Journal of Contemporary Applied Mathematics
V. 2, No 2, 2012, December
ISSN 2222-5498

Integral Mean Estimates for the Polar Derivative of a Polynomials

N. A. Rather*, Suhail Gulzar

Abstract. Let $P(z)$ be a polynomial of degree n having all zeros in $|z| \leq k$ where $k \leq 1$, then it was proved by Dewan *et al* [6] that for every real or complex number α with $|\alpha| \geq k$ and each $r \geq 0$

$$n(|\alpha| - k)\left\{\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} \left|1 + ke^{i\theta}\right|^r d\theta\right\}^{\frac{1}{r}} \underset{|z|=1}{Max}|D_\alpha P(z)|.$$

In this paper, we shall present a generalization of above result and also extend it to the class of polynomials $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, having all its zeros in $|z| \leq k$ where $k \leq 1$ and thereby obtain certain generalizations of above and many other known results.

Key Words and Phrases: Polynomials; Polar derivatives; Integral mean estimates. Bernstein's inequality.

2000 Mathematics Subject Classifications: 30A10, 30C10, 30E10, 30C15

1. Introduction and statement of results

Let $P(z)$ be a polynomial of degree n. It was shown by Turán [12] that if $P(z)$ has all its zeros in $|z| \leq 1$, then

$$n\underset{|z|=1}{Max}|P(z)| \leq 2\underset{|z|=1}{Max}\left|P'(z)\right|. \tag{1}$$

Inequality (1) is best possible with equality holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$. The above inequality (1) of Turán [12] was generalized by Malik [10], who proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then

$$\underset{|z|=1}{Max}\left|P'(z)\right| \geq \frac{n}{1+k}\underset{|z|=1}{Max}|P(z)|. \tag{2}$$

where as for $k \geq 1$, Govil [7] showed that

$$\underset{|z|=1}{Max}\left|P'(z)\right| \geq \frac{n}{1+k^n}\underset{|z|=1}{Max}|P(z)|, \tag{3}$$

*Corresponding author.

Both the above inequalities (2) and (3) are best possible, with equality in (2) holding for $P(z) = (z+k)^n$, where $k \geq 1$. While in (3) the equality holds for the polynomial $P(z) = \alpha z^n + \beta k^n$ where $|\alpha| = |\beta|$.

As a refinement of (2), Aziz and Shah [4] proved if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then

$$\underset{|z|=1}{Max}\left|P'(z)\right| \geq \frac{n}{1+k}\left\{\underset{|z|=1}{Max}\left|P(z)\right| + \frac{1}{k^{n-1}}\underset{|z|=1}{Min}|P(z)|\right\}. \quad (4)$$

Let $D_\alpha P(z)$ denotes the polar derivative of the polynomial $P(z)$ of degree n with respect to the point α, then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\underset{\alpha\to\infty}{Lim}\left[\frac{D_\alpha P(z)}{\alpha}\right] = P'(z).$$

Aziz and Rather [2] extends (2) to polar derivatives of a polynomial and proved that if all the zeros of $P(z)$ lie in $|z| \leq k$ where $k \leq 1$ then for every real or complex number α with $|\alpha| \geq k$,

$$\underset{|z|=1}{Max}\left|D_\alpha P(z)\right| \geq n\left(\frac{|\alpha|-k}{1+k}\right)\underset{|z|=1}{Max}\left|P(z)\right|. \quad (5)$$

For the class of polynomials $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, Aziz and Rather [3] proved that if α is real or complex number with $|\alpha| \geq k^\mu$ then

$$\underset{|z|=1}{Max}\left|D_\alpha P(z)\right| \geq n\left(\frac{|\alpha|-k^\mu}{1+k^\mu}\right)\underset{|z|=1}{Max}\left|P(z)\right|. \quad (6)$$

Malik [11] obtained a generalization of (1) in the sense that the left-hand side of (1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$. In fact he proved that if $P(z)$ has all its zeros in $|z| \leq 1$, then for each $q > 0$,

$$n\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi}\left|1+e^{i\theta}\right|^q d\theta\right\}^{1/q}\underset{|z|=1}{Max}|P'(z)|. \quad (7)$$

If we let q tend to infinity in (7), we get (1).

