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Theorems on Carlitz's twisted q-Euler polynomials under S_5

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Abstract. In this paper, we consider Carlitz's twisted q-Euler polynomials and present some new symmetric identities for Carlitz's twisted q-Euler polynomials using the fermionic p-adic invariant integral on $\mathbb{Z}_p$ under symmetric group of degree five.

Key Words and Phrases: Symmetric identities; Carlitz's twisted q-Euler polynomials; p-adic invariant integral on $\mathbb{Z}_p$; Invariant under S_5.

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1. Introduction

The Euler polynomials $E_n(x)$ are defined by means of the Taylor expansion about $t = 0$:

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt} \text{ with } (|t| < \pi). \tag{1}$$

When $x = 0$, we have $E_n(0) := E_n$ that are known as Euler numbers (see, e.g., [3], [5], [6], [7], [8], [9]).

Let $\mathbb{Z}$ be the set of the all integers, $\mathbb{C}$ be the set of all complex numbers and p be chosen as a fixed odd prime number. Throughout this paper, the symbols $\mathbb{Z}_p$, $\mathbb{Q}$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ shall denote topological closure of $\mathbb{Z}$, the field of rational numbers, topological closure of $\mathbb{Q}$ and the field of p-adic completion of an algebraic closure of $\mathbb{Q}_p$, respectively.

For d an odd positive number with $(d,p) = 1$, let

$$X := X_d = \varprojlim_{n} \mathbb{Z}/dp^N\mathbb{Z} \;\; and \; X_1 = \mathbb{Z}_p$$

and

$$t + dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv t \left(mod \; dp^N\right)\right\}$$

where $t \in \mathbb{Z}$ lies in $0 \leq t < dp^N$. See, for more details, [1,2, 4-9].

The normalized absolute value according to the theory of p-adic analysis is given by $|p|_p = p^{-1}$. The notation q can be considered as an indeterminate, a complex number $q \in \mathbb{C}$ with $|q| < 1$, or a p-adic number $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ and $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

For fixed x, let us introduce the following notation

$$[x]_q = \frac{1-q^x}{1-q} \tag{2}$$

which is known as q-number of x (or q-analogue of x), see [1-11]. Notice that as $q \to 1$, the notation $[x]_q$ reduce to the x.

For

$$f \in UD(\mathbb{Z}_p) = \{f \,| f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

Kim [5] defined the p-adic invariant integral on $\mathbb{Z}_p$ as follows:

$$\lim_{q \to -1} I_q(f) = I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x. \tag{3}$$

By Eq. (3), we have

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2\sum_{k=0}^{n-1} (-1)^{n-k-1} f(k)$$

where $f_n(x)$ implies $f(x+n)$. For more details, take a look at the references [1], [2], [4], [5], [6], [7], [8], [9].

Let

$$T_p = \bigcup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N},$$

where $C_{p^N} = \left\{ w : w^{p^N} = 1 \right\}$ is the cyclic group of order p^N. For $w \in T_p$, we show by $\phi_w : \mathbb{Z}_p \to C_p$ the locally constant function $x \to w^x$. For $q \in C_p$ with $|1-q|_p < 1$ and $w \in T_p$, the Carlitz's twisted q-Euler polynomials are defined by the following p-adic invariant integral on $\mathbb{Z}_p$ in [9], with respect to μ_{-1}:

$$\mathcal{E}_{n,q,w}(x) = \int_{\mathbb{Z}_p} w^y [x+y]_q^n \, d\mu_{-1}(y) \quad (n \geq 0). \tag{4}$$

If we let $x = 0$ into the Eq. (4), we then get $\mathcal{E}_{n,q,w}(0) := \mathcal{E}_{n,q,w}$ called as n-th Carlitz's twisted q-Euler numbers.

