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An Operator Preserving L_p Inequality Between Polynomials

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Abstract. If $P(z)$ is a polynomial of degree at most n which does not vanish in $|z| < 1$, then it was recently formulated by Shah and Liman [Integral estimates for the family of B-operators, Operators and Matrices, **5**(2011), 79 - 87] that for every $R \geq 1$, $p \geq 1$,

$$\|B[P \circ \rho](z)\|_p \leq \frac{R^n|\Lambda| + |\lambda_0|}{\|1+z\|_p} \|P(z)\|_p ,$$

where B is a B_n-operator with parameters $\lambda_0, \lambda_1, \lambda_2$ in the sense of Rahman and Schmeisser [16], $\rho(z) = Rz$ and $\Lambda = \lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3(n-1)}{8}$. Unfortunately the proof of this result is not correct. In this paper, we present a refined L_p-inequality for B_n-operators which not only provide a correct proof of the above inequality as a special case but also extend the inequality for $0 \leq p < 1$ as well.
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1. Introduction and Statements of Results

Let $\mathcal{P}_n$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree at most n and let $\mathcal{P}_n(A)$ be the set of polynomials in $\mathcal{P}_n$ having all zeros in $A \subset \mathbb{C}$. We write $U = \{z \in \mathbb{C} : |z| = 1\}$, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, $\bar{\Delta}$ its closure, $\Delta^c = \mathbb{C} \setminus \Delta$ and $\Omega = \mathbb{C} \setminus \bar{\Delta}$. For $P \in \mathcal{P}_n$, define

$$\|P(z)\|_0 = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log\left|P(e^{i\theta})\right| d\theta\right\},$$

$$\|P(z)\|_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}, \quad 0 < p < \infty,$$

$$\|P(z)\|_\infty = \max_{z \in U} |P(z)|$$

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and denote for any complex function $\rho : \mathbb{C} \to \mathbb{C}$ the composite function of P and ρ, defined by $(P \circ \rho)(z) = P(\rho(z)) \quad (z \in \mathbb{C})$, as $P \circ \rho$.

If $P \in \mathcal{P}_n$, then

$$\left\|P'(z)\right\|_p \leq n \left\|P(z)\right\|_p, \quad p \geq 1 \tag{1}$$

and

$$\left\|P(Rz)\right\|_p \leq R^n \left\|P(z)\right\|_p, \quad R > 1, \; p > 0. \tag{2}$$

Inequality (1) was found out by Zygmund [20] whereas inequality (2) is a simple consequence of a result of Hardy [9]. Arestov [2] proved that (1) remains true for $0 \leq p < 1$ as well. For $p = \infty$, the inequality (1) is due to Bernstein (for reference, see [11], [15], [17]) whereas the case $p = \infty$ of inequality (2) is a simple consequence of the maximum modulus principle (see [12], [13], [16]). Both the inequalities (1) and (2) can be sharpened if we restrict ourselves to the class of
polynomials $\mathcal{P}_n^\circ = \mathcal{P}_n(\Delta^c)$. In fact, if $P \in \mathcal{P}_n^\circ$, then inequalities (1) and (2) can be respectively replaced by

$$\left\|P'(z)\right\|_p \leq n \frac{\left\|P(z)\right\|_p}{\left\|1+z\right\|_p}, \quad 0 \leq p \leq \infty \tag{3}$$

and

$$\left\|P(Rz)\right\|_p \leq \frac{\left\|R^n z + 1\right\|_p}{\left\|1+z\right\|_p} \left\|P(z)\right\|_p, \quad R > 1, \quad p > 0. \tag{4}$$

Inequality (3) is due to De-Bruijn [7] (see also [3]) for $p \geq 1$. Rahman and Schmeisser [15] extended it for $0 \leq p < 1$ whereas the inequality (4) was proved by Boas and Rahman [6] for $p \geq 1$ and later it was extended for $0 \leq p < 1$ by Rahman and Schmeisser [15]. For $p = \infty$, the inequality (3) was conjectured by Erdös and later verified by Lax [10] whereas inequality (4) was proved by Ankeny and Rivlin [1].

