Journal of Contemporary Applied Mathematics
V. 2, No 2, 2012, December
ISSN 2222-5498

On the Convergence of New Iterations with Errors for T-Zamfirescu Operators

Sherly George*, Shaini P.

Abstract. In this paper T-Mann iteration with errors and T-Ishikawa type iteration with errors for T-Zamferescu operators are introduced. Then we study the convergence of these iterations in the class of T-Zamfiresu operators in real Banach spaces. Our result improves the corresponding result proved by Jose R. Morales and Edixon Rojas in [6].

Key Words and Phrases: T-Mann iteration with errors, T-Ishikawa iteration with errors, Strong convergence, T- Zamfirescu operators, Banach spaces.

2000 Mathematics Subject Classifications: 47H09, 47H10

1. Introduction

The literature of the last five decades abounds with papers which establish fixed point theorems for self maps or non self maps satisfying a variety of contractive type conditions on several ambient spaces. One of the most studied class of quasi-contractive type operotor is that of Zamfirescu operators. Banach's contraction principle was the first well known result on fixed points for contractive type mapping. In 1962 Edelstein introduced a result on fixed point for contractive type mapping in the setting of compact metric space.

Recently A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh [1] introduced the T-Contraction and T-contractive mappings and then they extended the Banach contraction principle and the Edelstein's fixed point Theorem. S. Moradi[9] introduced the T-Kannan contractive type mappings, extending the well -known Kannan's fixed point theorem [8]. Followed by this Jose R. Morales and Edixon Rojas [5] obtained sufficient conditions for the existence of a unique fixed point of T-Chatterjea mappings for complete cone metric spaces. After this he introduced the concept of T-Zamfirescu operators and obtained sufficient conditions for the existence of a unique fixed point of T-Zamfirescu mapping in the frame work of complete cone metric spaces. They also studied the existence of fixed points for T-Zamfirescu operators in complete metic spaces and proved a covergence theorem for this class of operators [6].

*Corresponding author.

The idea of considering fixed point iteration procedures with errors comes from the practical numerical computation. The fixed piont iteration with errors was introduced by Liu, Lishan [2 pg 151]. After this many fixed point iteration processes including Mann and Ishikawa iterations processes with errors have been studied extensively by many authors. The aim of this paper is to establish strong convergence theorems for T-Mann iteration scheme with errors and T-Ishikawa type iteration scheme with errors.

2. Preliminaries

Definition 1. *Let (M,d) be a metric space and $T,S : M \to M$ be two functions. A mapping S is said to be* ***T-Banach contraction***
(TB- contraction) if there exist $a \in [0,1)$ such that
$d(TSx,TSy) \le ad(Tx,Ty) \qquad \forall x,y \in M.$
If we take $T = I$, the identity map, then we obtain the definition of Banach'contraction [2].

Definition 2. *Let (M,d) be a metric space and $T,S : M \to M$ be two functions. A mapping S is said to be* ***T-Kannan contraction***
(TK- contraction) if there exist $b \in [0,\frac{1}{2})$ such that
$d(TSx,TSy) \le b[d(Tx,TSx) + d(Ty,TSy)] \qquad \forall x,y \in M.$
If we take $T = I$, then we obtain the definition of Kannan mapping [8]

Definition 3. *Let (M,d) be a metric space and $T,S : M \to M$ be two functions. A mapping S is said to be* ***T-Chatterjea contraction***
(TC- contraction) if there exist $c \in [0,\frac{1}{2})$ such that
$d(TSx,TSy) \le c[d(Tx,TSy) + d(Ty,TSx)] \qquad \forall x,y \in M.$
If we take $T = I$, then we obtain the definition of Chatterjea [3].

Definition 4. *Let (M,d) be a metric space and $T,S : M \to M$ be two functions. A mapping S is said to be* ***T-Zamfirescu operator*** *(TZ operator) if there exist real numbers $0 \le a < 1, 0 \le b < \frac{1}{2}, 0 \le c < \frac{1}{2}$ such that for all $x,y \in M$ at least one of the following conditions is true:*
$(TZ_1): \quad d(TSx,TSy) \le ad(Tx,Ty) \qquad \forall x,y \in M.$
$(TZ_2): \quad d(TSx,TSy) \le b[d(Tx,TSx) + d(Ty,TSy)] \qquad \forall x,y \in M.$
$(TZ_3): \quad d(TSx,TSy) \le c[d(Tx,TSy) + d(Ty,TSx)] \qquad \forall x,y \in M.$
If we take $T = I$, then we obtain the definition of Z-operator [4].

