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New approach of (G'/G)-expansion method to solve the fractional differential equations arising in fluid mechanics

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Abstract. In this paper, an approach based on the generalized (G'/G)-expansion method for two nonlinear fractional physical models is proposed to achieve their solutions including three kinds of hyperbolic function solution and trigonometric function solution as well as the rational solution. These relations are the space-time fractional Whitham-Broer-Kaup (WBK), generalized Hirota-Satsuma coupled KdV equations. The fractional derivative in the present work is presented from the Caputo viewpoint, and precise solutions of the stated nonlinear fractional equations are obtained. Also, the generalized fractional complex transform is employed properly for conversion of this equation into an ordinary differential equation (ODE). Subsequently, some precise solutions are obtained.

Key Words and Phrases: Generalized (G'/G)-expansion method; space-time fractional Whitham-Broer-Kaup; Generalized Hirota-Satsuma coupled KdV; Hyperbolic function solution; Trigonometric function solution; Rational solution.

2010 Mathematics Subject Classifications: AMS classification codes

1. Introduction

The space-time fractional WBK relation is presented in the following form:

$$D_t^{\alpha}u + uD_x^{\alpha}u + D_x^{\alpha}v + \beta D_x^{2\alpha}u = 0, 0 < \alpha \leq 1,$$
$$D_t^{\alpha}\, v + \ D_x^{\alpha}(\ uv) - \beta\ D_x^{2\alpha}\ v + \gamma\ D_x^{3\alpha}\ u = 0. \tag{1}$$

Where, $u(x,\ t)$ and $v\ (x,\ t)$ are the horizontal velocity's field and the max deviation from the equilibrium location of the liquid; β and γ denote two real coefficients with fixed values presenting diferent difusion powers. Once $\alpha = 1$, Eq. (1) is the generalized format of WBK relations that is usable for description of the dispersive long wave in the shallow water [1, 2, 3]. The space-time fractional generalized Hirota-Satsuma coupled KdV relations [4] are given as:

$$D_t^{\alpha}u - \frac{1}{2}D_x^{3a}u + 3uD_x^{a}u - 3D_x^{\alpha}(vw) = 0,$$
$$D_t^{\alpha}v + D_x^{3\alpha}v - 3uD_x^{\alpha}v = 0, \tag{2}$$
$$D_t^{\alpha}w + D_x^{3\alpha}w - 3uD_x^{\alpha}w = 0.$$

There are many utilities for the fractional calculus in various scientific fields that are based on the mathematical modelling. The signal/image processing, physical/chemical fields, aerodynamics, economics, polymer rheology, biophysics, and control theory are some of these applications [5, 6, 7]. Various manners are studied by researchers to derive the fractional calculus like numerical methods that solve the fractional DEs by various numerical tools. Utilization of tan $(\Phi(\xi)/2)$ expansion approach is an efficient and analytical method to solve the fractional partial DEs (FPDEs). Various efficient approaches are presented in the recent years to search precise solutions of nonlinear evolution relations. For instance, Hirota's bilinear approach is proposed in [8], and homotopy analysis technique is presented in [9, 10]. In addition, some other methods are reported in the literature including variational iteration approach [11, 12], homotopy perturbation approach [13], sine-cosine and tanh-coth techniques in [14] and [15], Backlund transformation [16], (G'/G) expansion technique [17, 18], and Exp-function approach [**?**], as well as improved simple equation approach [22]. Two high-performance approaches are utilized in this paper to provide a series of precise solutions for the considered nonlinear partial DEs (NLPDEs). The original tanh approach is a famous analytical approach that is firstly introduced by Malffiet's [23] and is also presented in [23, 24]. A general version of the tanh-coth approach is proposed in [15] to solve some NLPDEs. Reference [25] suggested a novel version of the generalized (G'/G) expansion approach for achieving precise traveling wave solutions of NLEEs. Two powerful methods of the generalized tanh–coth and generalized (G'/G)-expansion are proposed in the present work for seeking the travelling wave solutions of nonlinear evolution relations. Precise solutions of the integrable sixth-order Drinfeld-Sokolov-Satsuma-Hirota are achieved in [26] using these two generalized approaches. Moreover, (G'/G) expansion approach is used in [27] to precisely solve a number of nonlinear evolution equations. More information of these approaches are presented in [28, 29, 30]. Present work is mainly aimed to analytically solve the space-time fractional Whitham-Broer-Kaup, and generalized Hirota-Satsuma coupled with KdV equations. Also, the accuracy of the generalized (G'/G) expansion approach is evaluated here for the considered problems. Some basic studies for different aspects of the fractional calculus are provided by Caputo [31], Debanth [32], Jafari and Seifi [33], Kemple and Beyer [34], Kirci et al. [35], Kilbas and Trujillo [36], Momani and Shawagfeh [37], Oldham and Spanier [38]. Additionally, multitude approaches are suggested for solving the FPDEs including Laplace transform and Fourier transform approaches [34], Adomian's decomposition [37], homotopy analysis [39], etc. Consequently, in recent years, various techniques have been devised by researchers to address these problems, including the fractional generalized CBS-BK equation [40], the generalized Bogoyavlensky-Konopelchenko equation [41], the generalized Hietarinta equation [42], the nonlinear vibration and dispersive wave systems [43], and the Van der Waals model [44]. The improved extended tanh-function approach is used in the present work to solve FPDEs in the sense of the modified Riemann-Liouville derivative that is introduced in [45]. We can reduce the considered equations to nonlinear ODEs with integer orders by a few fractional complex transformations. Jumarie's modified Riemann-Liouville derivative with order of α can be presented in the below form:

$$D_t^{\alpha} u(t) =$$

$$= \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t (t-\tau)^{-\alpha}(u(\tau)-u(0))d\tau & \text{if} \quad 0<\alpha\leq 1, \\ \left[u^{(n)}(t)\right]^{(\alpha-n)}, & \text{if} \quad n\leq\alpha<n+1,\ n\geq 1. \end{cases} \tag{3}$$