The corresponding generalization of (2) which is an extension of (7) was obtained by Aziz [1] by proving that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq$ where $k \geq 1$, then for each $q \geq 1$

$$n\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}\underset{|z|=1}{Max}|P'(z)|. \quad (8)$$

The result is best possible and equality in (5) holds for the polynomial $P(z) = \alpha z^n + \beta k^n$ where $|\alpha| = |\beta|$.

As a generalization of inequality 5, Dewan *et al* [6] obtained an L^p inequality for the polar derivative of a polynomial and proved the following:

Theorem 1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex number α with $|\alpha| \geq k$ and for each $r > 0$,*

$$n(|\alpha| - k)\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi}\left|1 + ke^{i\theta}\right|^r d\theta\right\}^{\frac{1}{r}} \underset{|z|=1}{Max}|D_\alpha P(z)|. \qquad (9)$$

In this paper, we consider the class of polynomials $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, having all its zeros in $|z| \leq k$ where $k \leq 1$ and establish some improvements and generalizations of inequalities (1),(2),(5),(8) and (9).

In this direction, we first present the following interesting results which yields (9) as a special case.

Theorem 2. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex α, β with $|\alpha| \geq k$, $|\beta| \leq 1$ and for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have*

$$n(|\alpha| - k)\left\{\int_0^{2\pi}\left|P(e^{i\theta}) + \beta\frac{m}{k^{n-1}}\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi}|1 + ke^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi}|D_\alpha P(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}} \qquad (10)$$

where $m = \underset{|z|=k}{Min}|P(z)|$.

If we take $\beta = 0$, we get the following result.

Corollary 1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex α, with $|\alpha| \geq k$ and for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have*

$$n(|\alpha| - k)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi}|1 + ke^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi}|D_\alpha P(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}}. \qquad (11)$$

Remark 1. Theorem 1 follows from (11) by letting $q \to \infty$ (so that $p \to 1$) in Corollary 1. If we divide both sides of inequality (11) by $|\alpha|$ and make $\alpha \to \infty$, we get (5).

Dividing the two sides of (10) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following result.

Corollary 2. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex β with $|\beta| \leq 1$ and for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have*

$$n\left\{\int_0^{2\pi}\left|P(e^{i\theta}) + \beta\frac{m}{k^{n-1}}\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi}|1 + ke^{i\theta}|^{pr}d\theta\right\}^{\frac{1}{pr}}\left\{\int_0^{2\pi}|P'(e^{i\theta})|^{qr}d\theta\right\}^{\frac{1}{qr}} \tag{12}$$

where $m = \underset{|z|=k}{Min}|P(z)|$.

If we let $q \to \infty$ in (12), we get the following corollary.

Corollary 3. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex β with $|\beta| \leq 1$ and for each $r > 0$, we have*

$$n\left\{\int_0^{2\pi}\left|P(e^{i\theta}) + \beta\frac{m}{k^{n-1}}\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi}|1 + ke^{i\theta}|^{r}d\theta\right\}^{\frac{1}{r}}\underset{|z|=1}{Max}|P'(z)|, \tag{13}$$

where $m = \underset{|z|=k}{Min}|P(z)|$.

Remark 2. If we let $r \to \infty$ in (13) and choosing argument of β suitably with $|\beta| = 1$, we obtain (4).

Next, we extend (9) to the class of polynomials $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu}z^{n-\nu}$, $1 \leq \mu \leq n$, having all its zeros in $|z| \leq k$, $k \leq 1$ and thereby obtain the following result.