Rahman [14] (see also Rahman and Schmeisser [16, p. 538]) introduced a class B_n of operators B that maps $P \in \mathcal{P}_n$ into itself. That is, the operator B carries $P \in \mathcal{P}_n$ into

$$B[P](z) = \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) \frac{P'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!}, \tag{5}$$

where λ_0, λ_1 and λ_2 are real or complex numbers such that all the zeros of

$$u(z) = \lambda_0 + n\lambda_1 z + \frac{n(n-1)}{2}\lambda_2 z^2, \tag{6}$$

lie in the half plane

$$|z| \leq |z - n/2| \tag{7}$$

and proved that if $P \in \mathcal{P}_n$, then

$$|B[P \circ \rho](z)| \leq R^n |\Lambda| \left\|P(z)\right\|_\infty \quad \text{for} \quad z \in U \tag{8}$$

and if $P \in \mathcal{P}_n^{\circ}$, then as a special case of Corollary 14.5.6 in [16, p. 539], we have

$$|B[P \circ \rho](z)| \leq \frac{1}{2}\left\{R^n|\Lambda| + |\lambda_0|\right\}\|P(z)\|_\infty \quad \text{for} \quad z \in U, \tag{9}$$

where $\rho(z) = Rz,\ R \geq 1$ and

$$\Lambda = \lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3(n-1)}{8}. \tag{10}$$

Inequality (9) also follows by combining the inequalities (5.2) and (5.3) due to Rahman [14].

As an extension of inequality (9) to L_p-norm, recently Shah and Liman [19, Theorem 1] proved:

Theorem A. *If* $P \in \mathcal{P}_n$, *then for every* $R \geq 1$ *and* $p \geq 1$,

$$\|B[P \circ \rho](z)\|_p \leq R^n|\Lambda|\,\|P(z)\|_p\,,$$

where $B \in B_n$, $\rho(z) = Rz$ *and* Λ *is defined by* (10).

While seeking the desired extension of inequality (9) to L_p-norm, they [19, Theorem 2] have made an incomplete attempt by claiming to have proved:

Theorem B. *If* $P \in \mathcal{P}_n^{\circ}$, *then for each* $p \geq 1$, $R \geq 1$,

$$\|B[P \circ \rho](z)\|_p \leq \frac{R^n|\Lambda| + |\lambda_0|}{\|1+z\|_p}\|P(z)\|_p\,, \tag{11}$$

where $B \in B_n$, $\rho(z) = Rz$ *and* Λ *is defined by* (10).

Further, it has been claimed in [19] to have proved the inequality (11) for self-inversive polynomials as well.

Unfortunately the proof of inequality (11) and other related results including the key lemma [19, Lemma 4] given by Shah and Liman is not correct. The reason being that the authors in [19] deduce:
line 10 from line 7 on page 84, line 19 on page 85 from Lemma 3 [19] and line 16 from line 14 on page 86, by using the fact that if $P^\star(z) = z^n\overline{P(1/\overline{z})}$, then for $\rho(z) = Rz$, $R \geq 1$ and $z \in U$,

$$|B[P^\star \circ \rho](z)| = |B[(P^\star \circ \rho)^\star](z)|,$$

which is not true, in general, for every $R \geq 1$ and $z \in U$. To see this, let

$$P(z) = a_n z^n + \cdots + a_k z^k + \cdots + a_1 z + a_0$$

be an arbitrary polynomial of degree n, then

$$P^\star(z) = z^n\overline{P(1/\overline{z})} = \bar{a_0}z^n + \bar{a_1}z^{n-1} + \cdots + \bar{a_k}z^{n-k} + \cdots + \bar{a_n}.$$

Now with $\Lambda_1 = \lambda_1 n/2$ and $\Lambda_2 = \lambda_2 n^2/8$, we have

$$B[P^\star \circ \rho](z) = \sum_{k=0}^{n} (\lambda_0 + \Lambda_1(n-k) + \Lambda_2(n-k)(n-k-1))\, \bar{a_k} z^{n-k} R^{n-k},$$

and in particular for $z \in U$, we get

$$B[P^\star \circ \rho](z) = R^n z^n \sum_{k=0}^{n} (\lambda_0 + \Lambda_1(n-k) + \Lambda_2(n-k)(n-k-1))\, \overline{a_k \left(\frac{z}{R}\right)^k},$$

whence

$$|B[P^\star \circ \rho](z)| = R^n \left| \sum_{k=0}^{n} \overline{(\lambda_0 + \Lambda_1(n-k) + \Lambda_2(n-k)(n-k-1))} a_k \left(\frac{z}{R}\right)^k \right|.$$