Some important features of the modified Riemann-Liouville derivative are given by:

$$\begin{gathered} (1) D^{\alpha}[f(t)g(t)] = f(t)D^{\alpha}g(t) + g(t)D^{\alpha}f(t), \\ (2) D^{\alpha}[f(g(t))] = f'_g(g(t))D^{\alpha}g(t), \\ (3) D^{\alpha}[f(g(t))] = D_g^{\alpha}f(g(t))[g'(t)]^{\alpha}, \\ (4)\ D_t^{\alpha}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(1+\alpha-\gamma)}t^{\gamma-\alpha}, \gamma>0. \end{gathered} \tag{4}$$

Where G is the Gamma function. Rest of this study is structured in the following form: Section 2 presented the preliminaries and Notations. The fractional complex transform is described briefly in Section 3. Section 4 contains the generalized (G'/G) expansion approach explanation. Also, the space-time fractional WBK, space-time fractional generalized Hirota-Satsuma coupled KdV equations are evaluated in Section 4. At the last, the paper is concluded in Section 5.

2. The generalized (G'/G)-expansion method

The diverse type of PDE, in through the nonlinear segment and the uppermost order of differential are involved.

Step 1. It's assumed that the considered nonlinear FPDE for $u(x,\ t)$ can be presented as follows:

$$\mathcal{N}\left(u, D_t^{\alpha}u, D_x^{\alpha}u, \ldots\right) = 0, \qquad 0<\alpha\leq 1. \tag{5}$$

Where $D_t^a u$ and $D_x^a u$ are the modified Riemann-Liouville derivatives and $\mathcal{N}$ is a polynomial in $u = u(x,\ t)$ and its fractional derivatives. $\mathcal{N}$ can be converted to an ODE:

$$\mathcal{Q}(u, -cu^{'}, ku^{'}, \dots) = 0. \tag{6}$$

By the transformation $\xi = \frac{kx^{\alpha}}{\Gamma(\alpha+1)} - \frac{ct^{\alpha}}{\Gamma(\alpha+1)}$, is wave variable. Also, $k,\ c$ are constants that will be obtained later.

Step 2. Assume that the traveling wave solution of Eq. (6) is given by:

$$u(\xi) = S(\Phi(\xi)) = \sum_{k=0}^{m} A_k (p + \Phi(\xi))^k + \sum_{k=1}^{m} B_k (p + \Phi(\xi))^{-k}. \tag{7}$$

Where, $A_k (0 \le k \le m)$ are constant values and should be determined, in a way that $A_m \neq 0$, $B_m \neq 0$ and $\Phi(\xi) = G'(\xi)/G(\xi)$ meets the below ODE:

$$k_1 G G^{''} - k_2 G G^{'} - k_3 (G^{'})^2 - k_4 G^2 = 0. \tag{8}$$

The special solutions of above equation will be considered here:

Family 1: When $k_2 \neq 0,\ r = k_1 - k_3$ and $s = k_2^2 + 4k_4(k_1 - k_3) > 0$, then

$$\Phi(\xi) = \frac{k_2}{2r} + \frac{\sqrt{s}}{2r} \frac{C_1\ sinh\xi(\frac{\sqrt{s}}{2k_1}\xi) + C_2\ cosh\xi(\frac{\sqrt{s}}{2k_1}\xi)}{C_1\ cosh\xi(\frac{\sqrt{s}}{2k_1}\xi) + C_2\ sinh\xi(\frac{\sqrt{s}}{2k_1}\xi)}.$$

Family 2: When $k_2 \neq 0,\ r = k_1 - k_3$ and $s = k_2^2 + 4k_4(k_1 - k_3) < 0$, then

$$\Phi(\xi) = \frac{k_2}{2r} + \frac{\sqrt{-s}}{2r} \frac{-C_1\ sin\xi(\frac{\sqrt{-s}}{2k_1}\xi) + C_2\ cos\xi(\frac{\sqrt{-s}}{2k_1}\xi)}{C_1\ cos\xi(\frac{\sqrt{-s}}{2k_1}\xi) + C_2\ sin\xi(\frac{\sqrt{-s}}{2k_1}\xi)}.$$

Family 3: When $k_2 \neq 0,\ r = k_1 - k_3$ and $s = k_2^2 + 4k_4(k_1 - k_3) = 0,$ then $\Phi(\xi) = \frac{k_2}{2r} + \frac{C_2}{C_1 + C_2\xi}$.

Family 4: When $k_2 = 0,\ r = k_1 - k_3$ and $q = rk_4 > 0$, then

$$\Phi(\xi) = \frac{\sqrt{q}}{r} \frac{C_1\ sinh\xi(\frac{\sqrt{q}}{k_1}\xi) + C_2\ cosh\xi(\frac{\sqrt{q}}{k_1}\xi)}{C_1\ cosh\xi(\frac{\sqrt{q}}{k_1}\xi) + C_2\ sinh\xi(\frac{\sqrt{q}}{k_1}\xi)}.$$

Family 5: When $k_2 = 0,\ r = k_1 - k_3$ and $q = rk_4 < 0$, then

$$\Phi\,(\xi) = \frac{\sqrt{-q}}{r} \frac{-C_1\ sin\xi(\frac{\sqrt{-q}}{k_1}\xi) + C_2\ cos\xi(\frac{\sqrt{-q}}{k_1}\xi)}{C_1\ cos\xi(\frac{\sqrt{-q}}{k_1}\xi) + C_2\ sin\xi(\frac{\sqrt{-q}}{k_1}\xi)}.$$

Family 6: When $k_4 = 0 \; and \; r = k_1 - k_3$, then

$$\Phi(\xi) = \frac{C_1 k_2^2 \; exp\xi(\frac{-k_2}{k_1}\xi)}{rk_1 + C_1 k_1 k_2 \; exp\xi(\frac{-k_2}{k_1}\xi)}.$$

Family 7: When $k_2 \neq 0 \; and \; r = k_1 - k_3 = 0$, then

$$\Phi\left(\xi\right) = -\frac{k_4}{k_2} + C_1 \exp\left(\frac{k_2}{k_1}\xi\right),$$

where $A_j \left(j = 0, \; 1, \; \ldots, \; m\right), \; B_j \left(j = 1, \; \ldots, \; m\right), \; a, \; b$ and c are fixed values that should be determined. While, m is a positive integer which can be integrated by consistent the highest order of derivatives to the highest order of nonlinear portions, perform in Eq. (8).

