applying Poincaré's inequality and incorporating (20) yields

$$\begin{aligned}\int_{\{|u_\epsilon|\ge\kappa\}}(\log(1+\kappa))^2\,dx &\le \int_{\{|u_\epsilon|\ge\kappa\}}(\log(1+|u_\epsilon|))^2\,dx \\ &\le \int_\Omega(\log(1+|u_\epsilon|))^2\,dx \\ &\le C(f,\Omega,\mathbf{v}),\end{aligned}$$

this means

$$meas\{|u_\epsilon|\ge\kappa\} \le \frac{C(f,\Omega,\mathbf{v})}{(\log(1+\kappa))^2},$$

which proves (21).
From the inequality (22), we set $\Phi(\kappa,\ell) = \text{meas}\{|\nabla u_\epsilon|^2 > \ell,\ |u_\epsilon| > \kappa\}$, for all $\kappa, \ell \geq 0$. According to (21), we have

$$\Phi(\kappa, 0) \leq \frac{C}{(\log(1+\kappa))^2}, \text{ for all } \kappa > 0 \text{ large enough.} \tag{27}$$

As $\Phi(\kappa)$ is non-increasing with respect to ℓ for $\kappa, \ell > 0$, we have the following

$$\begin{aligned} \Phi(0,\ell) &= \tfrac{1}{\ell}\int_0^\ell \Phi(0,\ell)dx \leq \frac{1}{\ell}\int_0^\ell \Phi(0,\beta)dx \\ &\leq \tfrac{1}{\ell}\int_0^\ell \big[\Phi(0,\beta) + \big(\Phi(\kappa,0) - \Phi(\kappa,\beta)\big)\big]dx \\ &\leq \Phi(\kappa,0) + \tfrac{1}{\ell}\int_0^\ell \big(\Phi(0,\beta) - \Phi(\kappa,\beta)\big)dx. \end{aligned}$$

Given that

$$\Phi(0,\beta) - \Phi(\kappa,\beta) = \text{meas}\{|u_\epsilon| \leq \kappa, |\nabla u_\epsilon|^2 > \beta\},$$

we can conclude that

$$\int_0^\infty \big(\Phi(0,\beta) - \Phi(\kappa,\beta)\big)dx = \int_{\{|u_\epsilon|\leq\kappa\}} |\nabla u_\epsilon|^2 dx. \tag{28}$$

Thanks to (18), we get

$$\int_{\{|u_\epsilon|\leq k\}} |\nabla u_\epsilon|^2 dx \leq C. \tag{29}$$

Combining (28) and (29) yields

$$\int_0^\infty \big(\Phi(0,\beta) - \Phi(\kappa,\beta)\big)dx \leq C. \tag{30}$$

Returning to (27) and using (30), we get

$$\Phi(0,\ell) \leq \frac{C}{\ell} + \frac{C}{(\log(1+\kappa))^2}, \quad \text{for all } \ \kappa \geq 1, \ell > 0.$$

In particular,

$$\Phi(0,\ell) \leq \frac{C}{\ell} + \frac{C}{(\log(1+\kappa))^2}, \quad \text{for all } \kappa \geq 1 \ \text{ and } \ell \geq 1. \tag{31}$$

By setting $\ell = \kappa$ in (31), leads to (22).

3.3. Convergence results

Here the aim is to introduce the convergence results which are necessary for the proof of the existence of the solution.

Lemma 6.
(**i**) *For all* $\kappa > 0, T_\kappa(u_\epsilon) \to T_\kappa(u)$ *in* $L^2(\Omega)$ *and a.e. in* Ω*, as* $\epsilon \to 0$.
(**ii**) *There exists a measurable function* u *such that* $u_\epsilon \to u$ *a.e. in* Ω*, as* $\epsilon \to 0$.

