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An Optimality Condition of Pontryagin Maximum Principle Type in a Control Problem for 2d Linear Fractional-Order Difference Equations

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Abstract. In this paper, an optimal control problem described by a system of linear nonhomogeneous two-dimensional fractional-order difference equations with a control function in the boundary condition is studied. The representation of the solution to the boundary-value problem for the system of linear nonhomogeneous two-dimensional fractional-order difference equations is obtained. The necessary and sufficient optimality conditions in the form of a discrete maximum principle are proved. In the case of a nonlinear but convex functional, a sufficient optimality condition is established.

Key Words and Phrases: admissible control, optimal control, Pontryagin Maximum principle, control problem, fractional operator, difference equation

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1. Introduction

In recent decades, the theory of fractional calculus has developed intensively and has become an important tool for the mathematical modeling of processes with memory and hereditary effects [1, 2, 3, 4, 5]. Unlike the classical operators of differentiation and integration of integer order, fractional-order operators have a nonlocal character and allow one to take into account the memory effect of the system. This property makes them a more adequate tool for modeling in such fields as viscoelasticity, diffusion processes, biophysics, economic dynamics, and control theory.

The discrete analog of fractional calculus-the theory of fractional difference equations-has formed as an independent direction comparatively recently [4, 5, 6, 7]. It combines methods of the classical theory of difference equations and fractional analysis.

Two-dimensional difference equations arising in the modeling of processes depending on two independent discrete variables, such as time and spatial coordinates, are of particular interest.

These models are used in the study of systems with distributed parameters, in the theory of net processes, and in numerical methods. A significant contribution to the development of the qualitative theory of multiparameter systems was made by I. V. Gayshun [8].

The introduction of fractional order into two-dimensional discrete systems allows for the incorporation of spatio-temporal memory and expands the possibilities for modeling complex dynamic phenomena [9, 10, 11]. The most popular models of two-dimensional linear systems are those proposed by Rosser, Fornasini-Marchesini, and Kurek.

In optimal control theory, a particular interest lies in the study of fractional-order dynamic systems. In many real processes, the current state of the system depends not only on the present control but also on previous states. This requires a generalization of the classical models of optimal control. For fractional differential systems, the necessary optimality conditions based on Pontryagin's maximum principle and variational methods have been widely investigated [9, 10, 13, 14, 15, 16, 21].

In this paper, the system of linear nonhomogeneous two-dimensional fractional-order difference equations is considered. In the case of a linear functional, the necessary and sufficient optimality conditions are obtained. For this, we use representation formulas for the solutions of linear nonhomogeneous one-dimensional and two-dimensional difference equations. In the case of a nonlinear and convex functional, a sufficient optimality condition is obtained.

The relevance of this study is supported by both the development of the theory of fractional discrete operators and the need to construct adequate mathematical models of processes with memory in applied problems of analysis and control.

2. Basic concepts

First, we present some concepts and definitions [17, 18, 19, 20]. Let N be the set of natural numbers with zero. For $a \in Z$ we introduce the following notations: ${N_a}^+ = \{a, a+1, a+2, ...,\}$, $\sigma(t) = t+1$, $\rho(t) = t-1$.

Definition 1. *The extended binomial coefficient* $\begin{pmatrix} a \\ n \end{pmatrix}$ *is defined as follows:*

$$\begin{pmatrix} a \\ n \end{pmatrix} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n+1)}, & n > 0, \\ 1, & n = 0, \\ 0, & n < 0. \end{cases}$$

Let for any $x, y \in R$, $x^{(y)} = \frac{\Gamma(x+1)}{\Gamma(x+1-y)}$, *where* $\Gamma-$*gamma function, for which* $\Gamma(x+1) = x\Gamma(x)$ *holds.*