Theorem 3. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu}z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for every real or complex α with $|\alpha| \geq k^{\mu}$ and for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have*

$$n(|\alpha| - k^{\mu})\left\{\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi}|1 + k^{\mu}e^{i\theta}|^{pr}d\theta\right\}^{\frac{1}{pr}}\left\{\int_0^{2\pi}|D_\alpha P(e^{i\theta})|^{qr}d\theta\right\}^{\frac{1}{qr}}. \tag{14}$$

Remark 3. We let $r \to \infty$ and $p \to \infty$ (so that $q \to 1$) in (14), we get inequality (6).

If we divide both sides of (14) by $|\alpha|$ and make $\alpha \to \infty$, we get the following result.

Corollary 4. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu}z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have*

$$n\left\{\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi}|1 + k^{\mu}e^{i\theta}|^{pr}d\theta\right\}^{\frac{1}{pr}}\left\{\int_0^{2\pi}|P'(e^{i\theta})|^{qr}d\theta\right\}^{\frac{1}{qr}}. \tag{15}$$

Letting $q \to \infty$ (so that $p \to 1$) in (14), we get the following result:

Corollary 5. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, where $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k$, where $k \le 1$, then for every real or complex number α with $|\alpha| \ge k^\mu$ and for each $r > 0$,*

$$n(|\alpha| - k^\mu)\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi}\left|1 + k^\mu e^{i\theta}\right|^r d\theta\right\}^{\frac{1}{r}} \underset{|z|=1}{Max}|D_\alpha P(z)|. \tag{16}$$

As a generalization of Theorem 3, we present the following result:

Theorem 4. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$ where $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k$ where $k \le 1$, then for every real or complex α with $|\alpha| \ge k^\mu$ and for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have*

$$n(|\alpha| - k^\mu)\left\{\int_0^{2\pi}|P(e^{i\theta}) + \beta m|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi}|1 + k^\mu e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}}\left\{\int_0^{2\pi}|D_\alpha P(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}} \tag{17}$$

where $m = \underset{|z|=k}{Min}|P(z)|$.

If we divide both sides by $|\alpha|$ and make $\alpha \to \infty$, we get the following result:

Corollary 6. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k$ where $k \le 1$, then for for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have*

$$n\left\{\int_0^{2\pi}|P(e^{i\theta}) + \beta m|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi}|1 + k^\mu e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}}\left\{\int_0^{2\pi}|P'(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}} \tag{18}$$

where $m = \underset{|z|=k}{Min}|P(z)|$.

Letting $q \to \infty$ (so that $p \to 1$) in (14), we get the following result:

Corollary 7. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$ where $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k$, where $k \le 1$, then for every real or complex number α with $|\alpha| \ge k^\mu$ and for each $r > 0$,*

$$n(|\alpha| - k^\mu)\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right) + \beta m\right|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi}\left|1 + k^\mu e^{i\theta}\right|^r d\theta\right\}^{\frac{1}{r}} \underset{|z|=1}{Max}|D_\alpha P(z)| \tag{19}$$

where $m = \underset{|z|=k}{Min}|P(z)|$.

2. Lemmas

For the proofs of the theorems, we need the following Lemmas:

Lemma 1. *If $P(z)$ is a polynomial of degree almost n having all its zeros in in $|z| \leq k$ $k \leq 1$ then for $|z| = 1$,*

$$|Q'(z)| + \frac{nm}{k^{n-1}} \leq k|P'(z)|, \tag{20}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$ and $m = \underset{|z|=k}{Min}|P(z)|$.

The above Lemma is due to Govil and McTume [8].

Lemma 2. *Let $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu}z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n, which does not vanish for $|z| < k$, where $k \geq 1$ then for $|z| = 1$,*

$$k^{\mu}|P'(z)| \leq |Q'(z)|, \tag{21}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.

The above Lemma is due to Chan and Malik [5]. By applying Lemma 2 to the polynomial $z^n\overline{P(1/\overline{z})}$, one can easily deduce:

Lemma 3. *Let $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu}z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n, having all its zeros in $|z| \leq k$, where $k \leq 1$ then for $|z| = 1$*

$$k^{\mu}|P'(z)| \geq |Q'(z)|, \tag{22}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.

