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Inverse Problem of Finding Right-Hand Sides for a System of Reactıon-Diffusion Type Parabolic Equations in Varıable Boundary Domain

A. Akhundov, A. Habibova, A. Hasanova

Abstract. The well-posedness of the inverse problem of finding time-dependent unknown functions on the right-hand side of the system of reaction-diffusion type parabolic equations in a domain with time-dependent boundary is investigated in this work. Existence, uniqueness and stability theorems for the generalized solution of the inverse problem with additional condition in integral form are proved.

Key Words and Phrases: inverse problem, hyperbolic heat equation, uniqueness,existence.

2020 Mathematics Subject Classifications: 35R30, 35L70, 65M06

1. Introduction

System of reaction-diffusion type parabolic equations occurs in modelling of many processes, such as spread of the epidemic, "predator-prey" interactions in biology, burning and autowave processes in chemistry, diffusion in perturbed media in physics.

Different aspects of direct problems for this kind of systems have been considered in [1-6,20].

Inverse problems for the systems of parabolic equations have been treated in [7,8] (linear equations plus additional local condition), while those for the systems of reaction-diffusion type parabolic equations have been considered in [9-14] (for cylindrical domains) and [15] (for non-cylindrical domains).

Let $D = (0, \gamma(t)) \times (0, T], T = const > 0$,

$1^0 . \gamma(t) \in C^{1+\alpha}[0, T],\ \gamma'(t) > 0,\ t \in [0, T],\ 0 < \gamma(0) \le \gamma(t) \le \gamma(T) < +\infty.$

2^0. $g_k(x,t,p) \in C^{\alpha,\alpha/2}(\mathrm{N} = \overline{D} \times R^m), R^m$ be an m-dimensional Euclidean space, $p = (p_1, ..., p_m)$, $g_* \leq |g_k(x,t,p)| \leq g*$, $(x,t,p) \in N$, $g_*, g^* = const > 0$, the function $g_k(x,t,p)$ satisfy Lipschitz condition with respect to the variable p in every bounded subset of the space R^m:

$$\left|g_k(x,t,p^1) - g_k(x,t,p^2)\right| \leq const \sum_{i=1}^{m} \left|p_i^1 - p_i^2\right|, p^1,\ p^2 \in R^m;$$

$$3^0 . \varphi_k(x) \in C^{2+\alpha}[0, \gamma(0)],\ \ \varphi_k^{'}(\gamma(0)) = \varphi_k^{''}(\gamma(0)) = 0;$$

$$4^0 . \psi_{1k}(t), \psi_{2k}(t) \in C^{1+\alpha}[0,T],\ \ \psi_{1k}(0) = \varphi_k(0),\ \ \psi_{2k}(0) = \varphi_k(\gamma(0));$$

$$5^0 . h_k(t) \in C^{1+\alpha}[0,T],\ \ \int_e^d \varphi_k(x)dx = h_k(0), 0 < e < d < \gamma(0),\ \ e, d = const$$

$$|h_k^{'}(t)| \leq h^*,\ \ h^* = const, t \in [0,T],\ \ k = \overline{1,m},$$

The function spaces $C^l(\cdot), C^{l+\alpha}(\cdot)$, $C^{l,l/2}(\cdot)$, $C^{l+\alpha,(l+\alpha)/2}$, $\alpha \in (0,1)$, $l = 0,1,2$, and the norms in them are defined in a usual way, for example, as in [16, pp.12-30]:

$$\|u\|_D^{(l)} = \sum_{i=1}^{m} \sum_{j=0}^{l} \sup_{\overline{D}} \left|\frac{\partial^j u_i}{\partial x^j}\right|,\ \ \|q\|_T^{(l)} = \sum_{i=1}^{m} \sum_{j=0}^{l} \sup_{[0,T]} \left|\frac{d^j q(t)}{dt^j}\right|,\ \ u_{kt}(x,t) = \frac{\partial u_k(x,t)}{\partial t},$$

$$u_{kx}(x,t) = \frac{\partial u_k(x,t)}{\partial x}, u_{kxx}(x,t) = \frac{\partial^2 u_k(x,t)}{\partial x^2},\ \ q_k^{'}(t) = \frac{dq_k(t)}{dt}.$$