Step 3. Inserting Eq. (7) into Eq. (6) by the determined m amount in prior step. Through gathering the parameters of $(p \; + \; \Phi\left(\xi\right))^k$, $(p \; + \; \Phi\left(\xi\right))^{-k}$ $(k = 0, \; 1, \; 2, \; ...)$ and setting them to zero, a series of over-determined partial DEs will be obtained for $A_0, \; A_k, \; B_k \left(k = 1, \; \ldots, \; m\right) k_1, \; k_2, \; k_3, \; k_4 \; and \; p$ using Maple 18.

Step 4. Solving the algebraic equations of prior step, and inserting $A_j, \; B_j \left(j = 1, \; \ldots, \; m\right), \; k \;, c, \; p$ in Eq. (7).

3. Illustrative examples

Multitude examples are presented in this part of the paper in order to demonstrate the efficiency and ability of the generalized (G'/G)-expansion approach in solving nonlinear FPDEs introduced in Section 1.

3.0.1. The space-time fractional WBK equations

Consider the nonlinear space-time fractional WBK in the following form:

$$\begin{aligned} D_t^{\alpha} u + u D_x^{\alpha} u + D_x^{\alpha} u + \beta D_x^{2\alpha} u = 0, \\ D_t^{\alpha} v + D_x^{\alpha}(uv) - \beta D_x^{2\alpha} v + \gamma D_x^{3\alpha} u = 0. \end{aligned} \tag{9}$$

By using the wave variable $\xi \; = \frac{kx^a}{\Gamma(\alpha+1)} \; - \frac{ct^{\alpha}}{\Gamma(\alpha+1)}$, it can be reduced into an ordinary DE in the following form:

$$\begin{aligned} -cu + \tfrac{k}{2} u^2 + kv + \beta k^2 u^{'} = 0, \\ -cv + k(uv) - \beta k^2 v^{'} + \gamma k^3 u^{''} = 0. \end{aligned} \tag{10}$$

Where Eq. (10) is achieved via integration based on ξ and ignoring the constant of this integration. Below result can be achieved by regarding the balance between u'

and u^2 in Eq. (10) as $M+1 = 2M \;\; \rightarrow \;\; M = 1$. In a similar way, below result is reached by regarding the homogeneous balance between u'' and uv in Eq. (10) as $M+2 = M+N \;\rightarrow\; N = 2$. Then the trail solution is:

$$u = \; A_0 + \; A_1(\, p+\Phi) + \frac{B_1}{p+\Phi},\; v(\xi) =$$

$$= \; E_0 + \; E_1(\, p+\Phi) + \; E_2(\, p+\Phi)^2 + \frac{D_1}{p+\Phi} + \frac{D_2}{(\, p+\Phi)^2}. \tag{11}$$

Inserting (11) into Eq. (10) and with the aid of Maple computational tool, below sets of non-trivial solutions can be achieved.

Set I:

$$k = \frac{A_1\, k_1}{2\, k_3\sqrt{\beta^2+\gamma}},$$

$$c = \frac{A_1^2\, k_1\sqrt{k_2^2 - \; k_3\, k_4}}{4\, k_3^2\sqrt{\beta^2+\gamma}},\; A_1 = \; A_1,\; B_1 = 0,\; D_1 = 0,\; D_2 = 0, \tag{12}$$

$$p = 0,\; z = 1 + \frac{\beta}{\sqrt{\beta^2+\gamma}},\; E_0 = -\frac{k_4\, A_1^2\, s}{2\, k_3},\; E_1 = -\frac{k_2\, A_1^2\, s}{2\, k_3},\; E_2 = -\frac{A_1^2\, s}{2},$$

$$A_0 = \frac{A_1}{2\, k_3}(\, k_2 + \sqrt{\, k_2^2 - 4\, k_3\, k_4}),\; u(\xi) = \; A_0 + \, A_1\Phi(\xi),\; v(\xi) = \; E_0 + \, E_1\Phi(\xi) + \, E_2\Phi^2(\xi). \tag{13}$$

Where $k_1,\; k_2,\; k_3,\; \gamma$ and m are arbitrary fixed values. Considering Eq. (15) and Family 1 (if $C_1 = 0$ but $C_2 \neq 0$, $C_1 \neq 0$ but $C_2 = 0$) we get respectively:

$$\begin{aligned}
u_{1_1}(\xi) &= \frac{A_1}{2\, k_3}(\, k_2 + \sqrt{\, k_2^2 - 4\, k_3\, k_4}) + \; A_1\left[\frac{k_2}{2\, r} + \frac{\sqrt{s}}{2\, r}\, coth(\frac{\sqrt{s}}{2\, k_1}\xi)\right],\\
v_{1_1}(\xi) &= -\frac{k_4\, A_1^2\, z}{2\, k_3} - \frac{k_2\, A_1^2\, z}{2\, k_3}\left[\frac{k_2}{2\, r} + \frac{\sqrt{s}}{2\, r}\, coth(\frac{\sqrt{s}}{2\, k_1}\xi)\right] - \\
&\quad -\frac{A_1^2\, z}{2}\left[\frac{k_2}{2\, r} + \frac{\sqrt{s}}{2\, r}\, coth(\frac{\sqrt{s}}{2\, k_1}\xi)\right]^2,
\end{aligned} \tag{14}$$