Definition 5. *Let (M,d) be a metric space. A mapping $T : M \to M$ is said to be* ***subsequentially convergent****, if we have for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ has a convergent subsequence.*

Definition 6. *Let (M,d) be metric space, $x_0 \in M$ be arbitrary and $T,S : M \to M$ be two mappings. The sequence $Tx_n \in M$ defined by*

$$Tx_{n+1} = TSx_n = TS^n x_0, \qquad n = 1, 2....$$

is called the T-Picard iteration associated to S.

We need the following results to prove our results.

Theorem 1. *[6, Th 3.1] Let (M,d) be a complete metric space and $T,S : M \to M$ be two mappings such that T is continuous, one to one and subsequentially convergent. If S is a TZ operator, then S has a unique fixed point.*

Lemma 1. [6]. *Let (M,d) be a metric space and $T,S : M \to M$ be two functions. If S is a TZ-operator, then there exist $0 \le \delta < 1$ such that*
$d(TSx, TSy) \le \delta d(Tx, Ty) + 2\delta d(Tx, TSx) \qquad \forall x, y \in M.$

Lemma 2. [1, p.13]. *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be the sequences of non-negative numbers satisfying*
$a_{n+1} \le (1-\omega_n)a_n + b_n + c_n \qquad \forall n \ge 0$ *where* $\{\omega_n\}_{n=0}^{\infty} \subset [0,1]$.
If $\sum_{n=0}^{\infty} \omega_n = \infty, b_n = O(\omega_n)$ *and* $\sum_{n=0}^{\infty} c_n < \infty$ *then* $\lim_{n\to\infty} a_n = 0$

In this paper we introduce two new iteration schemes with errors, namely T-Mann iteration with errors and T-Ishikwa type iteration with errors.
We define the iterations as follows.

Definition 7. *Let E be a Banach space, $x_0 \in E$ and $T,S : E \to E$ be two mappings. The sequence $Tx_n \in E$ defined by*

$$Tx_{n+1} = (1-\alpha_n)Tx_n + \alpha_n TSx_n + u_n \qquad n = 0, 1, 2.... \tag{1}$$

where $\alpha_n \in [0,1]$ and $\{u_n\}$ is a sequence of real numbers satisfying
$\sum_{n=0}^{\infty} \|u_n\| < \infty$ *is called T-Mann iteration with errors associated to S.*

When $\alpha_n = 1, u_n = 0$ the iteration defined by (1) reduces to T-Picard associated to S. When we put $T = I$, the identity map, and $u_n = 0$, the iteration(1) reduces to

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Sx_n \qquad n = 0, 1, 2...$$

which is Mann iteration. When $\alpha_n = \lambda$, this iteration becomes

$$x_{n+1} = (1-\lambda)x_n + \lambda Sx_n \qquad n = 0, 1, 2...$$

which is Krasnoselskij iteration.

Definition 8. *Let E be a Banach space, $x_0 \in E$ and $T, S : E \to E$ be two mappings. The sequence $Tx_n \in E$ defined by*

$$Tx_{n+1} = (1-\alpha_n)Tx_n + \alpha_n STy_n + u_n \tag{2}$$

$$Ty_n = (1-\beta_n)Tx_n + \beta_n TSx_n + v_n \qquad n = 0,1,2...$$

where $\{\alpha_n\}, \{\beta_n\}$ in$[0,1]$ and $\{u_n\}, \{v_n\}$ are sequences of real numbers satisfying $\sum_{n=0}^{\infty} \|u_n\| < \infty, \sum_{n=0}^{\infty} \|v_n\| < \infty$ is called T-Ishikawa type iteration with errors associated to S.

When $T = I, u_n = 0, v_n = 0$,the iteration defined by (2) reduces to

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Sy_n$$

$$y_n = (1-\beta_n)x_n + \beta_n Sx_n \qquad n = 0,1,2...$$

which is Ishikawa iteration.