Definition 2. *Fractional sum of* $a-$*order is defined as follows:*

$$\Delta^{-\alpha}u(n) = \sum_{j=0}^{n-1} \begin{pmatrix} j+\alpha-1 \\ j \end{pmatrix} u(n-j) = \sum_{j=1}^{n} \begin{pmatrix} n-j+\alpha-1 \\ n-j \end{pmatrix} u(j),$$

and the fractional operator of α *-order is defined as:*

$$\Delta^{-\alpha}u(n) = \sum_{j=0}^{n-1} \begin{pmatrix} j-\alpha \\ j \end{pmatrix} \Delta u(n-j) =$$

$$= \sum_{j=1}^{n} \begin{pmatrix} n-j-\alpha-1 \\ n-j \end{pmatrix} u(j) - \begin{pmatrix} n-\alpha-1 \\ n-1 \end{pmatrix} u(0).$$

At the same time, the fractional sum and fractional operator of order α *- can also be defined as follows.*

Let a *be an arbitrary real number and* $b = k + a$, *where* $k \in N$, $k \geq 2$. $T = \{a, a+1, ..., b\}$. *We denote by* T *the set of functions defined on* T.

Definition 3. *Let* $f \in T$. *The left and the right fractional sums of order* $\alpha > 0$, *are respectively defined as follows:*

$$a\Delta_t{}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t+\alpha-\sigma(s))^{(\alpha-1)} f(s),$$

$$_t\Delta_b^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b} (s+\alpha-\sigma(t))^{(\alpha-1)} f(s).$$

Definition 4. *Let* $0 < \alpha \leq 1$ *and* $\mu = 1 - a$. *Then for a function* $f \in T$, *the left and right fractional difference operators of order* α*- are defined as follows:*

$$a\Delta_t{}^{\alpha} f(t) = \Delta\left(a\Delta_t{}^{-\mu} f(t)\right),$$

$$t\Delta_b{}^{\alpha} f(t) = -\Delta\left(t\Delta_b{}^{-\mu} f(t)\right).$$

Let us introduce some properties of the fractional sum and fractional difference:

1.$\Delta^{\alpha}\Delta^{\beta} f(t) = \Delta^{\alpha+\beta} f(t)$,

2.$\Delta^{-\alpha}\Delta^{\alpha} f(t) = f(t) - f(0)$,

3.$\Delta^{\alpha}\Delta^{-\alpha} f(t) = f(t)$,

4.$\Delta^{\alpha} f(0) = 0$ and $\Delta^{a} f(1) = f(1) - f(0) = \Delta f(1)$.

3. The problem statement

Let us consider the problem of minimizing the linear functional

$$S(u) = c^{'}a(x_1) + d^{'}z(t_1, x_1) \tag{1}$$

subject to the constraints

$$u(x) \in U \subset R^r, x \in X = \{x_0, x_0 + 1, ..., x_1\}. \tag{2}$$

$$\Delta^{\alpha} z(t+1, x+1) =$$
$$= A(t,x)\, z(t,x) + B(t,x)\, z(t+1,x) + C(t,x)\, z(t,x+1) + D(t,x), \tag{3}$$

$$z(t_0, x) = a(x), x \in X \cup x_1, \tag{4}$$

$$z(t, x_0) = b(t), t \in T \cup t_1,\ T = \{t_0, t_0 + 1, ..., t_1 - 1\} \tag{5}$$

$$a(x_0) = b(t_0) = a_0,$$

$$\Delta^{\beta} a(x+1) = K(x)\, a(x) + g(x, u(x)), x \in X, \tag{6}$$

$$a(x_0) = a_0. \tag{7}$$

Here $A(t,x), B(t,x), C(t,x), K(x)-$ are given $(n \times n)-$ discrete matrix functions, and $D(t,x)-$ is a given $n-$dimensional discrete vector-function. The function $b(t)-$ is a given discrete vector-function, a_0, t_0, t_1, x_0, x_1- are given. The function $g(x,u)-$ is a given continuous $n-$ dimensional vector-function. The control $u(x)-$is an $r-$ dimensional vector of control actions taking values in a given nonempty and bounded set U (admissible control), while c and d- are given constant $n-$ dimensional vectors. The operators Δ^a and Δ^{β}, with $0 < \alpha, \beta < 1-$ are fractional operators of order a and β, respectively. [18, 19].