$$\begin{aligned}
u_{1_2}(\xi) &= \frac{A_1}{2\, k_3}(\, k_2 + \sqrt{\, k_2^2 - 4\, k_3\, k_4}) + \; A_1[\frac{k_2}{2\, r} + \frac{\sqrt{s}}{2\, r}\, tanh\left(\frac{\sqrt{s}}{2\, k_1}\xi\right)],\\
v_{1_2}(\xi) &= -\frac{k_4\, A_1^2\, z}{2\, k_3} - \frac{k_2\, A_1^2\, z}{2\, k_3}\left[\frac{k_2}{2\, r} + \frac{\sqrt{s}}{2\, r}\, tanh\left(\frac{\sqrt{s}}{2\, k_1}\xi\right)\right] -
\end{aligned}$$

$$-\frac{A_1^2\,z}{2}\left[\frac{k_2}{2\,r}+\frac{\sqrt{s}}{2\,r}\,tanh\left(\frac{\sqrt{s}}{2\,k_1}\xi\right)\right]^2, \tag{15}$$

$$u_{1_3}(\xi)=\frac{A_1}{2\,k_3}(\,k_2+\sqrt{k_2^2-4\,k_3\,k_4})+\,A_1\left[\frac{k_2}{2\,r}+\frac{\sqrt{-\,s}}{2\,r}\,cot\left(\frac{\sqrt{-\,s}}{2\,k_1}\xi\right)\right],$$

$$v_{1_3}(\xi)=-\frac{k_4\,A_1^2\,z}{2\,k_3}-\frac{k_2\,A_1^2\,z}{2\,k_3}\left[\frac{k_2}{2\,r}+\frac{\sqrt{-\,s}}{2\,r}\,cot\left(\frac{\sqrt{-\,s}}{2\,k_1}\xi\right)\right]-$$

$$-\frac{A_1^2\,z}{2}\left[\frac{k_2}{2\,r}+\frac{\sqrt{-\,s}}{2\,r}\,cot\left(\frac{\sqrt{-\,s}}{2\,k_1}\xi\right)\right]^2. \tag{16}$$

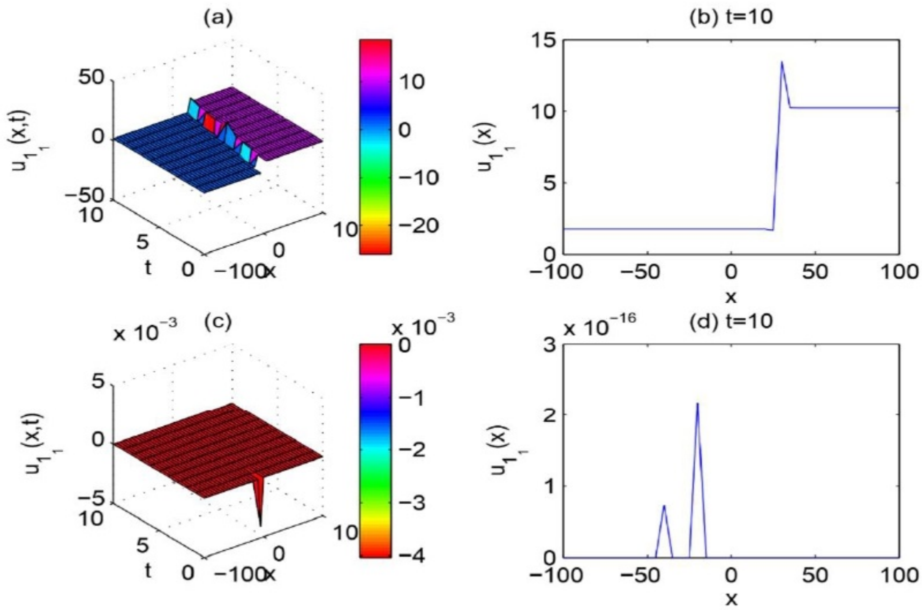

Fig.1. Dynamical diversity of the function of Eq. (14) are demonstrated at $A_1=3,\ k_1=2,\ k_2=2,\ k_3=1,\ k_4=1,\ \beta=2,\ \gamma=2,\ a=0.9$, and $t=10$ with the changes of fractional and free parameters.

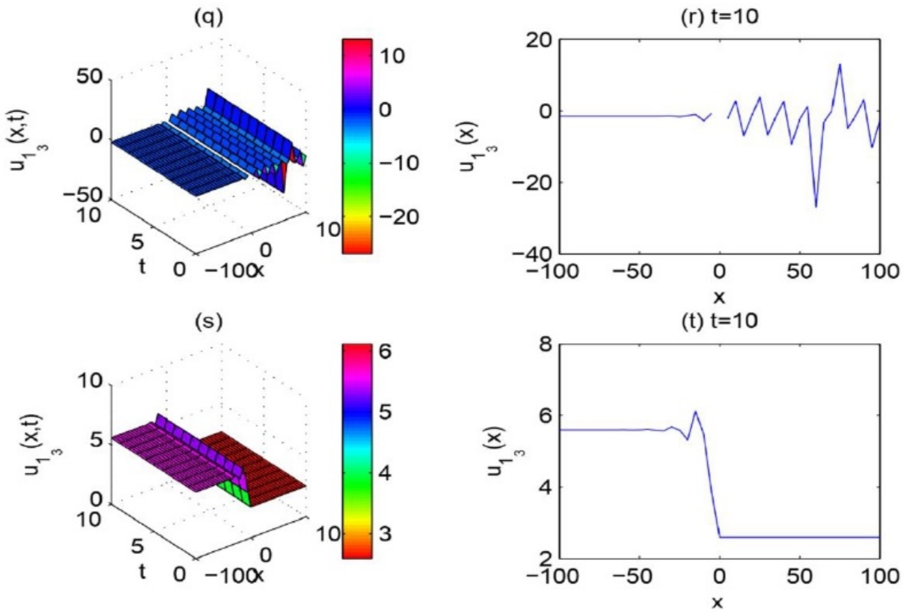

Fig.2. Dynamical diversity of the function of Eq. (15) are demonstrated at $A_1 = 3,\ k_1 = 1,\ k_2 = 2,\ k_3 = 2,\ k_4 = 2,\ \beta = 2,\ \gamma = 2,\ a = 0.9$, and $t = 10$ with the changes of fractional and free parameters.