Proof. Thanks to Lemma 4 and the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, the Rellich theorem allows us to extract a sub-sequence $(T_\kappa(u_\epsilon))_\epsilon$ such that $(T_\kappa(u_\epsilon))_\epsilon$ converges strongly to $T_\kappa(u)$ in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$. Thus, for any $\kappa > 0$, there exists $w_\kappa \in H_0^1(\Omega)$ such that

$$T_\kappa(u_\epsilon) \to w_\kappa \text{ strongly in } L^2(\Omega) \text{ and a.e. in } \Omega, \tag{32}$$

$$T_\kappa(u_\epsilon) \rightharpoonup w_\kappa \text{ weakly in } H_0^1(\Omega). \tag{33}$$

We will prove that the sequence $(u_\epsilon)_{\epsilon>0}$ is Cauchy in measure. Let $j > 0$. For all $\epsilon > 0$ and for all $\epsilon', \kappa > 0$, we define

$$E_\epsilon := \{|u_\epsilon| > \kappa\}, \quad E_{\epsilon'} := \{|u_{\epsilon'}| > k\} \text{ and } E_{\epsilon,\epsilon'} := \{|T_\kappa(u_\epsilon) - T_\kappa(u_{\epsilon'})| > j\},$$

where $\kappa > 0$ is to be fixed. We note that $\{|u_\epsilon - u_{\epsilon'}| > j\} \subset E_\epsilon \cup E_{\epsilon'} \cup E_{\epsilon,\varepsilon'}$, and therefore

$$\text{meas}\{|u_\epsilon - u_{\epsilon'}| > j\} \leq \text{ meas}(E_\epsilon) + \text{meas}(E_\epsilon) + \text{meas}(E_{\epsilon,\epsilon'}). \tag{34}$$

We choose $\kappa = \kappa(\varepsilon)$ such that

$$\text{meas}(E_\epsilon) \leq \frac{\varepsilon}{3} \text{ and meas}(E_{\epsilon'}) \leq \frac{\varepsilon}{3}. \tag{35}$$

Since $\big(T_\kappa(u_\epsilon)\big)_\epsilon$ converges strongly in $L^2(\Omega)$, then it is a Cauchy sequence in $L^2(\Omega)$. Thus,

$$\text{meas}(E_{\epsilon,\epsilon'}) \leq \frac{1}{j^2} \int_\Omega \big|T_\kappa(u_\epsilon) - T_\kappa(u_{\epsilon'})\big|^2 dx \leq \frac{\varepsilon}{3}, \tag{36}$$

for all $\epsilon, \epsilon' \geq \epsilon_0(j, \varepsilon)$.
From (34)-(36), we finally derive

$$\text{meas}\{|u_\epsilon - u_{\epsilon'}| > j\} \leq \ \varepsilon \text{ for all } \epsilon, \epsilon' \geq \epsilon_0(j, \varepsilon). \tag{37}$$

Hence, the sequence $(u_\epsilon)_\epsilon$ is a Cauchy sequence in measure and there exists a function u, which is finite almost everywhere on Ω, such that $u_\epsilon \to u$ in measure.

We can then extract a subsequence such that $u_\epsilon \to u$ a.e. in Ω. Since T_κ is continuous, then $T_\kappa(u_\epsilon) \to T_\kappa(u)$ a.e. in Ω, $w_\kappa = T_\kappa(u)$ a.e. in Ω and $T_\kappa(u) \in H_0^1(\Omega)$. Moreover, we have

$$\nabla T_\kappa(u_\epsilon) \rightharpoonup \nabla T_\kappa(u) \text{ weakly in } L^2(\Omega)^d. \tag{38}$$

3.4. Passing to the limit

To pass to the limit in the approximate problem, we will use the following Lemma, whose the proof is similar to that in [10, Lemma 6].