An admissible control that yields the minimum value of the functional (1) under boundary conditions (2)-(6) is called an optimal control.

4. Increment formula of the quality functional

Let $u(x), \overline{u}(x) = u(x) + \Delta u(x)-$ be two admissible controls. We denote the corresponding solutions of the system (2)–(7) by $(a(x),\ z(t,x))$, $(\overline{a}(x) = a(x) + \Delta a(x), \overline{z}(t,x) = z(t,x) + \Delta z(t,x))$.

Then the increment of the functional (1) takes the form:

$$\Delta S(u) = c^{'}\Delta a(x_1) + d^{'}\Delta z(t_1, x_1). \tag{8}$$

In this case, $\Delta a(x)$, $\Delta z(t,x)$ are the solutions of the following problems

$$\Delta^{\beta}\Delta a(x) = K(x)\Delta a(x) + \Delta_{\overline{u}(x)} g[x], \tag{9}$$

$$\Delta a(x_0) = 0, \tag{10}$$

$$\Delta^{\alpha}\Delta z(t+1,x+1) = A(t,x)\Delta z(t,x) + B(t,x)\Delta z(t+1,x) + \\ +C(t,x)\Delta z(t,x+1), \tag{11}$$

$$\Delta z(t_0,x) = \Delta a(x), x \in X \cup x_1, \\ \Delta z(t,x_0) = 0, t \in T \cup t_1 \tag{12}$$

respectively.

Here, by definition:

$$\Delta_{\overline{u}(x)} g[x] = g(x, \overline{u}(x)) - g(x, u(x)).$$

As we can see, the equations (9) and (11) are the systems of linear nonhomogeneous difference equations relative $\Delta a(x)$ and $\Delta z(t,x)$ respectively.

Auxiliary facts. First, we obtain representation formula for the solution of a system of nonlinear fractional-order difference equations [18, 19, 20].

Let's consider a system of linear nonhomogeneous difference equations of the following form:

$$\Delta^{a} y(t+1) = A(t)y(t) + g(t), \tag{13}$$

$$t \in T = \{t_0, t_0+1, t_0+2, ..., t_1-1\},$$

$$y(t_0) = y_0. \tag{14}$$

Here $A(t)-$ is a given $n\times n-$ dimensional discrete matrix function, $g(t)-$ is a given $n-$ dimensional discrete vector-column, $y = (y_1, y_2, ..., y_n)^{'} -$ is a given $n-$ dimensional discrete vector-column, $y_0 = (y_{1_0}, y_{2_0}, ..., y_{n_0})^{'} -$ is a given constant vector-column, t_0, t_1, are natural numbers $t_1 - t_0$ are natural numbers, Δ^{α} $(0 < \alpha < 1)$ is fractional α order.

Let us find the representation formula for the solution of problem (13)–(14).

Assume that $y(\tau)$ is a solution of the equation (13) satisfying initial condition (14).

It means that:

$$\Delta^{\alpha} y(\tau+1) = A(\tau)y(\tau) + g(\tau), \tag{15}$$

Let $0 < a < 1$, and $\mu = 1 - a$.

Applying the fractional sum $\Delta^{-\alpha}$ to both sides of the equation (15) we obtain

$$\Delta^{-\alpha}(\Delta^{\alpha} y(\tau+1)) = \Delta^{-a}\left(A(\tau)\, y(\tau) + g(\tau)\right). \tag{16}$$

Using the definition of the fractional difference [17], we transform the expression $\Delta^{-\alpha}\left(\Delta^{\alpha} y(\tau+1)\right)$.