But

$$|B[(P^\star \circ \rho)^\star](z)| = R^n \left| \sum_{k=0}^{n} (\lambda_0 + \Lambda_1 k + \Lambda_2 k(k-1))\, a_k \left(\frac{z}{R}\right)^k \right|,$$

so the asserted identity does not hold in general for every $R \geq 1$ and $z \in U$ as e.g. the immediate counterexample of $P(z) = z^n$ demonstrates in view of $P^\star(z) = 1$, $|B[P^\star \circ \rho](z)| = |\lambda_0|$ and

$$|B[(P^\star \circ \rho)^\star](z)| = |\lambda_0 + \lambda_1(n^2/2) + \lambda_2 n^3(n-1)/8| \quad (z \in U).$$

The main aim of this paper is to establish a sharp L_p extension of inequality (9) for $0 \leq p < \infty$ which includes the correct proof of inequality (11) as a special case. In this direction, we present the following compact generalization inequalities (1), (2) and (9), which is also a refinement of inequality (11) and extends it for $0 \leq p < 1$ as well.

Theorem 1. *If $P \in \mathcal{P}_n^\circ$ and $m = \min_{|z|=1} |P(z)|$, then for every real or a complex number δ, with $|\delta \leq 1, R > 1$ and $0 \leq p < \infty$,*

$$\left\| B[P \circ \rho](z) + \delta \frac{(R^n|\Lambda| - |\lambda_0|)m}{2} \right\|_p \leq \frac{\|R^n \Lambda z + \lambda_0\|_p}{\|1+z\|_p} \|P(z)\|_p, \tag{12}$$

where $B \in B_n$, $\rho(z) = Rz$ and Λ is defined by (10). *The result is sharp, as is shown by the extremal polynomial $P(z) = az^n + b$, $|a| = |b| \neq 0$.*

Remark 1. If we choose $\lambda_0 = 0 = \lambda_2$ in (12), we get for every real or a complex number δ, with $|\delta \leq 1, R > 1$ and $0 \leq p < \infty$,

$$\left\| P'(Rz) + \delta \frac{nR^{n-1}m}{2} \right\|_p \leq \frac{nR^{n-1}}{\|1+z\|_p} \|P(z)\|_p,$$

which, in particular, yields inequality (3). Next if we take $\lambda_1 = 0 = \lambda_2$ and $\delta = 0$ in (12), we get inequality (4).

By the triangle inequality, the following result immediately follows from Theorem 1.

Corollary 1. *If $P \in \mathcal{P}_n^\circ$, and $m = \min_{|z|=1} |P(z)|$, then for every real or a complex number δ, with $|\delta \leq 1, R > 1$ and $0 \leq p < \infty$,*

$$\left\| B[P \circ \rho](z) + \delta \frac{(R^n|\Lambda| - |\lambda_0|)m}{2} \right\|_p \leq \frac{R^n|\Lambda| + |\lambda_0|}{\|1+z\|_p} \|P(z)\|_p \,, \tag{13}$$

where $B \in B_n$, $\rho(z) = Rz$ and Λ is defined by (10).

Letting $p \to \infty$ in (12) and choosing the argument of δ suitably, we get the following refinement of inequality (9).

Corollary 2. *If $P \in \mathcal{P}_n^\circ$, and $m = \min_{|z|=1} |P(z)|$, then for every real or a complex number δ, with $|\delta \leq 1, R > 1$,*

$$|B[P \circ \rho](z)| \leq \frac{1}{2} \left\{ (R^n|\Lambda| + |\lambda_0|) \|P(z)\|_\infty - \delta (R^n|\Lambda| - |\lambda_0|) m \right\} \quad \text{for} \quad z \in U, \tag{14}$$

For $\delta = 0$, inequality (14) reduces to inequality (9)

For $\delta = 0$, Theorem 1 reduces to the following result:

Corollary 3. *If $P \in \mathcal{P}_n^\circ$, then for every $R > 1$ and $0 \leq p < \infty$,*

$$\|B[P \circ \rho](z)\|_p \leq \frac{\|R^n \Lambda z + \lambda_0\|_p}{\|1+z\|_p} \|P(z)\|_p \,, \tag{15}$$

Remark 2. Corollary 2 not only validates Theorem B for $p \geq 1$ but also extends it for $0 \leq p < 1$ as well.