Let's consider the following problem of finding the pairs of unknown functions $\left\{f_k(t), u_k(x,t), k = \overline{1,m}\right\}$:

$$u_{kt} - u_{kxx} = f_k(t) g_k(x,t,u),\ \ (x,t) \in D, \tag{1}$$

$$u_k(x,0) = \varphi_k(x),\ \ x \in [0, \gamma(0)], \tag{2}$$

$$u_k(0,t) = \psi_{1k}(t),\ \ u_k(\gamma(t),t) = \psi_{2k}(t),\ \ t \in [0,T], \tag{3}$$

$$\int_e^d u_k(x,t)dx = h_k(t),\ \ t \in [0,T], \tag{4}$$

where $\gamma(t)$, $g_k(x,t,u)$, $\varphi_k(x)$, $\psi_{1k}(t)$, $\psi_{2k}(t)$, $h_k(t)$ are the given functions.

Definition 1. *The pairs of functions* $\left\{f_k(t), u_k(x,t), k = \overline{1,m}\right\}$ *are called the classical solution of the problem* (1)-(4) *(problem I for short) if: 1)* $f_k(t) \in C^{\alpha}[0,T]$*; 2)* $u_k(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{D})$*; 3) the relations* (1)-(4) *are satisfied in the usual sense for these functions.*

The problem I is incorrect in the sense of Hadamard. We can give examples where the uniqueness or the stability of the solution of problem I is violated. So we will investigate the well-posedness of problem I in the sense of Tikhonov [17].

To treat problem I, we will reduce it to the equivalent one.

Lemma 1. *Let the initial data satisfy the conditions 1^0-5^0. Then, for the pairs $\left\{f_k(t), u_k(x,t), k=\overline{1,m}\right\}$ to be the solutions of problem I in the sense of Definition 1, it is necessary and sufficient that these pairs satisfy the relations* (1),(2), (3) *and*

$$f_k(t) = \left[h_k^{'}(t) - u_{kx}(d,t) + u_{kx}(e,t)\right] \Big/ \int_e^d g_k(x,t,u)dx, \;\; t \in [0,T], \qquad (5)$$

Proof. *Necessity.* Let the pairs $\left\{f_k(t), u_k(x,t), k=\overline{1,m}\right\}$ be the solutions of problem I in the sense of Definition 1. Integrating the equation (1) over the interval (l,d) with respect to x, we obtain

$$\int_e^d u_{kt}dx - \int_e^d u_{kxx}dx = \int_e^d f_k(t)g_k(x,t,u)dx.$$

Hence, by the conditions of Lemma 1, it follows that the formula (5) is valid.

Sufficiency. Let the pairs $\left\{\tilde{f}_k(t), \tilde{u}_k(x,t), k=\overline{1,m}\right\}$ be the classical solution of the problem (1)-(3), (5). Rewrite the formula (5) for these pairs:

$$\tilde{f}_k(t) = \left[h_k^{'}(t) - \tilde{u}_{kx}(d,t) + \tilde{u}_{kx}(e,t)\right] \Big/ \int_e^d g_k(x,t,\tilde{u})dx, \;\; k=\overline{1,m}, \; t \in [0,T]. \qquad (6)$$

Denote $\int_e^d \tilde{u}_k(x,t)dx = \tilde{h}_k(t), \quad t \in [0,T], \;\; k=\overline{1,m}.$

Writing the equation (1) for the functions $\tilde{u}_k(x,t), \tilde{f}_k(t), k=\overline{1,m}$ and calculating as we did to derive (5), we obtain

$$\tilde{f}_k(t) = \left[\tilde{h}_k^{'}(t) - \tilde{u}_{kx}(d,t) + \tilde{u}_k(e,t)\right] \Big/ \int_e^d g_k(x,t,\tilde{u})dx, \;\; t \in [0,T], \;\; k=\overline{1,m}. \qquad (7)$$

Subtracting the relations (6) and (7) from each other, we obtain the following relation for the function $\theta_k(t) = h_k(t) - \tilde{h}_k(t)$:
$\theta_k^{'}(t) = 0, \;\; \theta_k(0) = 0,$ i.e. $\theta_k(t) \equiv 0, \;\; h_k(t) \equiv \tilde{h}_k(t), \;\; t \in [0,T], \;\; k=\overline{1,m}.$

Lemma 1 is proved.