3. Main results

Theorem 2. *Let E be a real Banach space, K a closed, convex subset of E and $T, S : K \to K$ be two mappings such that T is continuous, one to one, subsequentially convergent and S is a TZ operator. Let $\{Tx_n\}_{n=0}^{\infty}$ be the sequence defined by (1) where $\{\alpha_n\} \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{u_n\}$ is a sequence of real numbers satisfying $\sum_{n=0}^{\infty} \|u_n\| < \infty$. Then $\{Tx_n\}_{n=0}^{\infty}$ converges strongly to Tx^* where x^* is the fixed point of S.*

Proof

By theorem 2.7, we get that S has a unique fixed point say x^* in K.
Since S is a TZ-operator, by applying lemma 2.8 we get that there exist $0 \le \delta < 1$ such that

$$\|TSx - TSy\| \le \delta \|Tx - Ty\| + 2\delta \|Tx - TSx\| \qquad \forall x, y \in K. \tag{3}$$

Let $\{Tx_n\}_{n=0}^{\infty}$ be the T-Mann iteration with errors defined by (1).
Then

$$\begin{aligned}
\|Tx_{n+1} - Tx^*\| &= \|(1-\alpha_n)Tx_n + \alpha_n TSx_n + u_n - Tx^*\| \\
&= \|(1-\alpha_n)(Tx_n - Tx^*) + \alpha_n(TSx_n - Tx^*) + u_n\| \\
&\le (1-\alpha_n)\|Tx_n - Tx^*\| + \alpha_n \|TSx_n - Tx^*\| + \|u_n\|. \qquad (4)
\end{aligned}$$

Taking $x = x^*, y = x_n$ in (3), we get

$\|TSx^* - TSx_n\| \leq \delta \|Tx^* - Tx_n\| + 2\delta \|Tx^* - TSx^*\|,$
which gives

$$\|Tx^* - TSx_n\| \leq \delta \|Tx^* - Tx_n\|. \tag{5}$$

Using (4) and (5)

$$\begin{aligned} \|Tx_{n+1} - Tx^*\| &\leq (1-\alpha_n)\|Tx_n - Tx^*\| + \alpha_n\delta\|Tx^* - Tx_n\| + \|u_n\| \\ &\leq (1-\alpha_n+\alpha_n\delta)\|Tx_n - Tx^*\| + \|u_n\| \\ &\leq [1-\alpha_n(1-\delta)]\|Tx_n - Tx^*\| + \|u_n\|. \end{aligned} \tag{6}$$

Since $0 \leq \delta < 1, \alpha_n \in [0,1], \sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \|u_n\| < \infty$
and setting $a_n = \|Tx_n - Tx^*\|, \omega_n = \alpha_n(1-\delta)$, $c_n = \|u_n\|$, by lemma 2.9
we get

$$\lim_{n\to\infty} \|Tx_{n+1} - Tx^*\| = 0.$$

Hence $\{Tx_n\}_{n=0}^{\infty}$ converges strongly to Tx^* where x^* is the fixed point of S.

Corollary 1. *[6 Th 3.1] Let (M,d) be a complete matric space and $T,S : M \to M$ be two mappings such that T is continuous, one to one and subsequentially convergent. If S is a TZ operator,then S has a unique fixed point. Moreover, if T is sequentially convergent then for every $x_o \in M$ the T-Picard iteration associated to $S, (TS^n x_o)$ converges to Tx^* where x^* is the fixed point of S.*

When $u_n = 0$ we obtain the following convergence theorem for TZ-operators.

Corollary 2. *Let E be a real Banach space, K a closed, convex subset of E and T,S : $K \to K$ be two mappings such that T is continuous, one to one, subsequentially convergent and S is a TZ-operator. Let $\{Tx_n\}_{n=0}^{\infty}$ be the sequence defined by*

$$Tx_{n+1} = (1-\alpha_n)Tx_n + \alpha_n TSx_n \tag{7}$$

where $\{\alpha_n\} \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{Tx_n\}_{n=0}^{\infty}$ converges strongly to Tx^* where x^* is the fixed point of S.

remarks

Since T-Kannan and T-Chatterjea contractive conditions are both included in the class of TZ-operators, by above corollary we obtain the convergence theorems for Tmann iterations in these classes of operators.

Corollary 3. *Let E be a real Banach space, K a closed, convex subset of E and T,S : $K \to K$ be two mappings such that T is continuous, one to one, subsequentially convergent and S is a TK-contraction. Let $\{Tx_n\}_{n=0}^{\infty}$ be the sequence defined by (7) where*

$\{\alpha_n\} \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{Tx_n\}_{n=0}^{\infty}$ converges strongly to Tx^* where x^* is the fixed point of S.