3. Proof of Theorems

Proof. [**Proof of Theorem 2**] **Let** $Q(z) = z^n\overline{P(1/\overline{z})}$ **then** $P(z) = z^n\overline{Q(1/\overline{z})}$ **and it can be easily verified that for** $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)| \quad \text{and} \quad |P'(z)| = |nQ(z) - zQ'(z)|. \tag{23}$$

By Lemma (1)**, we have for every** β **with** $|\beta| \leq 1$ **and** $|z| = 1$,

$$\left|Q'(z) + \bar{\beta}\frac{nmz^{n-1}}{k^{n-1}}\right| \leq |Q'(z)| + \frac{nm}{k^{n-1}} \leq k|P'(z)|. \tag{24}$$

Using (23) **in** (24)**, for** $|z| = 1$ **we have**

$$\left|Q'(z) + \bar{\beta}\frac{nmz^{n-1}}{k^{n-1}}\right| \leq k|nQ(z) - zQ'(z)|. \tag{25}$$

By Lemma 3 with $\mu = 1$, for every real or complex number α with $|\alpha| \geq k$ and $|z| = 1$, we have

$$\begin{aligned} |D_\alpha P(z)| &\geq |\alpha||P'(z)| - |Q'(z)| \\ &\geq (|\alpha| - k)|P'(z)|. \end{aligned} \tag{26}$$

Since $P(z)$ has all its zeros in $|z| \leq k \leq 1$, it follows by Gauss-Lucas Theorem that all the zeros of $P'(z)$ also lie in $|z| \leq k \leq 1$. This implies that the polynomial

$$z^{n-1}\overline{P'(1/\overline{z})} \equiv nQ(z) - zQ'(z)$$

does not vanish in $|z| < 1$. Therefore, it follows from (25) that the function

$$w(z) = \frac{z\left(Q'(z) + \bar{\beta}\dfrac{nmz^{n-1}}{k^{n-1}}\right)}{k\left(nQ(z) - zQ'(z)\right)}$$

is analytic for $|z| \leq 1$ and $|w(z)| \leq 1$ for $|z| = 1$. Furthermore, $w(0) = 0$. Thus the function $1 + kw(z)$ is subordinate to the function $1 + kz$ for $|z| \leq 1$. Hence by a well known property of subordination [9], we have

$$\int_0^{2\pi} \left|1 + kw(e^{i\theta})\right|^r d\theta \leq \int_0^{2\pi} \left|1 + ke^{i\theta}\right|^r d\theta, \quad r > 0. \tag{27}$$

Now

$$1 + kw(z) = \frac{n\left(Q(z) + \bar{\beta}\dfrac{mz^n}{k^{n-1}}\right)}{nQ(z) - zQ'(z)},$$

and

$$|P'(z)| = |z^{n-1}\overline{P'(1/\overline{z})}| = |nQ(z) - zQ'(z)|, \ \ \textbf{for} \ \ |z| = 1,$$

therefore for $|z| = 1$,

$$n\left|Q(z) + \bar{\beta}\frac{mz^n}{k^{n-1}}\right| = |1 + kw(z)||nQ(z) - zQ'(z)| = |1 + kw(z)||P'(z)|.$$

equivalently,

$$n\left|z^n\overline{P(1/\overline{z})} + \bar{\beta}\frac{mz^n}{k^{n-1}}\right| = |1 + kw(z)||P'(z)|.$$

This implies

$$n\left|P(z) + \beta\frac{m}{k^{n-1}}\right| = |1 + kw(z)||P'(z)| \ \ \text{for} \ \ |z| = 1. \tag{28}$$

From (25) **and** (28)**, we deduce that for** $q > 0$,

$$n^r(|\alpha| - k)^r \int_0^{2\pi} \left|P(e^{i\theta}) + \beta\frac{m}{k^{n-1}}\right|^r d\theta \leq \int_0^{2\pi} |1 + kw(e^{i\theta})|^r |D_\alpha P(e^{i\theta})|^r d\theta.$$