Notice that when $w=1$ and $q\to 1$ in the Eq. (4), we observe that

$$\mathcal{E}_{n,q,w}(x)\to E_n(x):=\int_{\mathbb{Z}_p}(x+y)^n d\mu_{-1}(y)\,.$$

Recently, symmetric identities of some special polynomials, such as q-Genocchi polynomials of higher order under third Dihedral group D_3 in [1], q-Frobenious-Euler polynomials under symmetric group of degree five in [2], weighted q-Genocchi polynomials under the symmetric group of degree four in [4], Carlitz's q-Euler polynomials invariant under the symmetric group of degree five in [6], have been studied extensively.

In the next section, we give some new symmetric identities for Carlitz's twisted q-Euler polynomials associated with the fermionic p-adic invariant integral on $\mathbb{Z}_p$ symmetric group of degree five denoted by S_5.

2. Results on $\mathcal{E}_{n,q,w}(x)$ under S_5

Let $w\in T_p$ and w_i are odd natural numbers where $i\in\{1,2,3,4,5\}$. By the Eqs. (3) and (4), we acquire

$$\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y} \tag{5}$$

$$\times e^{[w_1w_2w_3w_4y+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s]_q t}d\mu_{-1}(y) \tag{6}$$

$$=\lim_{N\to\infty}\sum_{y=0}^{p^N-1}(-1)^y w^{w_1w_2w_3w_4y} \tag{7}$$

$$\times e^{[w_1w_2w_3w_4y+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s]_q t}$$

$$=\lim_{N\to\infty}\sum_{l=0}^{w_5-1}\sum_{y=0}^{p^N-1}(-1)^{l+y}w^{w_1w_2w_3w_4(l+w_5y)}$$

$$\times e^{[w_1w_2w_3w_4(l+w_5y)+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s]_q t}.$$

Taking

$$\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1}(-1)^{i+j+k+s}w^{w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s}$$

on the both sides of Eq. (6) gives

$$\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1}(-1)^{i+j+k+s}w^{w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s} \tag{8}$$

$$\times\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y}$$

$$\times e^{[w_1w_2w_3w_4y+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s]_q t}d\mu_{-1}(y)$$
$$= \lim_{N\to\infty}\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1}\sum_{l=0}^{w_5-1}\sum_{y=0}^{p^N-1}(-1)^{i+j+k+s+y+l}$$
$$\times w^{w_1w_2w_3w_4(l+w_5y)+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s}$$
$$\times e^{[w_1w_2w_3w_4(l+w_5y)+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s]_q t}.$$

Observe that the equation (8) is invariant for any permutation $\sigma \in S_5$. Therefore, we obtain the following theorem.

Theorem 2.1. *Let $w \in T_p$ and w_i are odd natural numbers where $i \in \{1,2,3,4,5\}$. Then the following*

$$\sum_{i=0}^{w_{\sigma(1)}-1}\sum_{j=0}^{w_{\sigma(2)}-1}\sum_{k=0}^{w_{\sigma(3)}-1}\sum_{s=0}^{w_{\sigma(4)}-1}(-1)^{i+j+k+s}$$
$$\times w^{w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k+w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(1)}w_{\sigma(2)}s}$$
$$\times \int_{\mathbb{Z}_p} w^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}\left(l+w_{\sigma(5)}y\right)}$$
$$\times \exp\big([w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}y + w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}w_{\sigma(5)}x + w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i$$
$$+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j + w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k + w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(1)}w_{\sigma(2)}s]_q t\big)\, d\mu_{-1}(y)$$

holds true for any $\sigma \in S_5$.

By using Eq. (2), we obtain the following equality:

$$[w_1w_2w_3w_4y + w_1w_2w_3w_4w_5x + w_5w_4w_2w_3i + w_5w_4w_1w_3j + w_5w_4w_1w_2k + w_5w_3w_1w_2s]_q$$
$$= [w_1w_2w_3w_4]_q\left[y + w_5x + \frac{w_5}{w_1}i + \frac{w_5}{w_2}j + \frac{w_5}{w_3}k + \frac{w_5}{w_4}s\right]_{q^{w_1w_2w_3w_4}}. \tag{9}$$