2. Existence of Generalized Solution

It is easy to verify that for the function

$$F_k(x,t) = \varphi_k^*(x) + \frac{\gamma(t)-x}{\gamma(t)}\left[\psi_{1k}(t)-\psi_{1k}(0)\right] + \frac{x}{\gamma(t)}\left[\psi_{2k}(t)-\psi_{2k}(0)\right] \quad (8)$$

we have $F_k(x,0)=\varphi_k^*(x),\ x\in[0,\gamma(T)],\ F_k(0,t)=\psi_{1k}(t),\ F_k(\gamma(t),t)=\psi_{2k}(t)$, $t\in[0,T]$ and $F_k(x,t)\in C^{2+\alpha,1+\alpha/2}(\overline{D})\ \ k=\overline{1,m}$, where

$$\varphi_k^* = \begin{cases} \varphi_k(x), & x\in[0,\gamma(0)], \\ \varphi_k(\gamma(0)), & x\in[\gamma(0),\gamma(T)],\ k=\overline{1,m}. \end{cases}$$

Assume that problem I has a solution in the sense of Definition 1. Then, the solution of the problem of finding the functions $u_k(x,t)$ from the relations (1),(2),(3) for the given functions $f_k(t)\in C^\alpha[0,T], k=\overline{1,m}$ can be represented as follows [18, p.230]:

$$u_k(x,t) = F_k(x,t) + \upsilon_k(x,t) + y_k(x,t), \quad (9)$$

where

$$\upsilon_k(x,t) = \int_0^t G_x(x,t-\tau)\rho_{1k}(\tau)d\tau + \int_0^t G_x(x-\gamma(\tau),t-\tau),\rho_{2k}(\tau)d\tau, k=\overline{1,m} \quad (10)$$

is a solution of the problem

$$\begin{aligned} &\upsilon_{kt}-\upsilon_{kxx}=0, \quad (x,t)\in D \\ &\upsilon(x,0)=0, \qquad x\in[0,\gamma(T)] \\ &\upsilon_k(0,t)=-y_k(0,t),\ \ \upsilon_k(\gamma(t),t)=-y_k(\gamma(t),t),\ \ t\in[0,T],\ \ k=\overline{1,m}, \end{aligned}$$

and the potential

$$y_k(x,t) = \int_0^t\int_{-\infty}^{+\infty} G(x-\xi;t-\tau)\tilde{\Phi}_k(\xi,\tau)d\xi d\tau,\ \ k=\overline{1,m} \quad (11)$$

is a solution of the equation

$$y_{kt}-y_{kxx}=\tilde{\Phi}_k(x,t)(x,t)\in Q=(-\infty,+\infty)\times[0,T].$$

Note that

$$\Phi_k(x,t) = f_k(t)g_k(x,t,u) - F_{kt}(x,t) + F_{kxx}(x,t)$$

$$\tilde{\Phi}_k(x,t) = \begin{cases} \Phi_k(0,t), & -\infty < x < 0, \quad 0 \le t \le T \\ \Phi_k(x,t) & 0 \le x \le \gamma(t), \quad 0 \le t \le T \\ \Phi_k(\gamma(t),t) & \gamma(t) < x < +\infty, \quad 0 \le t \le T, \end{cases} \tag{12}$$

the function $G(x,t) = \frac{1}{\sqrt{4t}} \exp\left(-\frac{x^2}{4t}\right)$, $t > 0$ is a fundamental solution of the equation $y_t - y_{xx} = 0$ and [18]

$$G(x,t) \le const(t-\tau)^{1/2},$$

$$\begin{gathered} |G_x(x-\gamma(\tau), t-\tau)|, |G_x(\gamma(t)-\gamma(\tau), t-\tau)| \le const(t-\tau)^{-1/2} \\ |G_{xx}(x-\gamma(\tau), t-\tau)|, |G_{xx}(\gamma(t)-\gamma(\tau), t-\tau)| \le const(t-\tau)^{-3/2}. \end{gathered} \tag{13}$$