2. Lemmas

For the proofs of these theorems, we need the following lemmas. The first lemma follows from Corollary 18.3 of [11, p. 65].

Lemma 1. *If* $B \in B_n$ and $P \in \mathcal{P}_n(\bar{\Delta})$, *then* $B[P] \in \mathcal{P}_n(\bar{\Delta})$.

Lemma 2. *If $P \in \mathcal{P}_n(\Delta^c)$, then*

$$|B[P \circ \rho](z)| \leq |B[P^\star \circ \rho](z)| \quad \text{for} \quad z \in U, \tag{16}$$

where $B \in B_n$ and $\rho(z) = Rz$ with $R > 1$ arbitrary.

Lemma 2 is due to Rahman [14].

Lemma 3. *If* $P \in \mathcal{P}_n(\bar{\Delta})$, *then*

$$|B[P \circ \rho](z)| \geq R^n |\Lambda| m \quad \text{for} \quad z \in U, \tag{17}$$

where $B \in B_n$, $m = \underset{|z|=1}{Min}|P(z)|$ *and* $\rho(z) = Rz$ with $R > 1$ arbitrary.

Proof of Lemma 3. By hypothesis all the zeros of $P(z)$ lie in $\bar{\Delta}$ and

$$m|z|^n \leq |P(z)| \quad \text{for} \quad z \in U.$$

we first show that the polynomial $G(z) = P(z) - \alpha m z^n$ has all its zeros in $\bar{\Delta}$ for every real or complex number α with $\alpha \in \Delta$. This is obvious if $m = 0$, that if $P(z)$ has a zero on U. Henceforth, we assume $P(z)$ has all its zeros in Δ, then $m > 0$ and it follows by Rouche's theorem that the polynomial $G(z) = P(z) - \alpha m z^n$ has all its zeros in Δ for every complex number $\alpha \in \Delta$ and hence all the zeros of $G(Rz) = P(Rz) - \alpha m R^n z^n$ lie in Δ. Applying Lemma 1 to $G(Rz)$, we conclude that $B[G \circ \rho](z) = B[P \circ \rho](z) - \alpha m R^n \Lambda z^n$ has all its zeros in Δ. This implies

$$|B[P \circ \rho](z)| \geq R^n |\Lambda| |z|^n m \quad \text{for} \quad z \in \Delta^c, \tag{18}$$

which proves Lemma 3.

Lemma 4. *If* $P \in \mathcal{P}_n(\Delta^c)$, *and* $m = \underset{|z|=1}{Min}|P(z)|$ *then*

$$|B[P \circ \rho](z)| \leq |B[P^\star \circ \rho](z)| - (R^n \Lambda - |\lambda_0|)m \quad \text{for} \quad z \in U, \tag{19}$$

where $B \in B_n$ *and* $\rho(z) = Rz$ with $R > 1$ arbitrary.

Proof of Lemma 4. By hypothesis all the zeros of $P(z)$ lie in Δ^c and

$$m \leq |P(z)| \quad \text{for} \quad z \in U. \tag{20}$$

We show $F(z) = P(z) + \lambda m$ does not vanish in Δ for $\lambda \in \bar{\Delta}$. This is obvious if $m = 0$, that if $P(z)$ has a zero on U. So, we assume $P(z)$ has all its zeros in Ω, then $m > 0$ then by maximum modulus principle, it follows from (20),

$$m < |P(z)| \quad \text{for} \quad z \in \Delta. \tag{21}$$

Now if $F(z) = P(z) + \lambda m = 0$ for some $z = z_0$ with $z_0 \in \Delta$, then

$$P(z_0) + \lambda m = 0.$$

This implies

$$P(z_0)| = |\lambda| m \leq m \quad \text{for} \quad z_0 \in \Delta,$$

which is clearly a contradiction to (21). Thus the polynomial $F(z)$ does not vanish for $z \in \Delta$ for every $\lambda \in \Delta$. Applying Lemma 2 to $F(z)$, we get

$$|B[F \circ \rho](z)| \leq |B[F^\star \circ \rho](z)| \quad \text{for} \quad z \in U,$$