By virtue of Eqs. (6) and (9), we derive

$$\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y}$$
$$e^{[w_1w_2w_3w_4y+w_1w_2w_3w_4w_5x+w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s]_q t}d\mu_{-1}(y) \tag{10}$$
$$= \sum_{n=0}^{\infty}[w_1w_2w_3w_4]_q^n$$
$$\times\left(\int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y}\left[y + w_5x + \frac{w_5}{w_1}i + \frac{w_5}{w_2}j + \frac{w_5}{w_3}k + \frac{w_5}{w_4}s\right]^n_{q^{w_1w_2w_3ww_4}} d\mu_{-1}(y)\right)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} [w_1 w_2 w_3 w_4]_q^n \, \mathcal{E}_{n,q^{w_1 w_2 w_3 w_4}, w^{w_1 w_2 w_3 w_4}} \left(w_5 x + \frac{w_5}{w_1} i + \frac{w_5}{w_2} j + \frac{w_5}{w_3} k + \frac{w_5}{w_4} s \right) \frac{t^n}{n!}.$$

On account of Eq. (10), we have

$$\begin{aligned} &\int_{\mathbb{Z}_p} w^{w_1 w_2 w_3 w_4 y} \qquad (11) \\ &\times [w_1 w_2 w_3 w_4 y + w_1 w_2 w_3 w_4 w_5 x + w_5 w_4 w_2 w_3 i + w_5 w_4 w_1 w_3 j + \\ &+ w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 s]_q^n \, d\mu_{-1}(y) \\ &= [w_1 w_2 w_3 w_4]_q^n \, \mathcal{E}_{n,q^{w_1 w_2 w_3 w_4}, w^{w_1 w_2 w_3 w_4}} \left(w_5 x + \frac{w_5}{w_1} i + \frac{w_5}{w_2} j + \frac{w_5}{w_3} k + \frac{w_5}{w_4} s \right), \ (n \geq 0). \end{aligned}$$

Herewith, from Theorem 2.1 and Eq. (11), we have the following theorem.

Theorem 2.2. *Let $w \in T_p$ and w_i are odd natural numbers where $i \in \{1, 2, 3, 4, 5\}$. The following expression*

$$\begin{aligned} &\left[w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} w_{\sigma(4)}\right]_q^n \sum_{i=0}^{w_{\sigma(1)}-1} \sum_{j=0}^{w_{\sigma(2)}-1} \sum_{k=0}^{w_{\sigma(3)}-1} \sum_{s=0}^{w_{\sigma(4)}-1} (-1)^{i+j+k+s} \\ &\times w^{w_{\sigma(5)} w_{\sigma(4)} w_{\sigma(2)} w_{\sigma(3)} i + w_{\sigma(5)} w_{\sigma(4)} w_{\sigma(1)} w_{\sigma(3)} j + w_{\sigma(5)} w_{\sigma(4)} w_{\sigma(1)} w_{\sigma(2)} k + w_{\sigma(5)} w_{\sigma(3)} w_{\sigma(1)} w_{\sigma(2)} s} \\ &\times \mathcal{E}_{n,q^{w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} w_{\sigma(4)}}, w^{w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} w_{\sigma(4)}}} \left(w_{\sigma(5)} x + \frac{w_{\sigma(5)}}{w_{\sigma(1)}} i + \frac{w_{\sigma(5)}}{w_{\sigma(2)}} j + \frac{w_{\sigma(5)}}{w_{\sigma(3)}} k + \frac{w_{\sigma(5)}}{w_{\sigma(4)}} s \right) \end{aligned}$$

holds true for any $\sigma \in S_5$.