Lemma 7. *For all $S(u_\epsilon)\psi$ and $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$,*

$$\nabla[S(u_\epsilon)\psi] \to \nabla[S(u)\psi] \textit{ strongly in } (L^2(\Omega))^d, \textit{ as } \epsilon \to 0.$$

Using $S(u_\epsilon)\psi$ as a test function in (8) with $\psi \in \mathcal{D}(\Omega)$ and $S \in W^{1,\infty}(\mathbb{R})$, we get

$$\begin{aligned}\int_\Omega k_\epsilon(T_{1/\epsilon}(u_\epsilon))\, \nabla u_\epsilon\, \nabla[S(u_\epsilon)\psi]dx + \int_\Omega b\, u_\epsilon\, S(u_\epsilon)\, \psi dx \\ = \int_\Omega u_\epsilon \mathbf{v}.\nabla[S(u_\epsilon)\psi]dx \int_\Omega f_\epsilon S(u_\epsilon)\psi dx.\end{aligned} \tag{39}$$

Since S has compact support, there exists a positive real number κ such that $\operatorname{supp}(S) \subset [-\kappa, \kappa]$. Thus, we can replace u_ϵ with its truncation $T_\kappa(u_\epsilon)$. Moreover, we have a bound on u_ϵ (uniform in ϵ), for ϵ small enough, $k_\epsilon(T_{1/\epsilon}(u_\epsilon)) = k_\epsilon(u_\epsilon)$. Thus, we have

$$\begin{aligned}\int_\Omega k_\epsilon(u_\epsilon)\nabla T_\kappa(u_\epsilon)\, \nabla[S(u_\epsilon)\psi]dx + \int_\Omega b\, u_\epsilon\, S(u_\epsilon)\, \psi dx \\ = \int_\Omega u_\epsilon \mathbf{v}.\nabla[S(u_\epsilon)\psi]dx \int_\Omega f_\epsilon S(u_\epsilon)\psi dx.\end{aligned} \tag{40}$$

Let us begin with the first term on the left-hand side of (39). Given that $k_\epsilon(\cdot)$ is continuous and bounded, it follows that $k_\epsilon(u_\epsilon)$ converges to $k(u)$ in the weak-$\star$ topology of $L^\infty(\Omega)$. By applying (38) and Lemma 7, we can conclude that

$$\int_\Omega k_\epsilon(u_\epsilon)\nabla T_\kappa(u_\epsilon)\, \nabla[S(u_\epsilon)\psi]dx \to \int_\Omega k(u)\nabla T_\kappa(u)\, \nabla[S(u)\psi]dx, \text{ as } \epsilon \to 0. \tag{41}$$

According to Lemma 7, $\nabla[S(u_\epsilon)\psi]$ converges strongly to $\nabla[S(u)\psi]$ in $(L^2(\Omega))^d$, while $\nabla T_\kappa(u_\epsilon)$ converges weakly to $\nabla T_\kappa(u)$ in $(L^2(\Omega))^d$. Next, as stated in [3, Lemma A.1], we can conclude that the sequence $(u_\epsilon \mathbf{v})_{\epsilon>0}$ converges to $u\mathbf{v}$ in $L^2(\Omega)^d$. Moreover, by Lemma 7, it follows that

$$\int_\Omega T_\kappa(u_\epsilon)\mathbf{v}.\nabla[S(u_\epsilon)\psi]dx \longrightarrow \int_\Omega T_\kappa(u)\mathbf{v}.\nabla[S(u)\psi]dx = \int_\Omega u\mathbf{v}.\nabla[S(u)\psi]dx. \quad (42)$$

Since $S(u_\epsilon)\psi$ converges weak-$*$ to $S(u)\psi$ in $L^\infty(\Omega)$ and given that $b \in L^2(\Omega)$, we find that $T_\kappa(u_\epsilon)$ converges to $T_\kappa(u)$ in $L^2(\Omega)$ and almost everywhere in Ω. Consequently, $bT_\kappa(u_\epsilon) \longrightarrow bT_\kappa(u)$ in $L^1(\Omega)$. Additionally, f_ϵ converges strongly to f in $L^1(\Omega)$. Thus, as ϵ tends zero, we obtain the following convergence of integrals:

$$\int_\Omega bT_\kappa(u_\epsilon)S(u_\epsilon)\psi dx \longrightarrow \int_\Omega bT_\kappa(u)S(u)\psi dx = \int_\Omega buS(u)\psi dx \quad (43)$$

and

$$\int_\Omega f_\epsilon h(u_\epsilon)\psi dx \longrightarrow \int_\Omega fh(u)\psi dx. \quad (44)$$

Thus, passing to the limit in (40), we get that u verifies equality (5).
In the following lemma, we prove that the function u satisfies the estimate (6).