Corollary 4. *Let E be a real Banach space, K a closed, convex subset of E and $T, S : K \to K$ be two mappings such that T is continuous, one to one, subsequentially convergent and S is a TC-contraction. Let $\{Tx_n\}_{n=0}^{\infty}$ be the sequence defined by (7) where $\{\alpha_n\} \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{Tx_n\}_{n=0}^{\infty}$ converges strongly to Tx^* where x^* is the fixed point of S.*

Theorem 3. *Let E be a real Banach space, K a closed, convex subset of E. $T, S : K \to K$ be two mappings such that T is continuous, one to one, subsequentially convergent, S is a TZ- operator and S, T are commuting mappings. Let $\{Tx_n\}_{n=0}^{\infty}$ be the sequence defined by (2) where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty, \{u_n\}$ and $\{v_n\}$ are sequenes of real numbers satisfying $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\lim_{n\to\infty} \|v_n\| = 0$.*
Then $\{Tx_n\}_{n=0}^{\infty}$ converges to Tx^* where x^* is the fixed point of S.

Proof

By theorem 2.7, we get that S has a unique fixed point, say x^* in K. Since S is a TZ-operator, by lemma 2.8, there is $0 \leq \delta < 1$ such that

$$\|TSx - TSy\| \leq \delta \|Tx - Ty\| + 2\delta \|Tx - TSx\| \qquad \forall x, y \in K. \tag{8}$$

Let $\{Tx_n\}_{n=0}^{\infty} \in K$ be the T-Ishikawa iteration with errors associated to S defined by (2) and
$x_0 \in K$. Then

$$\begin{aligned} \|Tx_{n+1} - Tx^*\| &= \|(1-\alpha_n)Tx_n + \alpha_n STy_n + u_n - Tx^*\| \\ &= \|(1-\alpha_n)(Tx_n - Tx^*) + \alpha_n(STy_n - Tx^*) + u_n\|. \end{aligned}$$

Thus we get

$$\|Tx_{n+1} - Tx^*\| \leq (1-\alpha_n) \|Tx_n - Tx^*\| + \alpha_n \|STy_n - Tx^*\| + \|u_n\|. \tag{9}$$

Taking $x = x^*$ and $y = y_n$ in (8),

$$\|TSx^* - TSy_n\| \leq \delta \|Tx^* - Ty_n\| + 2\delta \|Tx^* - TSx^*\|,$$

which implies

$$\|Tx^* - TSy_n\| \leq \delta \|Tx^* - Ty_n\|. \tag{10}$$

Since S and T are commuting mappings the above equality becomes

$$\|Tx^* - STy_n\| \leq \delta \|Tx^* - Ty_n\|. \tag{11}$$

Using (11) in (9), we obtain

$$\|Tx_{n+1} - Tx^*\| \leq (1 - \alpha_n) \|Tx_n - Tx^*\| + \alpha_n \delta \|Tx^* - Ty_n\| + \|u_n\| . \tag{12}$$

Now from (2), $\|Ty_n - Tx^*\| = \|(1 - \beta_n)Tx_n + \beta_n TSx_n + v_n - Tx^*\|$
$= \|(1 - \beta_n)(Tx_n - Tx^*) + \beta_n(TSx_n - Tx^*) + v_n\|,$
which implies

$$\|Ty_n - Tx^*\| \leq (1 - \beta_n) \|(Tx_n - Tx^*)\| + \beta_n \|(TSx_n - Tx^*)\| + \|v_n\| . \tag{13}$$

Taking $x = x^*$ and $y = x_n$ in (8), we get
$$\|TSx^* - TSx_n\| \leq \delta \|Tx^* - Tx_n\| + 2\delta \|Tx^* - TSx^*\|.$$
Thus we obtain

$$\|Tx^* - TSx_n\| \leq \delta \|Tx^* - Tx_n\| \tag{14}$$

Using (14) in (13), we get
$\|Ty_n - Tx^*\| \leq (1 - \beta_n) \|(Tx_n - Tx^*)\| + \beta_n \delta \|(Tx_n - Tx^*)\| + \|v_n\|,$
which implies

$$\|Ty_n - Tx^*\| \leq (1 - \beta_n + \beta_n \delta) \|(Tx_n - Tx^*)\| + \|v_n\| \tag{15}$$