We observe that

$$\begin{aligned} &\int_{\mathbb{Z}_p} w^{w_1 w_2 w_3 w_4 y} \left[y + w_5 x + \frac{w_5}{w_1} i + \frac{w_5}{w_2} j + \frac{w_5}{w_3} k + \frac{w_5}{w_4} s \right]_{q^{w_1 w_2 w_3 w_4}}^n d\mu_{-1}(y) \qquad (12) \\ &= \sum_{m=0}^{n} \binom{n}{m} \left(\frac{[w_5]_q}{[w_1 w_2 w_3 w_4]_q} \right)^{n-m} [w_2 w_3 w_4 i + w_1 w_3 w_4 j + w_1 w_2 w_4 k + w_1 w_2 w_3 s]_{q^{w_5}}^{n-m} \\ &\times q^{m(w_5 w_4 w_2 w_3 i + w_5 w_4 w_1 w_3 j + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 s)} \\ &\int_{\mathbb{Z}_p} w^{w_1 w_2 w_3 w_4 y} [y + w_5 x]_{q^{w_1 w_2 w_3 w_4}}^m \, d\mu_{-1}(y) \\ &= \sum_{m=0}^{n} \binom{n}{m} \left(\frac{[w_5]_q}{[w_1 w_2 w_3 w_4]_q} \right)^{n-m} [w_2 w_3 w_4 i + w_1 w_3 w_4 j + w_1 w_2 w_4 k + w_1 w_2 w_3 s]_{q^{w_5}}^{n-m} \\ &\times q^{m(w_5 w_4 w_2 w_3 i + w_5 w_4 w_1 w_3 j + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 s)} \mathcal{E}_{m,q^{w_1 w_2 w_3 w_4}, w^{w_1 w_2 w_3 w_4}} (w_5 x). \end{aligned}$$

In view of the Eq. (12), we acquire

$$[w_1w_2w_3w_4]_q^n \sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1}(-1)^{i+j+k+s} w^{w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s} \tag{13}$$

$$\times \int_{\mathbb{Z}_p} w^{w_1w_2w_3w_4y}\left[y+w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}s\right]_{q^{w_1w_2w_3w_4}}^n d\mu_{-1}(y)$$

$$= \sum_{m=0}^{n}\binom{n}{m}[w_1w_2w_3w_4]_q^m [w_5]_q^{n-m}\mathcal{E}_{m,q^{w_1w_2w_3w_4},w^{w_1w_2w_3w_4}}(w_5x)\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}$$

$$\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1}(-1)^{i+j+k+s} w^{w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s}$$

$$\times q^{m(w_5w_4w_2w_3i+w_5w_4w_1w_3j+w_5w_4w_1w_2k+w_5w_3w_1w_2s)}$$

$$\times [w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k+w_1w_2w_3s]_{q^{w_5}}^{n-m}$$

$$= \sum_{m=0}^{n}\binom{n}{m}[w_1w_2w_3w_4]_q^m [w_5]_q^{n-m}\mathcal{E}_{m,q^{w_1w_2w_3w_4},w^{w_1w_2w_3w_4}}(w_5x)\, U_{n,q^{w_5},w^{w_5}}(w_1,w_2,w_3,w_4 \mid m),$$

where

$$U_{n,q,w}(w_1,w_2,w_3,w_4 \mid m) \tag{14}$$

$$= \sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{s=0}^{w_4-1}(-1)^{i+j+k+s} w^{w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k+w_1w_2w_3s}$$

$$\times q^{m(w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k+w_1w_2w_3s)}[w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k+w_1w_2w_3s]_q^{n-m}.$$

As a result of the Eq. (13), we conclude the following theorem.

Theorem 2.3. *Let $w \in T_p$ and w_i are odd natural numbers where $i \in \{1,2,3,4,5\}$. For $n \geq 0$, the following*

$$\sum_{m=0}^{n}\binom{n}{m}\left[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}\right]_q^m \left[w_{\sigma(5)}\right]_q^{n-m}$$
$$\times \mathcal{E}_{m,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}},w^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}}}\left(w_{\sigma(5)}x\right)$$

$$U_{n,q^{w_{\sigma(5)}},w^{w_{\sigma(5)}}}(w_{\sigma(1)},w_{\sigma(2)},w_{\sigma(3)},w_{\sigma(4)} \mid m)$$

holds true for some $\sigma \in S_5$.

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