Lemma 8. *For all $m > 0$,*

$$\lim_{m\to\infty} \limsup_{\epsilon\to 0} \int_{\{m<|T_\kappa(u_\epsilon)|<m+1\}} k_\epsilon(u_\epsilon)|\nabla T_\kappa(u_\epsilon)|^2\, dx = 0. \quad (45)$$

Proof. Take $\phi = T_1\big(T_\kappa(u_\epsilon) - T_m(T_\kappa(u_\epsilon))\big)$ as test function in (8), using the fact that b is nonnegative, T_1 is nondecreasing and $\nabla\big[T_1\big(T_\kappa(u_\epsilon) - T_m(T_\kappa(u_\epsilon))\big)\big] = \nabla T_\kappa(u_\epsilon)\, \chi_{\{m<|T_\kappa(u_\epsilon)|<m+1\}}$, we obtain

$$\begin{aligned} \int_{\{m<|T_\kappa(u_\epsilon)|<m+1\}} k_\epsilon(u_\epsilon)|\nabla T_\kappa(u_\epsilon)|^2\, dx \le & \int_{\{m<|T_\kappa(u_\epsilon)|<m+1\}} |T_\kappa(u_\epsilon)\mathbf{v}\cdot\nabla T_\kappa(u_\epsilon)|\, dx \\ & + \int_\Omega |f_\epsilon\, T_1\big(T_\kappa(u_\epsilon) - T_m(T_\kappa(u_\epsilon))\big)|\, dx. \quad (46) \end{aligned}$$

We now evaluate each term of (46).
Regarding the second term on the right-hand side of (46), we use the fact that $f_\epsilon\, T_1\big(T_\kappa(u_\epsilon) - T_m(T_\kappa(u_\epsilon))\big) \to f\, T_1\big(T_\kappa(u) - T_m(T_\kappa(u))\big)$ almost everywhere in Ω as $\epsilon \to 0$, with $|f_\epsilon\, T_1\big(T_\kappa(u_\epsilon) - T_m(T_\kappa(u_\epsilon))\big)| \le |f_\epsilon| \in L^1(\Omega)$ and $f_\epsilon \to f$ strongly in $L^1(\Omega)$. Therefore, we can apply the Lebesgue dominated convergence theorem to obtain

$$\lim_{\epsilon\to 0} \int_\Omega |f_\epsilon\, T_1\big(T_\kappa(u_\epsilon) - T_m(T_\kappa(u_\epsilon))\big)|\, dx = \int_\Omega |f\, T_1\big(T_\kappa(u) - T_m(T_\kappa(u))\big)|\, dx.$$

On the other hand, $f\, T_1\big(T_\kappa(u) - T_m(T_\kappa(u))\big) \to 0$ a.e. in Ω as $m \to \infty$, and $|f\, T_1\big(T_\kappa(u) - T_m(T_\kappa(u))\big)| \leq |f| \in L^1(\Omega)$. Therefore, the Lebesgue dominated convergence theorem yields

$$\lim_{m\to\infty}\lim_{\epsilon\to 0}\int_\Omega |f_\epsilon\, T_1\big(T_\kappa(u_\epsilon) - T_m(T_\kappa(u_\epsilon))\big)|\, dx = 0. \tag{47}$$

Next, we estimate the first integral on the right-hand side of (46). Using Lemma 6 and the Sobolev embedding $L^p(\Omega)^d \hookrightarrow L^2(\Omega)^d$, we conclude that

$$T_\kappa(u_\epsilon)\, \mathbf{v}\, \chi_{\{m<|T_\kappa(u_\epsilon)|<m+1\}} \to T_\kappa(u)\, \mathbf{v}\, \chi_{\{m<|T_\kappa(u)|<m+1\}} \text{ in } L^2(\Omega)^d \text{ as } \epsilon \to 0.$$