Replacing $F(z)$ by $P(z)+\lambda m$, we get

$$|B[P\circ\rho](z)+\lambda m\lambda_0| \le \left|B[P^\star\circ\rho](z)+\bar{\lambda}mR^n\Lambda z^n\right| \quad \text{for} \quad z\in U. \tag{22}$$

Choosing argument of λ, with $|\lambda|=1$ in the right hand side of (22) such that

$$|B[P^\star\circ\rho](z)+\bar{\lambda}mR^n\Lambda z^n| = |B[P^\star\circ\rho](z)| - mR^n|\Lambda||z|^n \quad \text{for} \quad z\in U, \tag{23}$$

which is possible by Lemma 3, we get

$$|B[P\circ\rho](z)| - m|\lambda_0| \le |B[P^\star\circ\rho](z)| - mR^n|\Lambda||z|^n \quad \text{for} \quad z\in U,$$

Equivalently, for $z\in U$, we have

$$|B[P\circ\rho](z)| \le |B[P^\star\circ\rho](z)| - (R^n|\Lambda| - |\lambda_0|)m.$$

This completes the proof of Lemma 4.

Next we describe a result of Arestov [2].
For $\boldsymbol{\gamma}=(\gamma_0,\gamma_1,\ldots,\gamma_n)\in\mathbb{C}^{n+1}$ and $P(z)=\sum_{j=0}^{n}a_jz^j\in\mathcal{P}_n$, we define

$$C_{\boldsymbol{\gamma}}P(z)=\sum_{j=0}^{n}\gamma_ja_jz^j.$$

The operator $C_{\boldsymbol{\gamma}}$ is said to be admissible if it preserves one of the following properties:
(i) $P\in\mathcal{P}_n(\bar{\Delta})$,
(ii) $P\in\mathcal{P}_n(\Delta^c)$. The result of Arestov may now be stated as follows.

Lemma 5. [2, Th.2] *Let $\phi(x)=\psi(\log x)$ where ψ is a convex nondecreasing function on $\mathbb{R}$. Then for all $P\in\mathcal{P}_n$ and each admissible operator $C_{\boldsymbol{\gamma}}$,*

$$\int_0^{2\pi}\phi\left(|C_{\boldsymbol{\gamma}}P(e^{i\theta})|\right)d\theta \le \int_0^{2\pi}\phi\left(c(\boldsymbol{\gamma})|P(e^{i\theta})|\right)d\theta$$

where $c(\boldsymbol{\gamma})=\max(|\gamma_0|,|\gamma_n|)$.

In particular Lemma 5 applies with $\phi: x\to x^p$ for every $p\in(0,\infty)$ and with $\phi: x\to \log x$ as well. Therefore, we have for $0\le p<\infty$,

$$\left\{\int_0^{2\pi}\left|C_{\boldsymbol{\gamma}}P(e^{i\theta})\right|^p d\theta\right\}^{1/p} \le c(\boldsymbol{\gamma})\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}. \tag{24}$$

From Lemma 5, we deduce the following result.

Lemma 6. *If $P\in P_n(\Delta^c)$, then for every $p>0$, $R>1$* and *α* real, $0\le\alpha<2\pi$,

$$\int_0^{2\pi}\left|B[P\circ\rho](e^{i\theta})\,e^{i\alpha}+B[P^\star\circ\rho]^\star(e^{i\theta})\right|^p d\theta \le \left|R^n\Lambda e^{i\alpha}+\bar{\lambda}_0\right|^p\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta, \tag{25}$$

where $B \in B_n$, $\rho(z) = Rz$, $B[P^\star \circ \rho]^\star(z) = (B[P^\star \circ \rho](z))^\star$ *and* Λ *is defined by* (10).