We obtain

$$\Delta^{-a}\left(\Delta^{a} y(\tau+1)\right) = y(\tau+1) - y(0).$$

Let denote the correctly unknown $n \times n$ dimensional matrix function by $F(t,\tau)$. Multiplying both sides of equation (16) by the matrix function $F(t,\tau)$, and then summing both sides with respect to t from t_0 to $(t-1)$, we obtain:

$$\sum_{t=t_0}^{t-1} F(t,\tau)\left(y(\tau+1) - y(0)\right) =$$

$$= \sum_{t=t_0}^{t-1} F(t,\tau)\left(\frac{1}{\Gamma(a)} \sum_{s=t_0}^{\tau} (\tau+\alpha-\sigma(s))^{(\alpha-1)} A(s) y(s)\right) +$$

$$+ \sum_{\tau=t_0}^{t-1} F(t,t)\left(\frac{1}{\Gamma(a)} \sum_{s=t_0}^{\tau} (\tau+\alpha-\sigma(s))^{(\alpha-1)} g(s)\right). \tag{17}$$

Using the substitution $\tau + 1 = \beta$ we obtain:

$$\sum_{\tau=t_0}^{t-1} F(t,\tau) y(\tau+1) = \sum_{\beta=t_0+1}^{t} F(t,\beta-1) y(\beta) =$$

$$= F(t,t-1)\, y(t) - F(t,t_0-1)\, y(t_0) + \sum_{\tau=t_0}^{t-1} F(t,\tau-1) y(\tau). \tag{18}$$

Taking into account formula (18) in formula (17) we obtain

$$F(t,t-1)\, y(t) - F(t,t_0-1)\, y(t_0) - \sum_{\tau=t_0}^{t-1} F(t,\tau) y(0) + \sum_{\tau=t_0}^{t-1} F(t,\tau-1) y(\tau) =$$

$$= \sum_{\tau=t_0}^{t-1} F(t,\tau)\left(\frac{1}{\Gamma(a)} \sum_{s=t_0}^{\tau} (t+\alpha-\sigma(s))^{(\alpha-1)} A(s)\, y(s)\right) +$$

$$+ \sum_{\tau=t_0}^{t-1} F(t,\tau)\left(\frac{1}{\Gamma(a)} \sum_{s=t_0}^{\tau} (\tau+\alpha-\sigma(s))^{(\alpha-1)} g(s)\right) =$$

$$= \sum_{\tau=t_0}^{t-1} \frac{1}{G(a)} \left(\sum_{s=\tau}^{t-1} (\tau+\alpha-\sigma(s))^{(a-1)} F(\tau,s) A(s) \right) y(t) +$$

$$+ \sum_{\tau=t_0}^{t-1} F(t,\tau) \left(\frac{1}{\Gamma(a)} \sum_{s=t_0}^{\tau} (\tau+\alpha-\sigma(s))^{(\alpha-1)} g(s) \right). \tag{19}$$

Let matrix function $F(t,\tau)$ be the solution of the following problem:

$$F(t,\tau-1) = \frac{1}{\Gamma(a)} \sum_{s=\tau}^{t-1} (\tau+\alpha-\sigma(s))^{(\alpha-1)} F(\tau,s) A(s),$$

$$F(t,t-1) = E.$$

Here E is the$-$ $n \times n-$ identity matrix.

From relation (19), it follows that:

$$y(t) = \left[F(t,t_0-1) + \sum_{\tau=t_0}^{t-1} F(t,\tau) \right] y(t_0) +$$

$$+ \sum_{\tau=t_0}^{t-1} F(t,\tau) \left(\frac{1}{\Gamma(a)} \sum_{s=t_0}^{\tau} (\tau+\alpha-\sigma(s))^{(\alpha-1)} g(s) \right). \tag{20}$$

Therefore, the vector function, defined by formula (20) is the solution of the system (13)-(14).

From formula (20), the solution of equation (13) with the initial condition (14) can be written in the form

$$\Delta a(x) = \sum_{\tau=x_0}^{x-1} F(x,\tau) \left(\frac{1}{\Gamma(a)} \sum_{s=x_0}^{\tau} (\tau+\alpha-\sigma(s))^{(\alpha-1)} \Delta_{\overline{u}} g[s] \right) =$$

$$= \sum_{\tau=x_0}^{x-1} \left(\frac{1}{\Gamma(a)} \sum_{s=\tau}^{x-1} F(s,\tau) (\tau+\alpha-\sigma(s))^{(\alpha-1)} \right) \Delta_{\overline{u}} g[\tau]. \tag{21}$$

Here we use fractional sums by parts [18].