Combining with the weak convergence of

$$\nabla T_\kappa(u_\epsilon)\, \chi_{\{m<|T_\kappa(u_\epsilon)|<m+1\}} \text{ to } \nabla T_\kappa(u)\, \chi_{\{m<|T_\kappa(u)|<m+1\}} \text{ in } L^2(\Omega)^d,$$

we deduce that

$$\lim_{\epsilon\to 0}\int_{\{m<|T_\kappa(u_\epsilon)|<m+1\}} |T_\kappa(u_\epsilon)\mathbf{v}.\nabla T_\kappa(u_\epsilon)|\, dx = \int_{\{m<|T_\kappa(u)|<m+1\}} |T_\kappa(u)\, \mathbf{v}\cdot\nabla T_\kappa(u)|\, dx. \tag{48}$$

We assert that

$$\lim_{m\to\infty}\int_{\{m<|T_\kappa(u)|<m+1\}} |T_\kappa(u)\, \mathbf{v}\cdot\nabla T_\kappa(u)|\, dx = 0. \tag{49}$$

Indeed, by the Sobolev embedding, we have

$$H_0^1(\Omega) \hookrightarrow L^{2*}(\Omega) \text{ with } 2* = \frac{2d}{d-2}, \quad \text{for } d \geq 3. \tag{50}$$

Using (50), we know that $H_0^1(\Omega) \subset L^{2^*}(\Omega)$. Since $T_\kappa(u) \in H_0^1(\Omega)$, it follows that $T_\kappa(u) \in L^{2^*}(\Omega)$. Therefore, $T_\kappa(u)\mathbf{v} \in (L^2(\Omega))^d$, with the relation $\frac{1}{2^*} + \frac{1}{d} = \frac{d-2}{2d} + \frac{1}{d} = \frac{1}{2}$. Thus, $T_\kappa(u)\, \mathbf{v}\cdot\nabla T_\kappa(u) \in L^1(\Omega)$.
Moreover, from Lemma 5, we know that $\text{meas}(\{T_\kappa(u) \geq m\}) \to 0$ as $m \to \infty$. Therefore, we can pass to the limit in the second integral of (48) to obtain

$$\lim_{m\to\infty}\int_{\{m<|T_\kappa(u)|<m+1\}} |T_\kappa(u)\, \mathbf{v}\cdot\nabla T_\kappa(u)|\, dx \leq \lim_{m\to\infty}\int_{\{|T_\kappa(u)|>m\}} |T_\kappa(u)\, \mathbf{v}\cdot\nabla T_\kappa(u)|\, dx = 0.$$

This proves (49). By combining (48) and (49), we obtain

$$\lim_{m\to\infty}\lim_{\epsilon\to 0}\int_{\{m<|T_\kappa(u_\epsilon)|<m+1\}} |T_\kappa(u_\epsilon)\, \mathbf{v}\cdot\nabla T_\kappa(u_\epsilon)|\, dx = 0. \tag{51}$$

Finally, using (47) and (51), we arrive at (45).

4. Concluding

The main contribution of this work is the establishment of the existence result stated in Theorem 1. Specifically, we have focused on the existence of renormalized solutions for the convection-diffusion problem (1). This builds upon the foundational work of DiPerna and Lions [5, 6], who introduced the notion of renormalized solutions in the context of the Boltzmann equation, and extends the results of [9] for the case $k \equiv 1$ and a vector field $\mathbf{v} \in (L^p(\Omega))^d$, where $2 < p < +\infty$ for $d = 2$, and $p = d$ for $d \geq 3$.

To prove the existence of renormalized solutions, we constructed a sequence of approximate solutions and derived suitable a priori estimates. This approach enabled us to extract a convergent subsequence whose limit function was rigorously shown to satisfy the definition of a renormalized solution. Looking forward, future research will focus on investigating the uniqueness of the renormalized solution to (1), particularly under the assumption that k is locally Lipschitz continuous.

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