By using of (13) and Family 2 (if $C_1 = 0$ but $C_2 \neq 0$, $C_1 \neq 0$ but $C_2 = 0$) we get respectively:

$$u_{1_4}(\xi) = \frac{A_1}{2\,k_3}(\,k_2 + \sqrt{k_2^2 - 4\,k_3\,k_4}) +\ A_1[\frac{k_2}{2\,r} - \frac{\sqrt{-\,s}}{2\,r}\,tan(\frac{\sqrt{-\,s}}{2\,k_1}\xi)],$$

$$v_{1_4}(\xi) = -\frac{k_4\,A_1^2\,z}{2\,k_3} - \frac{k_2\,A_1^2\,z}{2\,k_3}[\frac{k_2}{2\,r} - \frac{\sqrt{-\,s}}{2\,r}\,tan(\frac{\sqrt{-\,s}}{2\,k_1}\xi)] -$$
$$-\frac{A_1^2\,z}{2}[\frac{k_2}{2\,r} - \frac{\sqrt{-\,s}}{2\,r}\,tan(\frac{\sqrt{-\,s}}{2\,k_1}\xi)]^2. \tag{17}$$

By using of (15) and Family 3 we get:

$$u_{1\ 5}(\xi) = \frac{A_1}{2\,k_3}\left(k_2 + \sqrt{k_2^2 - 4\,k_3\,k_4}\right) +\ A_1\left[\frac{k_2}{2\,r} + \frac{C_2}{C_1 +\ C_2\xi}\right],$$

$$v_{1\ 5}(\xi) = -\frac{k_4\,A_1^2\,z}{2\,k_3} - \frac{k_2\,A_1^2\,z}{2\,k_3}\left[\frac{k_2}{2\,r} + \frac{C_2}{C_1 +\ C_2\xi}\right] -$$

$$
-\frac{A_1^2\, z}{2}\left[\frac{k_2}{2\, r}+\frac{C_2}{C_1+\, C_2\xi}\right]^2. \tag{18}
$$

By using of (13) and Family 4 (if $C_1 = 0$ but $C_2 \neq 0$, $C_1 \neq 0$ but $C_2 = 0$) we get respectively:

$$
\begin{aligned}
u_{1_6}(\xi) &= \frac{A_1\sqrt{-\, k_3\, k_4}}{k_3}+\frac{A_1\sqrt{q}}{r}\, coth\left(\frac{\sqrt{q}}{k_1}\xi\right),\\
v_{1_6}(\xi) &= -\frac{k_4\, A_1^2\, z}{2\, k_3}-\frac{A_1^2\, qz}{2\, r^2}\, coth^2\left(\frac{\sqrt{q}}{k_1}\xi\right),
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
u_{1_7}(\xi) &= \frac{A_1\sqrt{-4\, k_3\, k_4}}{2\, k_3}+\frac{A_1\sqrt{q}}{r}\, tanh\left(\frac{\sqrt{q}}{k_1}\xi\right),\\
v_{1_7}(\xi) &= -\frac{k_4\, A_1^2\, z}{2\, k_3}-\frac{A_1^2\, qz}{2\, r^2}\, tanh^2\left(\frac{\sqrt{q}}{k_1}\xi\right).
\end{aligned} \tag{20}
$$

By using of (13) and Family 5 (if $C_1 = 0$ but $C_2 \neq 0$, $C_1 \neq 0$ but $C_2 = 0$) we get respectively:

$$
\begin{aligned}
u_{1_8}(\xi) &= \frac{A_1\sqrt{-\, k_3\, k_4}}{k_3}+\frac{A_1\sqrt{-\, q}}{r}\, cot(\frac{\sqrt{-\, q}}{k_1}\xi), \qquad (21)\\
v_{1_8}(\xi) &= -\frac{k_4\, A_1^2\, z}{2\, k_3}+\frac{A_1^2\, qz}{2\, r^2}\, cot^2(\frac{\sqrt{-\, q}}{k_1}\xi),
\end{aligned}
$$

$$
\begin{aligned}
u_{1_9}(\xi) &= \frac{A_1\sqrt{-\, k_3\, k_4}}{k_3}-\frac{A_1\sqrt{-\, q}}{r}\, tan\left(\frac{\sqrt{-\, q}}{k_1}\xi\right),\\
v_{1_9}(\xi) &= -\frac{k_4\, A_1^2\, z}{2\, k_3}+\frac{A_1^2\, qz}{2\, r^2}\, tan^2\left(\frac{\sqrt{-\, q}}{k_1}\xi\right).
\end{aligned} \tag{22}
$$

By using of (13) and Family 6 we get:

$$
\begin{aligned}
u_{1_{10}}(\xi) &= \frac{A_1\, k_2}{k_3}+\frac{A_1\, C_1\, k_2^2\, e^{\frac{-\, k_2}{k_1}\xi}}{rk_1+\, C_1\, k_1\, k_2\, e^{\frac{-\, k_2}{k_1}\xi}},\\
v_{1_{10}}(\xi) &= -\frac{C_1\frac{k_2^3 A_{12}^2}{2k_3}e^{\frac{-k_2}{k_1}\xi}}{rk_1+C_1k_1k_2e^{\frac{-k_2}{k_1}\xi}}-\frac{A_1^2 z}{2}\left[\frac{C_1k_2^2e^{-\frac{k_2}{k_1}z}}{rk_1+C_1k_1k_2e^{\frac{-k_2}{k_1}}\xi}\right]^2.
\end{aligned} \tag{23}
$$

Remark 1. 2-D and 3-D plots for imaginary and real values of Eq. (14) and (17) are respectively captured in Figure 1 and Figure 2 that indicate the solutions' dynamics with proper parametric values. Based on the best knowledge of authors, the considered fractional solitary solutions have not been reported so far in the works of researchers. Achieved novel exact solutions in the present work offer a different physical interpretation for the nonlinear fractional WBK equation.