Proof of Lemma 6. Since $P \in \mathcal{P}_n(\Delta^c)$ and $P^\star(z) = z^n\overline{P(1/\overline{z})}$, by Lemma 2 , we have

$$|B[P \circ \rho](z)| \le |B[P^\star \circ \rho](z)| \quad \text{for} \quad z \in U. \tag{26}$$

Also, since $P^\star(Rz) = R^n z^n \overline{P(1/R\overline{z})}$,

$$\begin{aligned} B[P^\star \circ \rho](z) = {} & \lambda_0 R^n z^n \overline{P(1/R\overline{z})} \\ & + \lambda_1 \left(\frac{nz}{2}\right)\left(nR^n z^{n-1}\overline{P(1/R\overline{z})} - R^{n-1}z^{n-2}\overline{P'(1/R\overline{z})}\right) \\ & + \frac{\lambda_2}{2!}\left(\frac{nz}{2}\right)^2 \Big(n(n-1)R^n z^{n-2}\overline{P(1/R\overline{z})} \\ & - 2(n-1)R^{n-1}z^{n-3}\overline{P'(1/R\overline{z})} + R^{n-2}z^{n-4}\overline{P''(1/R\overline{z})}\Big) \end{aligned}$$

and therefore,

$$\begin{aligned} B[P^\star \circ \rho]^\star(z) &= (B[P^\star \circ \rho](z))^\star \\ &= \left(\bar{\lambda}_0 + \bar{\lambda}_1\frac{n^2}{2} + \bar{\lambda}_2\frac{n^3(n-1)}{8}\right) R^n P(z/R) \\ &- \left(\bar{\lambda}_1\frac{n}{2} + \bar{\lambda}_2\frac{n^2(n-1)}{4}\right) R^{n-1}zP'(z/R) + \bar{\lambda}_2\frac{n^2}{8}R^{n-2}z^2P''(z/R). \end{aligned}$$

Also,

$$|B[P^\star \circ \rho](z)| = |B[P^\star \circ \rho]^\star(z)| \quad \text{for} \;\; z \in U.$$

Using this in (26), we get

$$|B[P \circ \rho](z)| \le |B[P^\star \circ \rho]^\star(z)| \quad \text{for} \quad z \in U, \;\; R > 1.$$

Since $(P^\star \circ \rho) \in \mathcal{P}_n(\Delta)$, by Lemma 1, $B[P^\star \circ \rho] \in \mathcal{P}_n(\Delta)$, therefore, $B[P^\star \circ \rho]^\star \in \mathcal{P}_n(\Omega)$. Hence by the maximum modulus principle,

$$|B[P \circ \rho](z)| < |B[P^\star \circ \rho]^\star(z)| \quad \text{for} \quad z \in \Delta. \tag{27}$$

A direct application of Rouche's theorem shows that with $P(z) = a_n z^n + \cdots + a_0$,

$$\begin{aligned} C_\gamma P(z) &= B[P \circ \rho](z)\, e^{i\alpha} + B[P^\star \circ \rho]^\star(z), \\ &= \left\{R^n\left(\lambda_0 + \lambda_1\frac{n^2}{2} + \lambda_2\frac{n^3(n-1)}{8}\right)e^{i\alpha} + \bar{\lambda}_0\right\} a_n z^n \\ &+ \cdots + \left\{R^n\left(\bar{\lambda}_0 + \bar{\lambda}_1\frac{n^2}{2} + \bar{\lambda}_2\frac{n^3(n-1)}{8}\right) + e^{i\alpha}\lambda_0\right\} a_0, \end{aligned}$$

has all its zeros in Δ^c, that is, $C_\gamma P \in \mathcal{P}_n(\Delta^c)$. Therefore, C_γ is an admissible operator. Applying (24) of Lemma 5, the desired result follows immediately for each $p > 0$.

We also need the following lemma due to A.Aziz and N.A.Rather [5].