Let us introduce the following notation:

$$\Phi(x,\tau) = \left(\frac{1}{\Gamma(a)} \sum_{s=\tau}^{x-1} F(s,\tau) (\tau+\alpha-\sigma(s))^{(\alpha-1)} \right) \tag{22}$$

$$\Delta a(x) = \sum_{j=x_0}^{x-1} \Phi(x,j) \Delta_{\overline{u}} g[j] \tag{23}$$

In [19] the following theorem was proved.

Theorem 1. *The solution $z(t,x)$ of the system of linear two-dimensional fractional order difference equations* (3)–(5) *allows the following representation:*

$$z(t,x) = a(x_0) + \sum_{j=t_0}^{t-1} R_a(t-1, x-1; j, x_0-1)\, C(j, x_0-1)\, b(j) +$$

$$+ \sum_{s=x_0}^{x-1} R_a(t_0-1, x-1; t_0-1, s)\, B(t_0-1, s)\, a(s) +$$

$$+ \sum_{j=t_0}^{t-1} \sum_{s=x_0}^{x-1} R_a(t-1, j; s)\, D(j,s). \tag{24}$$

Here

$$R_a(t-1, x-1; j, s) = \begin{pmatrix} t-j+\alpha-1 \\ t-j \end{pmatrix} \begin{pmatrix} x-s+\alpha-1 \\ x-s \end{pmatrix}.$$

and $R_a(t-1, x-1; j, s)$ is a solution of the following problem:

$$R_a(t-1, x-1; j, s)\, A(j,s) = -R_\alpha(t-1, x-1; j-1, s)\, B(j-1, s) -$$

$$-R_a(t-1, x-1; j, s-1)\, C(j, s-1),\, j = t-1, ..., t_0, s = x-1, ..., x_0,$$

$$R_a(t, x; t-1, x-1) = E.$$

Then it is clear that the solution of the equation (11) with initial condition (12) has the form:

$$\Delta z(t,x) = \sum_{s=x_0}^{x-1} R_a(t-1, x-1; ; t_0-1, s)\, B(t_0-1, s)\, \Delta a(s). \tag{25}$$

By substituting (23) into (25), we obtain:

$$\Delta z(t,x) = \sum_{s=x_0}^{x-1} R_a(t-1, x-1; t_0-1, s)\, B(t_0-1, s)\, \Delta a(s).$$

Using the property of fractional summation by parts [18] we can rewrite the obtained relation in the following form:

$$\Delta z(t,x) = \sum_{s=x_0}^{x-1} R_a(t-1, x-1; t_0-1, s)\, B(t_0-1, s)\, \Delta a(s) =$$

$$= \sum_{s=x_0}^{x-1} R_a(t-1, x-1; t_0-1, s) B(t_0-1, s) \sum_{j=x_0}^{s-1} \Phi(s, j) \Delta_{\overline{u}} g[j].$$

Let us introduce the notation:

$$Q_1(t, x, s) = \sum_{j=s+1}^{x-1} R_a(t-1, x-1; t_0-1, j) B(t_0-1, j) \Phi(j, s). \tag{26}$$

Then

$$\Delta z(t, x) = \sum_{s=x_0}^{x-1} Q_1(t, x, s) \Delta_{\overline{u}} g[s]. \tag{27}$$

Taking into account (23), (27) we can write the increment formula (8) in the form:

$$\Delta S(u) = c^{'} \Delta a(x_1) + d^{'} \Delta z(t_1, x_1) =$$

$$= \sum_{x=x_0}^{x_1-1} c^{'} \Phi(x_1, x) \Delta_{\overline{u}} g[x] + \sum_{x=x_0}^{x_1-1} d^{'} Q_1(t_1, x_1, x) \Delta_{\overline{u}} g[x] =$$

$$= \sum_{x=x_0}^{x_1-1} \left[c^{'} \Phi(x_1, x) + d^{'} Q_1(t_1, x_1, x) \right] \Delta_{\overline{u}} g[x]. \tag{28}$$