3.1. The space-time fractional generalized Hirota-Satsuma coupled KdV equations

The fractional generalized Hirota-Satsuma coupled KdV equations are considered in this sub-section in the following form:

$$\begin{gathered} D_t^{\alpha}u-\frac{1}{2}D_x^{3\alpha}u+3uD_x^{\alpha}u-3D_x^{\alpha}(vw)=0,\\ D_t^{\alpha}v+D_x^{3\alpha}v-3uD_x^{\alpha}v=0,\\ D_t^{\alpha}w+D_x^{3\alpha}w-3uD_x^{\alpha}w=0. \end{gathered}\tag{24}$$

By utilizing the wave variable $\xi=\frac{kx^{\alpha}}{\Gamma(\alpha+1)}-\frac{ct^{\alpha}}{\Gamma(\alpha+1)}$ reduce it to an ODE system as follows:

$$\begin{gathered} -cu'+\frac{k^3}{2}u'''+3kuu'-3k(vw)'=0,-cv'+k^3v'''-3kuv'=0,\\ -cw'+k^3w'''-3kuw'=0. \end{gathered}\tag{25}$$

By taking the homogeneous balance between u''' and $(vw)'$, v''' and uv' , and also w''' and uw' in Eq. (25) we get $M+3=N+P+1,\ N+3=M+N+1,\ P+3=M+P+1$. Then, we obtain $M=N=P=2$.

$$u=\sum_{k=0}^{M}(\cdot),\ v=\sum_{k=0}^{N}(\cdot),\ w=\sum_{k=0}^{P}(\cdot).\tag{26}$$

Then, the trail solution is:

$$\begin{aligned} u(\xi)&=A_0+A_2(p+\Phi(\xi))^2+\frac{B_2}{(p+\Phi(\xi))^2},\\ v(\xi)&=C_0+C_2(p+\Phi(\xi))^2+\frac{D_2}{(p+\Phi(\xi))^2},\\ w(\xi)&=E_0+E_2(p+\Phi(\xi))^2+\frac{F_2}{(p+\Phi(\xi))^2}. \end{aligned}\tag{27}$$

Substituting (27) into Eq. (25) and by utilizing the well-known Maple software, we can get the below sets of non-trivial solutions as:

Set I:

$$k=k,\ c=-3\,k\,A_0,\ A_0=A_0,\ B_2=\frac{4\,k^2\,k_4^2}{k_1^2},\ E_0=E_0,$$

$$D_2=D_2,\ p=p,\ k_2=0,\ k_3=0,\tag{28}$$

$$C_0=-\frac{D_2^2\,k_1^2}{4\,k^4\,k_4^4}(8A_0\,k^2\,k_4^2-E_0D_2\,k_1^2),E_2=0,F_2=\frac{4\,k^4\,k_4^4}{D_2\,k_1^4},\ A_2=0,\ C_2=0,k_4=k_4,$$

$$u(\xi)=A_0+B_2(p+\Phi(\xi))^{-2},\ v(\xi)=C_0+D_2(p+\Phi(\xi))^{-2},\ w(\xi)=E_0+F_2(p+\Phi(\xi))^{-2}.$$

Where $k_4,\ k$ and p are arbitrary fixed values. Considering Eq. (28) and Family 1 (if $C_1 = 0$ but $C_2 \neq 0$, $C_1 \neq 0$ but $C_2 = 0$) we get respectively:

$$u_{1_1}(\xi) = A_0 + \frac{4\,k^2\,k_4^2}{k_1^2}\left[p + \frac{\sqrt{s}}{2\,r}\,coth(\frac{\sqrt{s}}{2\,k_1}\xi)\right]^{-2},$$

$$w_{1_1}(\xi) = E_0 + \frac{4\,k^4\,k_4^4}{D_2\,k_1^4}\left[p + \frac{\sqrt{s}}{2\,r}\,coth(\frac{\sqrt{s}}{2\,k_1}\xi)\right]^{-2},$$

$$v_{1_1}(\xi) = -\frac{D_2^2\,k_1^2}{4\,k^4\,k_4^4}(8\,A_0\,k^2\,k_4^2 - E_0\,D_2\,k_1^2) + D_2\left[p + \frac{\sqrt{s}}{2\,r}\,coth(\frac{\sqrt{s}}{2\,k_1}\xi)\right]^{-2}. \quad (29)$$