This gives with the help of Hölder's inequality and using (27)**, for** $p > 1,\ q > 1$ **with** $p^{-1} + q^{-1} = 1$,

$$n^r(|\alpha| - k)^r \int_0^{2\pi} \left|P(e^{i\theta}) + \beta\frac{m}{k^{n-1}}\right|^r d\theta \leq \left(\int_0^{2\pi} |1 + ke^{i\theta}|^{pr} d\theta\right)^{1/p} \left(\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta\right)^{1/q},$$

equivalently,

$$n(|\alpha| - k^\mu)\left\{\int_0^{2\pi} \left|P(e^{i\theta}) + \beta\frac{m}{k^{n-1}}\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} |1 + ke^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}}$$

which proves the desired result.

Proof. [**Proof of Theorem 3**] **Since** $P(z)$ **has all its zeros in** $|z| \leq k$**, therefore, by using Lemma 3 we have for** $|z| = 1$,

$$|Q'(z)| \leq k^\mu |nQ(z) - zQ'(z)|. \tag{29}$$

Now for every real or complex number α **with** $|\alpha| \geq k^\mu$**, we have**

$$\begin{aligned} |D_\alpha P(z)| &= |nP(z) + (\alpha - z)P'(z)| \\ &\geq |\alpha||P'(z)| - |nP(z) - zP'(z)|, \end{aligned}$$

by using (23) **and Lemma 3, for** $|z| = 1$**, we get**

$$\begin{aligned} |D_\alpha P(z)| &\geq |\alpha||P'(z)| - |Q'(z)| \\ &\geq (|\alpha| - k^\mu)|P'(z)|. \end{aligned} \tag{30}$$

Since $P(z)$ **has all its zeros in** $|z| \leq k \leq 1$**, it follows by Gauss-Lucas Theorem that all the zeros of** $P'(z)$ **also lie in** $|z| \leq k \leq 1$**. This implies that the polynomial**

$$z^{n-1}\overline{P'(1/\overline{z})} \equiv nQ(z) - zQ'(z)$$

does not vanish in $|z| < 1$**. Therefore, it follows from** (29) **that the function**

$$w(z) = \frac{zQ'(z)}{k^\mu\left(nQ(z) - zQ'(z)\right)}$$

is analytic for $|z| \leq 1$ **and** $|w(z)| \leq 1$ **for** $|z| = 1$. **Furthermore,** $w(0) = 0$. **Thus the function** $1 + k^{\mu}w(z)$ **is subordinate to the function** $1 + k^{\mu}z$ **for** $|z| \leq 1$. **Hence by a well known property of subordination [9], we have**

$$\int_0^{2\pi} \left|1 + k^{\mu}w(e^{i\theta})\right|^r d\theta \leq \int_0^{2\pi} \left|1 + k^{\mu}e^{i\theta}\right|^r d\theta, \quad r > 0. \tag{31}$$

Now

$$1 + k^{\mu}w(z) = \frac{nQ(z)}{nQ(z) - zQ'(z)},$$

and

$$|P'(z)| = |z^{n-1}\overline{P'(1/\overline{z})}| = |nQ(z) - zQ'(z)|, \;\; \textbf{for} \;\; |z| = 1,$$

therefore, for $|z| = 1$,

$$n|Q(z)| = |1 + k^{\mu}w(z)||nQ(z) - zQ'(z)| = |1 + k^{\mu}w(z)||P'(z)|. \tag{32}$$

From (30) **and** (32), **we deduce that for** $r > 0$,

$$n^r(|\alpha| - k^{\mu})^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \leq \int_0^{2\pi} |1 + k^{\mu}w(e^{i\theta})|^r |D_{\alpha}P(e^{i\theta})|^r d\theta.$$