Assuming

$$p(x) = -\left[c^{'} \Phi(x_1, x) + d^{'} Q_1(t_1, x_1, x) \right], \tag{29}$$

$$M^{'}(x, u, p) = p^{'} g(x, u),$$

$$\Delta_{\overline{u}} M[x] = p^{'} \Delta_{\overline{u}} g[x],$$

We rewrite the relation (28) in the form:

$$\Delta S(u) = -\sum_{x=x_0}^{x_1-1} \Delta_{\overline{u}} M[x]. \tag{30}$$

From (29) it follows that:

$$p(x-1) = -\left[c^{'} \Phi(x_1, x-1) + d^{'} Q_1(t_1, x_1, x-1) \right] \tag{31}$$

Further, from (26) we obtain

$$Q_1\left(t_1, x_1, x-1\right) =$$

$$= R_a\left(t_1-1, x_1-1; t_0-1, x\right) B\left(t_0-1, x\right) + Q_1\left(t_1, x_1, x\right). \tag{32}$$

Taking into account (22), and (32) in (31) we have

$$p\left(x-1\right) = p\left(x\right) + \psi\left(t_0-1, x\right) B^{'}\left(t_0-1, x\right) \ , \tag{33}$$

where, by definition

$$\psi\left(t, x\right) = -R^{'}{}_a\left(t_1, x_1; t, x\right) d. \tag{34}$$

From (33) it follows that

$$p\left(x_1-1\right) = -c. \tag{35}$$

Next, using (34) we prove that $\psi\left(t, x\right)$, defined by formula

$$\psi\left(t, x\right) = -R'{}_a\left(t_1, x_1; t, x\right) d,$$

is the solution of the boundary-value problem

$$\psi\left(t-1, x-1\right) = A^{'}\left(t, x\right) \psi\left(t, x\right) +$$

$$-B^{'}\left(t, x\right) \psi\left(t-1, x\right) - C^{'}\left(t, x\right) \psi\left(t, x-1\right),$$

$$\psi\left(t_1-1, x-1\right) = B^{'}\left(t_1-1, x\right) \psi\left(t_1-1, x\right),$$

$$\psi\left(t-1, x_1-1\right) = C^{'}\left(t, x_1-1\right) \psi\left(t, x_1-1\right),$$

$$\psi\left(t_1-1, x_1-1\right) = -d.$$

Optimality condition. With the help of representation (30) the following theorem is proved.

Theorem 2. *For the optimality of an admissible control* $u\left(x\right)$, $x \in X$, *in problem* (2)-(7) *it is necessary and sufficient that the following relation holds:*

$$\max_{v \in V} M\left(\xi, v, p(\xi)\right) = M\left(\xi, u(\xi), p(\xi)\right) \tag{36}$$

for all $v \in U$, $\xi \in X$.

Proof. (necessity) Let $u\left(x\right)$ be an optimal control. Then from the increment formula (30) it follows that for any admissible control $\overline{u}\left(x\right)=u\left(x\right)+\Delta u\left(x\right)$ we have:

$$\sum_{x=x_0}^{x_1-1}\Delta_{\overline{u}}M\left[x\right]\leq 0. \tag{37}$$

Using the arbitrariness of $\overline{u}(x)$, we define it as follows:

$$\overline{u}\left(x\right)=\begin{cases} v, & x=\xi\in X,\\ u\left(x\right), & x\neq\xi\in X,\end{cases}$$

where $\xi\in X-$ is an arbitrary point, $v\in U$ is an arbitrary vector.

Then inequality (37) takes the form: $\Delta_v M\left[\xi\right]\leq 0$.

From this, due to the arbitrariness of $v\in U$ and $\xi\in X$ the maximum condition (36) follows. Now let us prove the sufficiency of the maximum condition (36).