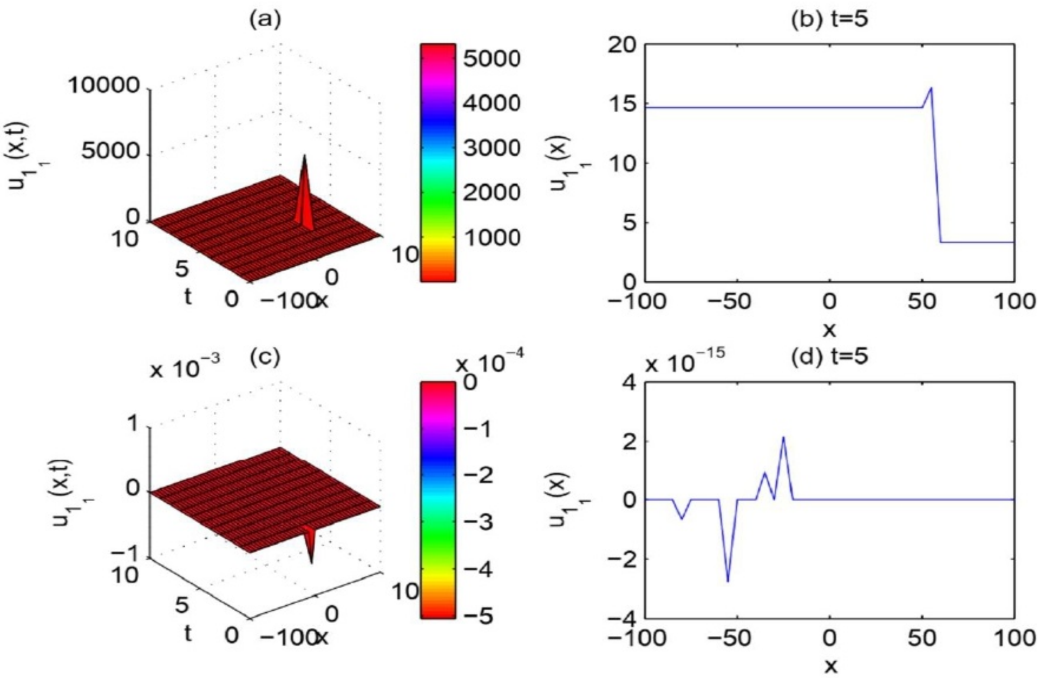

Fig.3. Dynamical diversity of the function of Eq. (29) are demonstrated at $p = 2,\ A_0 = 3,\ k = 2,\ k_1 = 2,\ k_2 = 2,\ k_3 = 1,\ k_4 = 1,\ \ a = 0.9$, and $t = 10$ with the changes of fractional and free parameters.

$$u_{1_2}(\xi) = A_0 + \frac{4\,k^2\,k_4^2}{k_1^2}[\,p + \frac{\sqrt{s}}{2\,r}\,tanh(\frac{\sqrt{s}}{2\,k_1}\xi)]^{-2},\ w_{1_2}(\xi) = E_0 + \frac{4\,k^4\,k_4^4}{D_2\,k_1^4}[\,p + \frac{\sqrt{s}}{2\,r}\,tanh(\frac{\sqrt{s}}{2\,k_1}\xi)]^{-2},$$

$$v_{1_2}(\xi) = -\frac{D_2^2 k_1^2}{4k^4 k_4^4}\left(8A_0 k^2 k_4^2 - E_0 D_2 k_1^2\right) + D_2\left[p + \frac{\sqrt{s}}{2r}\tanh\left(\frac{\sqrt{s}}{2k_1}\xi\right)\right]^{-2}. \quad (30)$$

By using of (28) and Family 2 (when $C_1 = 0$ but $C_2 \neq 0$, $C_1 \neq 0$ but $C_2 = 0$) we get respectively:

$$u_{1_3}(\xi) = A_0 + \frac{4\,k^2\,k_4^2}{k_1^2}\left[p + \frac{\sqrt{-\,s}}{2\,r}\cot\left(\frac{\sqrt{-\,s}}{2\,k_1}\xi\right)\right]^{-2},$$

$$w_{1_3}(\xi) = E_0 + \frac{4\,k^4\,k_4^4}{D_2\,k_1^4}\left[p + \frac{\sqrt{-\,s}}{2\,r}\cot\left(\frac{\sqrt{-\,s}}{2\,k_1}\xi\right)\right]^{-2},$$

$$v_{1_3}(\xi) = -\frac{D_2^2\,k_1^2}{4\,k^4\,k_4^4}\left(8\,A_0\,k^2\,k_4^2 - E_0\,D_2\,k_1^2\right) + D_2\left[p + \frac{\sqrt{-\,s}}{2\,r}\cot\left(\frac{\sqrt{-\,s}}{2\,k_1}\xi\right)\right]^{-2}, \tag{31}$$

$$u_{1_4}(\xi) = A_0 + \frac{4\,k^2\,k_4^2}{k_1^2}\left[p - \frac{\sqrt{-\,s}}{2\,r}\tan\left(\frac{\sqrt{-\,s}}{2\,k_1}\xi\right)\right]^{-2},$$

$$w_{1_4}(\xi) = E_0 + \frac{4\,k^4\,k_4^4}{D_2\,k_1^4}\left[p - \frac{\sqrt{-\,s}}{2\,r}\tan\left(\frac{\sqrt{-\,s}}{2\,k_1}\xi\right)\right]^{-2},$$

$$v_{1_4}(\xi) = -\frac{D_2^2k_1^2}{4k^4k_4^4}\left(8A_0k^2k_4^2 - E_0D_2k_1^2\right) + D_2\left[p - \frac{\sqrt{-s}}{2r}\tan\left(\frac{\sqrt{-s}}{2k_1}\xi\right)\right]^{-2}. \tag{32}$$

By using of (28) and Family 3 we get:

$$u_{1_5}(\xi) = A_0 + \frac{4\,k^2\,k_4^2}{k_1^2}\left[p + \frac{C_2}{C_1 + C_2\xi}\right]^{-2},$$

$$w_{1_5}(\xi) = E_0 + \frac{4\,k^4\,k_4^4}{D_2\,k_1^4}\left[p + \frac{C_2}{C_1 + C_2\xi}\right]^{-2},$$

$$v_{1_5}(\xi) = -\frac{D_2^2\,k_1^2}{4\,k^4\,k_4^4}(8\,A_0\,k^2\,k_4^2 - E_0\,D_2\,k_1^2) + D_2\left[p + \frac{C_2}{C_1 + C_2\xi}\right]^{-2}. \tag{33}$$