Using (15) in (12), we get

$\|Tx_{n+1} - Tx^*\| \leq (1-\alpha_n) \|Tx_n - Tx^*\| + \alpha_n \delta [(1 - \beta_n + \beta_n \delta) \|(Tx_n - Tx^*)\| + \|v_n\|] + \|u_n\|,$
which gives

$$\|Tx_{n+1} - Tx^*\| \leq [1-\alpha_n+\alpha_n\delta(1-\beta_n+\beta_n\delta)] \|Tx_n - Tx^*\| + \alpha_n\delta(1-\beta_n+\beta_n\delta) \|v_n\| + \|u_n\| . \tag{16}$$

Now
$1 - \alpha_n + \alpha_n \delta(1 - \beta_n + \beta_n \delta) = 1 - [\alpha_n(1 - \delta)(1 + \beta_n \delta)].$

Since $(1 + \beta_n \delta) \geq (1 - \delta)$, we get

$$1 - \alpha_n + \alpha_n \delta(1 - \beta_n + \beta_n \delta) \leq 1 - (1 - \delta)^2 \alpha_n. \tag{17}$$

Using (17) in (16), we obtain
$\|Tx_{n+1} - Tx^*\| \leq [1 - (1 - \delta)^2 \alpha_n] \|(Tx_n - Tx^*)\| + \alpha_n \delta (1 - \beta_n + \beta_n \delta) \|v_n\| + \|u_n\|.$
Now
$\alpha_n \delta (1 - \beta_n + \beta_n \delta) = \alpha_n \delta [1 - \beta_n(1 - \delta)] \leq \alpha_n \delta.$
So we have
$$\|Tx_{n+1} - Tx^*\| \leq [1 - (1 - \delta)^2 \alpha_n] \|(Tx_n - Tx^*)\| + \alpha_n \delta \|v_n\| + \|u_n\|$$

Since $0 \leq \delta < 1, \alpha_n \in [0, 1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

by setting $\omega_n = (1 - \delta)^2 \alpha_n, a_n = \|Tx_n - Tx^*\|, c_n = \|u_n\|$ and using the fact that

$\{\alpha_n\} \in [0,1], \sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \|v_n\| = 0, \sum_{n=1}^{\infty} \|u_n\| < \infty$, by applying lemma 2.9, we get that

$$\lim_{n\to\infty} \|Tx_{n+1} - Tx^*\| = 0.$$

Hence $\{Tx_n\}_{n=0}^{\infty}$ converges to Tx^* where x^* is the fixed point of S.
When $u_n = 0, v_n = 0$ we obtain the following convergence theorem for TZ-operators.

Corollary 5. *Let E be a real Banach space, K a closed, convex subset of E and $T, S : K \to K$ be two mappings such that T is continuous, one to one, subsequentially convergent and S is a TZ operator. Let $\{Tx_n\}_{n=0}^{\infty}$ be the sequence defined by*

$$Tx_{n+1} = (1-\alpha_n)Tx_n + \alpha_n TSy_n \tag{18}$$

$$Ty_n = (1-\beta_n)Tx_n + \beta_n TSx_n \qquad n = 1, 2...$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{Tx_n\}_{n=0}^{\infty}$ converges to Tx^* where x^* is the fixed point of S.

Since T-Kannan and T-Chatterjea contractive conditions are both included in the class of TZ-operators, by above corollary we obtain the following convergence theorems for T-Ishikwa type iterations in these classes of operators.

Corollary 6. *Let E be a real Banach space, K a closed, convex subset of E and $T, S : K \to K$ be two mappings such that T is continuous, one to one, subsequentially convergent and S is a TK- contraction. Let $\{Tx_n\}_{n=0}^{\infty}$ be the sequence defined by(18) where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{Tx_n\}_{n=0}^{\infty}$ converges to Tx^* where x^* is the fixed point of S.*

Corollary 7. *Let E be a real Banach space, K a closed, convex subset of E and $T, S : K \to K$ be two mappings such that T is continuous, one to one, subsequentially convergent and S is a TC- contraction. Let $\{Tx_n\}_{n=0}^{\infty}$ be the sequence defined by(18) where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{Tx_n\}_{n=0}^{\infty}$ converges to Tx^* where x^* is the fixed point of S.*

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Sherly George
Department of Mathematics Sam Higginbottom Institute of Agriculture, Technology and Sciences Allahabad-211007, India
E-mail: sherlygeorgepj@gmail.com

Shaini P.
Department of Mathematics Sam Higginbottom Institute of Agriculture, Technology and Sciences Allahabad-211007, India.