This gives with the help of Hölder's inequality and (31), **for** $p > 1$, $q > 1$ **with** $p^{-1} + q^{-1} = 1$,

$$n^r(|\alpha| - k^{\mu})^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \leq \left(\int_0^{2\pi} |1 + k^{\mu}e^{i\theta}|^{pr} d\theta\right)^{1/p} \left(\int_0^{2\pi} |D_{\alpha}P(e^{i\theta})|^{qr} d\theta\right)^{1/q},$$

equivalently,

$$n(|\alpha| - k^{\mu}) \left\{\int_0^{2\pi} |P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} |1 + k^{\mu}e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} |D_{\alpha}P(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}}$$

which proves the desired result.

Proof. [**Proof of Theorem 4**] **Let** $m = Min_{|z|=k}|P(z)|$, **so that** $m \leq |P(z)|$ **for** $|z| = k$. **If** $P(z)$ **has a zero on** $|z| = k$ **then** $m = 0$ **and result follows from Theorem 3. Henceforth we suppose that all the zeros of** $P(z)$ **lie in** $|z| < k$. **Therefore for every** β **with** $|\beta| < 1$, **we have** $|m\beta| < |P(z)|$ **for** $|z| = k$. **Since** $P(z)$ **has all its zeros in** $|z| < k \leq 1$, **it follows by Rouche's theorem that all the zeros of** $F(z) = P(z) + \beta m$ **lie in** $|z| < k \leq 1$. **If** $G(z) = z^n\overline{F(1/\overline{z})} = Q(z) + \bar{\beta}mz^n$, **then by applying Lemma 3 to polynomial** $F(z) = P(z) + \beta m$, **we have for** $|z| = 1$,

$$|G'(z)| \leq k^{\mu}|F'(z)|.$$

This gives

$$|Q'(z)+nm\bar{\beta}z^{n-1}|\leq k^{\mu}|P'(z)|. \tag{33}$$

Using (23) **in** (33)**, for** $|z|=1$ **we have**

$$|Q'(z)+nm\bar{\beta}z^{n-1}|\leq k^{\mu}|nQ(z)-zQ'(z)| \tag{34}$$

Since $P(z)$ **has all its zeros in** $|z|<k\leq 1$**, it follows by Gauss-Lucas Theorem that all the zeros of** $P'(z)$ **also lie in** $|z|<k\leq 1$**. This implies that the polynomial**

$$z^{n-1}\overline{P'(1/\overline{z})}\equiv nQ(z)-zQ'(z)$$

does not vanish in $|z|<1$**. Therefore, it follows from** (34) **that the function**

$$w(z)=\frac{z(Q'(z)+nm\bar{\beta}z^{n-1})}{k^{\mu}\,(nQ(z)-zQ'(z))}$$

is analytic for $|z|\leq 1$ **and** $|w(z)|\leq 1$ **for** $|z|=1$**. Furthermore,** $w(0)=0$**. Thus the function** $1+k^{\mu}w(z)$ **is subordinate to the function** $1+k^{\mu}z$ **for** $|z|\leq 1$**. Hence by a well known property of subordination [9], we have**

$$\int_0^{2\pi}\left|1+k^{\mu}w(e^{i\theta})\right|^r d\theta\leq\int_0^{2\pi}\left|1+k^{\mu}e^{i\theta}\right|^r d\theta,\ \ r>0. \tag{35}$$

Now

$$1+k^{\mu}w(z)=\frac{n(Q(z)+m\bar{\beta}z^{n})}{nQ(z)-zQ'(z)},$$

and

$$|P'(z)|=|z^{n-1}\overline{P'(1/\overline{z})}|=|nQ(z)-zQ'(z)|,\ \ \textbf{for}\ \ |z|=1,$$

therefore, for $|z|=1$,

$$n|Q(z)+m\bar{\beta}z^{n}|=|1+k^{\mu}w(z)||nQ(z)-zQ'(z)|=|1+k^{\mu}w(z)||P'(z)|.$$