Taking into account the inequalities (13), we can show the validity of the following properties for the functions $y_k(x,t)$, $k = \overline{1,m}$:

$$6^0 . y_k(x,t) \in C^{1,0}(Q);$$

$$7^0 . \, |y_k(x,t)| \le const \left\| \tilde{\Phi} \right\|_Q^{(0)} t, \;\; (x,t) \in Q;$$

$$8^0 . \, |y_{kx}(x,t)| \le const \left\| \tilde{\Phi} \right\|_Q^{(0)} t^{1/2}, \;\; (x,t) \in Q.$$

The functions $\rho_{1k}(t), \rho_{2k}(t)$, $k = \overline{1,m}$ appearing in (10) are the solution of the system of integral equations

$$\begin{gathered} \rho_{1k}(t) = 2\int_0^t G_x(-\gamma(\tau), t-\tau)\rho_{2k}(\tau)d\tau + 2y_k(0,t), \\ \rho_{2k}(t) = -2\int_0^t G_x(\gamma(t)-\gamma(\tau), t-\tau)\rho_{2k}(\tau)d\tau - \\ -2\int_0^t G_x(\gamma(t), t-\tau)\rho_{1k}(\tau)d\tau - 2y_k(\gamma(t),t) \end{gathered} \tag{14}$$

and $\rho_{1k}(t), \rho_{2k}(t) \in C[0,T]$.

Definition 2. *The pairs of functions* $\left\{ f_k(t), u_k(x,t), k = \overline{1,m} \right\}$ *will be called the generalized solution of the problem (5),(9) if 1)* $f_k(t) \in C[0,T]$; *2)* $u_k(x,t) \in C^{1,0}(\overline{D})$; *3) the relations (5),(9) are satisfied in the usual sense for these functions.*

Theorem 1. *Let the initial data of the problem (5),(9) satisfy the conditions* 1^0*-*5^0*. Then there exists the number* $T^*(0 < T^* \le T)$ *such that the problem (5),(9) has a unique solution in the sense of Definition 2 in the domain* $\overline{D}_* = [0,\gamma(t)] \times [0,T^*]$.

Proof. Denote $B = C^{1,0}(\overline{D})$, $E = C[0,T]$ and $A = B \times E$.

$$B' = \{u_k | u_k(x,t) \in B,\ |u_k(x,t)|, |u_{kx}(x,t)| \leq u_0 + \|F\|_D^{(1,0)},\ u_0 =$$

$$= const > 0, (x,t) \in \overline{D}, k = \overline{1,m}\Big\},$$

$E' = \big\{f_k | f_k(t) \in E, |f_k(t)| \leq f_0, f_0 = const > 0, t \in [0,T],\ k = \overline{1,m}\big\}$ and $A' = B' \times E'$ are the subsets of complete sets B, E and A, respectively.

Rewrite the problem (5), (9) in operator form $M[w] = M[(u,f)] = \{M_1[u(f_k)], M_2[f(u_k)]\}, M : A \to A$ and define the norm $\|w\|_A^{(1,0)} = \|u\|_D^{(1,0)} + \|f\|_T^{(0)}$, where

$$M_1[u(f_k)] = u_k, M_1 : B \to B\ k = \overline{1,m}.$$

$$M_1[u(f_k)] = F_k(x,t) + \upsilon_k(x,t) + y_k(x,t), \tag{15}$$

$$M_2[f(u_k)] = f_k,\ M_2 : E \to E.$$

$$M_2[f(u_k)] = [h'_k(t) - u_{kx}(d,t) + u_{kx}(e,t)] \backslash \int_e^d g_k(x,t,u)dx, \tag{16}$$

Let's show that there exists $T_1 (0 < T_1 \leq T)$ such that $M_1[u(f_k)] \in B'$, $M[B'] \subset B'$ for every $f_k(t) \in E'$.