By using of (28) and Family 4 (when $C_1 = 0$, but $C_2 \neq 0$, $C_1 \neq 0$ but $C_2 = 0$) we get respectively:

$$u_{1_6}(\xi) = A_0 + \frac{4\,k^2\,k_4^2}{k_1^2}\left[p + \frac{\sqrt{q}}{r}\,coth\left(\frac{\sqrt{q}}{k_1}\xi\right)\right]^{-2},$$

$$w_{1_6}(\xi) = E_0 + \frac{4\,k^4\,k_4^4}{D_2\,k_1^4}\left[p + \frac{\sqrt{q}}{r}\,coth\left(\frac{\sqrt{q}}{k_1}\xi\right)\right]^{-2},$$

$$v_{1_6}(\xi) = -\frac{D_2^2\,k_1^2}{4\,k^4\,k_4^4}(8\,A_0\,k^2\,k_4^2 - E_0\,D_2\,k_1^2) + D_2\left[p + \frac{\sqrt{q}}{r}\,coth\left(\frac{\sqrt{q}}{k_1}\xi\right)\right]^{-2}, \tag{34}$$

$$u_{1_7}(\xi) = A_0 + \frac{4\,k^2\,k_4^2}{k_1^2}\left[p + \frac{\sqrt{q}}{r}\,tanh\left(\frac{\sqrt{q}}{k_1}\xi\right)\right]^{-2},$$

$$w_{1_7}(\xi) = E_0 + \frac{4\,k^4\,k_4^4}{D_2\,k_1^4}\left[p + \frac{\sqrt{q}}{r}\,tanh\left(\frac{\sqrt{q}}{k_1}\xi\right)\right]^{-2},$$

$$v_{1_7}(\xi) = -\frac{D_2^2\,k_1^2}{4\,k^4\,k_4^4}(8\,A_0\,k^2\,k_4^2 - E_0\,D_2\,k_1^2) + D_2\left[p + \frac{\sqrt{q}}{r}\,tanh\left(\frac{\sqrt{q}}{k_1}\xi\right)\right]^{-2}. \tag{35}$$

By using of (28) and Family 5 (when $C_1 = 0$ but $C_2 \neq 0$, $C_1 \neq 0$ but $C_2 = 0$) we get respectively:

$$u_{1\,8}(\xi) = A_0 + \frac{4\,k^2\,k_4^2}{k_1^2}\left[p + \frac{\sqrt{-\,q}}{r}\,cot\left(\frac{\sqrt{-\,q}}{k_1}\xi\right)\right]^{-2},$$

$$w_{1\,8}(\xi) = E_0 + \frac{4\,k^4\,k_4^4}{D_2\,k_1^4}\left[p + \frac{\sqrt{-\,q}}{r}\,cot\left(\frac{\sqrt{-\,q}}{k_1}\xi\right)\right]^{-2},$$

$$v_{1_8}(\xi) = -\frac{D_2^2\,k_1^2}{4\,k^4\,k_4^4}(8\,A_0\,k^2\,k_4^2 - E_0\,D_2\,k_1^2) + D_2\left[p + \frac{\sqrt{-\,q}}{r}\,cot\left(\frac{\sqrt{-\,q}}{k_1}\xi\right)\right]^{-2}, \tag{36}$$

$$u_{1_9}(\xi) = A_0 + \frac{4\,k^2\,k_4^2}{k_1^2}\left[p - \frac{\sqrt{-\,q}}{r}\,tan\left(\frac{\sqrt{-\,q}}{k_1}\xi\right)\right]^{-2},$$

$$w_{1\,9}(\xi) = E_0 + \frac{4\,k^4\,k_4^4}{D_2\,k_1^4}\left[p - \frac{\sqrt{-\,q}}{r}\,tan\left(\frac{\sqrt{-\,q}}{k_1}\xi\right)\right]^{-2},$$

$$v_{1_9}(\xi) = -\frac{D_2^2k_1^2}{4k^4k_4^4}(8A_0k^2k_4^2 - E_0D_2k_1^2) + D_2\left[p - \frac{\sqrt{-q}}{r}\,tan\left(\frac{\sqrt{-q}}{k_1}\xi\right)\right]^{-2},$$

where $\xi = \frac{1}{\Gamma(\alpha+1)}(kx^\alpha - 3kA_0t^\alpha)$.

Remark 2. 2-D and 3-D plots for imaginary and real values of Eq. (29) are captured in Figure 3 indicating solution's' dynamics with proper parametric values. Based on the best knowledge of authors, the considered complex exponential function solutions have not been reported in the literature so far. Achieved analytical solutions figures offer a different physical interpretation for the considered nonlinear fractional generalized Hirota-Satsuma coupled KdV equation.

4. Conclusion

In this paper, a novel methods known as the generalized (G'/G)-expansion technique is proposed for achieving analytical solutions for the space-time fractional WBK equation as well as the generalized Hirota-Satsuma coupled KdV equation. It's demonstrated that, achieved solutions for mentioned equations are very suitable even in comparison with the used approaches in [3, 4]. Baed on obtained results, the proposed method offer high efficiency and reliability. In addition, new solutions are formally derived in the present work. Obtained precise solutions contain three kinds of hyperbolic function, and trigonometric function solutions as well as the rational one. Respect to the results, the suggested approach is highly efficient and reliable method to obtain precise solutions of broad types of problems. Precise solutions of fractional partial differential equations have an important significant disclose the internal mechanism of complex physical phenomena. Besides of the physical utility, the close-form results of nonlinear evolution equations can help the numerical solving tools for comparing the precision of their solutions and assist them in the stability study. It is very clear that our promoted technique is effective, reliable, and friendly applicable and deliver sufficient well-matched explanations to NLFEEs arise in engineering, applied mathematics, nonlinear dynamics and mathematical physics.

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