Lemma 7. *If A, B, C are non negative real numbers such that $B + C \leq A$ then for every real α,*

$$|(A - C)e^{i\alpha} + (B + C)| \leq |Ac^{i\alpha} + B|. \tag{28}$$

3. Proofs of the Theorems

Proof of Theorem 1. By hypothesis $P \in \mathcal{P}_n^\circ$, $\rho(z) = Rz$ and $R > 1$, therefore, by Lemma 4,we have

$$|B[P \circ \rho](z)| \leq |B[P^\star \circ \rho](z)| - (R^n|\Lambda| - |\lambda_0|)m \quad \text{for} \quad z \in U, \tag{29}$$

Since $B[P^\star \circ \rho]^\star(z)$ is the conjugate of $B[P^\star \circ \rho](z)$ and

$$|B[P^\star \circ \rho]^\star(z)| = |B[P^\star \circ \rho](z)|, \quad z \in U.$$

Therefore (29) can be written as

$$|B[P \circ \rho](z)| + \frac{(R^n|\Lambda| - |\lambda_0|)m}{2} \leq |B[P^\star \circ \rho]^\star(z)| - \frac{(R^n|\Lambda| - |\lambda_0|)m}{2} \quad \text{for} \quad z \in U, \tag{30}$$

Taking

$$A = |B[P^\star \circ \rho]^\star(z)|\,, \;\; B = |B[P \circ \rho](z)|$$

and

$$C = \frac{(R^n|\Lambda| - |\lambda_0|)m}{2}$$

in Lemma 7 and noting by (30) that

$$B + C \leq A - C \leq A,$$

we get for every real α,

$$\left|\left\{\,|B[P^\star \circ \rho]^\star(z)| - \frac{(R^n|\Lambda| - |\lambda_0|)m}{2}\right\}e^{i\alpha} + \left\{\,|B[P \circ \rho](z)| + \frac{(R^n|\Lambda| - |\lambda_0|)m}{2}\right\}\right|$$
$$\leq |\,|B[P^\star \circ \rho]^\star(z)|\, e^{i\alpha} + |B[P \circ \rho](z)|\,|.$$

This implies for each $p > 0$,

$$\int_0^{2\pi} \left|\left\{\,\left|B[P^\star \circ \rho]^\star(e^{i\theta})\right| - \frac{(R^n|\Lambda| - |\lambda_0|)m}{2}\right\}e^{i\alpha} + \left\{\,\left|B[P \circ \rho](e^{i\theta})\right| + \frac{(R^n|\Lambda| - |\lambda_0|)m}{2}\right\}\right| d\theta$$
$$\leq \int_0^{2\pi} \left|\,\left|B[P^\star \circ \rho]^\star(e^{i\theta})\right| e^{i\alpha} + \left|B[P \circ \rho](e^{i\theta})\right|\,\right| d\theta. \tag{31}$$

Integrating both sides of (31) with respect to α from 0 to 2π, we get with the help of Lemma 6 for each $p > 0$,

$$\int_0^{2\pi}\int_0^{2\pi}\left|\left\{\left|B[P^\star\circ\rho]^\star(e^{i\theta})\right|-\frac{(R^n|\Lambda|-|\lambda_0|)m}{2}\right\}+e^{i\alpha}\left\{\left|B[P\circ\rho](e^{i\theta})\right|+\frac{(R^n|\Lambda|-|\lambda_0|)m}{2}\right\}\right|^p d\theta d\alpha$$

$$\begin{aligned}
&\leq \int_0^{2\pi}\int_0^{2\pi}\left|\left|B[P^\star\circ\rho]^\star(e^{i\theta})\right|e^{i\alpha}+\left|B[P\circ\rho](e^{i\theta})\right|\right|^p d\theta d\alpha\\
&\leq \int_0^{2\pi}\left\{\int_0^{2\pi}\left|\left|B[P^\star\circ\rho]^\star(e^{i\theta})\right|e^{i\alpha}+\left|B[P\circ\rho](e^{i\theta})\right|\right|^p d\alpha\right\}d\theta\\
&\leq \int_0^{2\pi}\left\{\int_0^{2\pi}\left|B[P^\star\circ\rho]^\star(e^{i\theta})e^{i\alpha}+B[P\circ\rho](e^{i\theta})\right|^p d\alpha\right\}d\theta\\
&\leq \int_0^{2\pi}\left\{\int_0^{2\pi}\left|B[P^\star\circ\rho]^\star(e^{i\theta})e^{i\alpha}+B[P\circ\rho](e^{i\theta})\right|^p d\theta\right\}d\alpha\\
&\leq \int_0^{2\pi}\left|R^n\Lambda e^{i\alpha}+\bar{\lambda_0}\right|^p d\alpha\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta. \qquad (32)
\end{aligned}$$