Suppose that for an admissible control $u\left(x\right)$ Pontryagin maximum condition (36) holds. From this it follows that for any $\xi\in X$, $\overline{u}\left(\xi\right)=v\in U$

$$\Delta_{\overline{u}}M\left[\xi\right]\leq 0.$$

Since $\xi\in X$ is arbitrary, it follows that

$$\sum_{\xi=x_0}^{x_1-1}\Delta_{\overline{u}}M\left[\xi\right]\leq 0.$$

From this inequality and relation (30) it follows that for any admissible control $\overline{u}\left(x\right)$

$$\Delta S\left(u\right)=S\left(\overline{u}\right)-S\left(u\right)\geq 0,\quad i.e.$$

$$S\left(\overline{u}\right)\geq S\left(u\right).$$

From the last relation it follows that the control $u(x)$ is optimal. This completes the proof of the sufficiency of the Pontryagin discrete maximum condition.

The case of a convex quality criterion. Let us consider a more general case and see it the necessary conditions are obtained when the quality criterion is convex. Consider the problem of minimizing the functional

$$S\left(u\right)=\varphi_1\left(a\left(x_1\right)\right)+\varphi_2\left(z\left(t_1,x_1\right)\right) \tag{38}$$

subject to the constraints (2)–(7).

Here $\varphi_1\left(a\right)$, $\varphi_2\left(z\right)-$ are given continuously differentiable and convex scalar functions.

In the case of problem (2)-(7), using Taylor's formula, we can express the increment of the functional (36) corresponding to the admissible controls $u(x)$, $\overline{u}(x) = u(x) + \Delta u(x)$ in the form:

$$\Delta S(u) = S(\overline{u}) - S(u) = \frac{\partial\ f'_1(a(x_1))}{\partial a}\Delta a(x_1) +$$

$$+\frac{\partial f'_2(z(t_1,x_1))}{\partial z}\Delta z(t_1,x_1) + o_1(\|\Delta a(x_1)\|) + o_2(\|z(t_1,x_1)\|), \qquad (39)$$

where the values $o_i(\cdot)$, $i = 1,2$ are defined via the expansions:

$$\varphi_1(\overline{a}(x_1)) - \varphi_1(a(x_1)) = \frac{\partial\varphi_1(a(x_1))}{\partial a}\Delta a(x_1) + o_1(\|\Delta a(x_1)\|),$$

$$\varphi_2(\overline{z}(t_1,x_1)) - \varphi_2(z(t_1,x_1)) = \frac{\partial\varphi_2(z(t_1,x_1))}{\partial z}\Delta z(t_1,x_1) + o_2(\|z(t_1,x_1)\|).$$

Using the representations (23), (27) we can transform the increment formula (39) into the form:

$$\Delta S(u) = \sum_{x=x_0}^{x_1-1}\frac{\partial\ f_1(a(x_1))}{\partial a}\Phi(x_1,x)\,\Delta_{\overline{u}}g[x] +$$

$$+\sum_{x=x_0}^{x_1-1}\frac{\partial\varphi_2(z(t_1,x_1))}{\partial z}Q_1(t_1,x_1,x)\,\Delta_{\overline{u}}g[x] +$$

$$+o_1(\|\Delta a(x_1)\|) + o_2(\|z(t_1,x_1)\|) =$$

$$= \sum_{x=x_0}^{x_1-1}\left[\frac{\partial\varphi_1(a(x_1))}{\partial a}\,\Phi(x,x_1) + \frac{\partial\varphi_2(z(t_1,x_1))}{\partial z}Q_1(t_1,x_1,x)\right]\Delta_{\overline{u}}g[x] +$$

$$+o_1(\|\Delta a(x_1)\|) + o_2(\|z(t_1,x_1)\|). \qquad (40)$$

Assume that:

$$p'(x) = -\left[\frac{\partial\varphi_1(a(x_1))}{\partial a}\,\Phi(x,x_1) + \frac{\partial\varphi_2(z(t_1,x_1))}{\partial z}Q_1(t_1,x_1,x)\right], \qquad (41)$$

$$M(x,u,p) = p'g(x,u), \quad \psi(t,x) = -R'_\alpha(t_1,x_1;t,x)\,\frac{\partial\varphi_2(z(t_1,x_1))}{\partial z} \qquad (42)$$

we rewrite the increment formula (40) in the form

$$\Delta S(u) = -\sum_{x=x_0}^{x_1-1}\Delta_{\overline{u}}M[x] + o_1(\|\Delta a(x_1)\|) + o_2(\|z(t_1,x_1)\|), \qquad (43)$$