This implies

$$n|G(z)|=|1+k^{\mu}w(z)||nQ(z)-zQ'(z)|=|1+k^{\mu}w(z)||P'(z)|. \tag{36}$$

Since $|F(z)|=|G(z)|$ **for** $|z|=1$**, therefore, from** (36) **we get**

$$n|P(z)+\beta m|=|1+k^{\mu}w(z)||P'(z)|\ \ \text{for}\ \ |z|=1. \tag{37}$$

From (30) **and** (37)**, we deduce that for** $r>0$,

$$n^r(|\alpha|-k^{\mu})^r\int_0^{2\pi}|P(e^{i\theta})+\beta m|^r d\theta\leq\int_0^{2\pi}|1+k^{\mu}w(e^{i\theta})|^r|D_{\alpha}P(e^{i\theta})|^r d\theta.$$

This gives with the help of Hölder's inequality in conjunction with (35) **for** $p>1$, $q>1$ **with** $p^{-1}+q^{-1}=1$,

$$n^r(|\alpha|-k^\mu)^r\int_0^{2\pi}|P(e^{i\theta})+\beta m|^r d\theta \le \left(\int_0^{2\pi}|1+k^\mu e^{i\theta}|^{pr}d\theta\right)^{1/p}\left(\int_0^{2\pi}|D_\alpha P(e^{i\theta})|^{qr}d\theta\right)^{1/q},$$

equivalently,

$$n(|\alpha|-k^\mu)\left\{\int_0^{2\pi}|P(e^{i\theta})+\beta m|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi}|1+k^\mu e^{i\theta}|^{pr}d\theta\right\}^{\frac{1}{pr}}\left\{\int_0^{2\pi}|D_\alpha P(e^{i\theta})|^{qr}d\theta\right\}^{\frac{1}{qr}}$$

which proves the desired result.

References

[1] A. Aziz, Integral mean estimates polynomials with restricted zeros, *J. Approx. Theory* 55 (1988), 232-239.

[2] A. Aziz and N. A. Rather, A refinement of a theorem of Paul Turan concerning polynomials. *Math Ineq Appl.* 1, 231238 (1998).

[3] A. Aziz and N. A. Rather, Inequalities for the polar derivative of a polynomial with restricted zeros, *Mathematica Balkana,* 17 (2003), 15-28.

[4] A. Aziz and W. M. Shah, An integral mean estimate for polynomial, *Indian J. Pure and Appl. Math.,* 28 (1997) 1413-1419.

[5] T. N. Chan and M. A. Malik, On Erdös-Lax theorem, *Proc. Indian. Acad. Sci.*, 92 (1983), 191-193.

[6] K. K. Dewan et al., Some inequalities for the polar derivative of a polynomial, *South-east Asian Bull. Math.*,34 (2010), 69-77.

[7] N. K. Govil, On the derivative of a polynomial, *Proc. Amer. Math. Soc.* 41 (1973), 543-546.

[8] N. K. Govil and G. N. McTume, Some generalizations involving the polar derivative for an inequality of Paul Turán, *Acta Math. Hungar.,* 104 (2004) 115-126.

[9] E. Hille, *Analytic function theory*, Vol. II, Ginn and Company, New York, Toronto, 1962.

[10] M. A. Malik, On the derivative of a polynomial, *J. Lond. Math. Soc.*, Second Series 1 (1969), 57-60.

[11] M. A. Malik, An integral mean estimates for polynomials, *Proc. Amer. Math. Soc.*, 91 (1984), 281-284.

[12] P. Turan, Uber die Ableitung von Polynomen, *Compositio Mathematica* 7 (1939), 89-95 (German).

N. A. Rather
Department of Mathematics University of Kashmir Srinagar, Hazratbal 190006 India.
E-mail: dr.narather@gmail.com
Suhail Gulzar
Department of Mathematics University of Kashmir Srinagar, Hazratbal 190006 India.
E-mail: sgmattoo@gmail.com
Received 11 September 2012
Published 22 October 2012