Now it can be easily verified that for every real number α and $r \geq 1$,

$$\left|r+e^{i\alpha}\right| \geq \left|1+e^{i\alpha}\right|.$$

This implies for each $p > 0$,

$$\int_0^{2\pi}\left|r+e^{i\alpha}\right|^p d\alpha \geq \int_0^{2\pi}\left|1+e^{i\alpha}\right|^p d\alpha. \qquad (33)$$

If $\left|B[P\circ\rho](e^{i\theta})\right|+\dfrac{(R^n|\Lambda|-|\lambda_0|)m}{2} \neq 0$, we take

$$r=\frac{\left|B[P^\star\circ\rho]^\star(e^{i\theta})\right|-\dfrac{(R^n|\Lambda|-|\lambda_0|)m}{2}}{|B[P\circ\rho](e^{i\theta})|+\dfrac{(R^n|\Lambda|-|\lambda_0|)m}{2}},$$

then by (19), $r \geq 1$ and we get with the help of (33),

$$\int_0^{2\pi}\left|\left\{\left|B[P^\star\circ\rho]^\star(e^{i\theta})\right|-\frac{(R^n|\Lambda|-|\lambda_0|)m}{2}\right\}+e^{i\alpha}\left\{\left|B[P\circ\rho](e^{i\theta})\right|+\frac{(R^n|\Lambda|-|\lambda_0|)m}{2}\right\}\right|^p d\alpha$$

$$= \left|\left|B[P\circ\rho](e^{i\theta})\right| + \frac{(R^n|\Lambda|-|\lambda_0|)m}{2}\right|^p \int_0^{2\pi}\left|e^{i\alpha} + \frac{\left|B[P^\star\circ\rho]^\star(e^{i\theta})\right| - \frac{(R^n|\Lambda|-|\lambda_0|)m}{2}}{|B[P\circ\rho](e^{i\theta})| + \frac{(R^n|\Lambda|-|\lambda_0|)m}{2}}\right|^p d\alpha$$

$$= \left|\left|B[P\circ\rho](e^{i\theta})\right| + \frac{(R^n|\Lambda|-|\lambda_0|)m}{2}\right|^p \int_0^{2\pi}\left|e^{i\alpha} + \left|\frac{\left|B[P^\star\circ\rho]^\star(e^{i\theta})\right| - \frac{(R^n|\Lambda|-|\lambda_0|)m}{2}}{|B[P\circ\rho](e^{i\theta})| + \frac{(R^n|\Lambda|-|\lambda_0|)m}{2}}\right|\right|^p d\alpha$$

$$\geq \left|\left|B[P\circ\rho](e^{i\theta})\right| + \frac{(R^n|\Lambda|-|\lambda_0|)m}{2}\right|^p \int_0^{2\pi}|1+e^{i\alpha}|^p d\alpha \tag{34}$$

This inequality is trivially true for $\left|B[P\circ\rho](e^{i\theta})\right| + \frac{(R^n|\Lambda|-|\lambda_0|)m}{2} = 0$. Using this in (32), we conclude that for each $p > 0$,

$$\int_0^{2\pi}\left|\left|B[P\circ\rho](e^{i\theta})\right| + \frac{(R^n|\Lambda|-|\lambda_0|)m}{2}\right|^p d\theta \int_0^{2\pi}|1+e^{i\alpha}|^p d\alpha$$
$$\leq \int_0^{2\pi}\left|R^n\Lambda e^{i\alpha}+\lambda_0\right|^p d\alpha \int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta,$$

This gives for every real or complex number δ with$|\delta \leq 1$ and α real

$$\int_0^{2\pi}\left|B[P\circ\rho](e^{i\theta}) + \delta\frac{(R^n|\Lambda|-|\lambda_0|)m}{2}\right|^p d\theta \int_0^{2\pi}|1+e^{i\alpha}|^p d\alpha$$
$$\leq \int_0^{2\pi}\left|R^n\Lambda e^{i\alpha}+\lambda_0\right|^p d\alpha \int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta,$$

from which Theorem 1 follows for $p > 0$. To establish this result for $p = 0$, we simply let $p \to 0+$.

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