From (41) and (42) we obtain that $p(x)$ and $\psi(t,x)$ are respectively solutions of the following problems:

$$p(x-1) = p(x) + \psi(t_0-1,x)\, B^{'}(t_0-1,x\ ),$$

$$p(x_1-1) = -\frac{\partial\ \varphi_1(a(x_1))}{\partial a},$$

$$\psi(t-1,x-1) = A^{'}(t,x)\,\psi(t,x) +$$

$$-B^{'}(t,x)\,\psi(t-1,x) - C^{'}(t,x)\,\psi(t,x-1),$$

$$\psi(t_1-1,x-1) = B^{'}(t_1-1,x)\,\psi(t_1-1,x),$$

$$\psi(t-1,x_1-1) = C^{'}(t,x_1-1)\,\psi(t,x_1-1),$$

$$\psi(t_1-1,x_1-1) = -\frac{\partial\varphi_2(z(t_1,x_1))}{\partial z}.$$

Because of the convexity of the functions $f_1(a), f_2(z)$, it is clear that

$$o_1\|\Delta a(x_1)\| \geq 0,\ \ o_2\|z(t_1,x_1)\| \geq 0.$$

Therefore, from (43), the inequality

$$\Delta S(u) \geq -\sum_{x=x_0}^{x_1-1} \Delta_{\overline{u}} M[x]. \tag{44}$$

follows (44).

Theorem 3. *If the set*

$$g(x,U) = \{\beta : \beta = g(x,v), v \in U\} \tag{45}$$

is convex, then for the optimality of an admissible control $u(x)$ *in problem* (2)-(7), (38) *it is sufficient that the inequality*

$$\sum_{x=x_0}^{x_1-1} \Delta_{\overline{u}} M[x] \leq 0 \tag{46}$$

holds for all $v(x) \in U$*.*

Proof. Necessity. Let $u(x)$ be an optimal control in the problem (2)-(7), (38). Its special increment, due to the convexity of the set (45), can be defined by

$$\Delta u_\varepsilon(x) = v(x,\varepsilon) - u(x) \tag{47}$$

where $e \in [0,1]$ and $v(x,\varepsilon) \in U, x \in X$, is an admissible control such that

$$\Delta_{v(x,\varepsilon)} g(x,u(x)) = \varepsilon \Delta_{v(x)} g(x,u(x)).$$

Here $v(x) \in U, x \in X$ is an arbitrary admissible control.

Let us denote a special increment of the state $(a(x), z(t,x))$, corresponding to the increment (47) of the control $u(x)$ by $(\Delta a_\varepsilon(x), \Delta z_\varepsilon(t,x))$

From representations (23), (28) it follows that

$$\Delta a_\varepsilon(x) = \varepsilon \sum_{s=x_0}^{x-1} \Phi(x,s) \Delta_{\overline{u}} g[s] \; \Delta z_\varepsilon(t,x) = \varepsilon \sum_{s=x_0}^{x-1} Q_1(t,x,s) \Delta_{\overline{u}} g[s] \tag{48}$$

Relations (48) imply that $\|\Delta a_\varepsilon(x)\|$, $\|\Delta z_\varepsilon(t,x)\|$ have the order ε.

Therefore from (43), we obtain

$$-\varepsilon \sum_{s=x_0}^{x-1} \Delta_v M[x] + o(\varepsilon) \geq 0.$$

From the last relation, because of the arbitrariness of $\varepsilon \in [0,1]$ inequality (46) follows.

Sufficiency. Let the admissible control $u(x)$ satisfy the relation (46). Then, from inequality (46), it follows that for any admissible control $v(x)$

$$S(v) - S(u) \geq 0.$$

The last relation implies the optimality of the admissible control $u(x)$. This completes the proof of Theorem 3.

Thus, we have established, in two cases, the necessary and sufficient optimality conditions in the form of the Pontryagin discrete maximum principle.

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