Taking into account the relations (8), (10), (11), (12), (13) and the conditions 1^0-8^0, let's estimate $M_1[u(f_k)]$ defined by (15):

$$|M_1[u(f_k)]| \leq |F_k(x,t)| + |\upsilon_k(x,t)| + |y_k(x,t)|, \tag{17}$$

$$|F_k(x,t)| \leq \|F\|_D^{(1,0)},\ (x,t) \in \overline{D} \tag{18}$$

$$|\upsilon_k(x,t)| \leq c_1 \|\rho\|_T^{(0)} t^{1/2},$$

$c_1 > 0$ is a constant depending on the initial data of problem I. The positive constants which depend only on initial data will be denoted by $c_i, i = 1, 2....$

The norm $\|\rho\|_T^{(0)} = \|\rho_1\|_T^{(0)} + \|\rho_2\|_T^{(0)}$ is estimated by the system of integral equations (12) as follows:

$$|\rho_{1k}(t)| \leq c_2 \|\rho_2\|_T^{(0)} t^{1/2} + c_3 \left\|\tilde{\Phi}\right\|_Q^{(0)} t,\ \ t \in [0,T]$$
$$|\rho_{2k}(t)| \leq c_4 \|\rho\|_T^{(0)} t^{1/2} + c_5 \left\|\tilde{\Phi}\right\|_Q^{(0)} t,\ \ t \in [0,T],$$

or

$$\|\rho\|_T^{(0)} \leq c_6 \|\rho\|_T^{(0)} T^{1/2} + c_7 \left\|\tilde{\Phi}\right\|_Q^{(0)} T.$$

Let $T_2 (0 < T_2 \leq T)$ be a number such that $c_6 T_2^{1/2} < 1$. Then from the last inequality we obtain

$$\|\rho\|_T^{(0)} \leq c_8 \ \left\|\tilde{\Phi}\right\|_Q^{(0)} T, \tag{19}$$

and hence

$$|\upsilon_k(x,t)| \leq c_9 \ \left\|\tilde{\Phi}\right\|_Q^{(0)} T_2^{3/2}, \quad k = \overline{1,m}, \tag{20}$$

where

$$\left\|\tilde{\Phi}\right\|_Q^{(0)} \leq f_0 g^* + \|F\|_D^{(2,1)}.$$

By considering the relations (18), (20) and the property 7^0 in (17), we have

$$|M_1[u(f_k)]| \leq \|F\|_D^{(0)} + c_{10} T.$$

Let $T_3 (0 < T_3 \leq T)$ be a number such that $c_{10} T_3 < u_0$. Hence, $|M_1[u(f_k)]| \leq \|F\|_D^{(1,0)} + u_0$.

From (17) we have

$$|M_1[u(f_k)]_x| \leq |F_{kx}(x,t)| + |\upsilon_{kx}(x,t)| + |y_{kx}(x,t)|. \tag{21}$$

From (10) it follows that

$$|\upsilon_k(x,t)| \leq \int_0^t |G_{xx}(x, t-\tau)| \, |\rho_{1k}(\tau)| \, d\tau + \int_0^t |G_{xx}(x - \gamma(\tau), t-\tau)| \, |\rho_{2k}(\tau)| \, d\tau.$$

Taking into account (13), (19) and using the property 8^0, we obtain

$$|\upsilon_{kx}(x,t)| \leq c_{11} \ \left\|\tilde{\Phi}\right\|_Q^{(0)} T^{1/2}, \ (x,t) \in \overline{D}, \tag{22}$$

$$|y_{kx}(x,t)| \leq c_{12} \ \left\|\tilde{\Phi}\right\|_Q^{(0)} T^{1/2}, \ (x,t) \in \overline{D}. \tag{23}$$

So we have

$$|M_1[u(f_k)]_x| \leq \|F\|_D^{(1,0)} + c_{13} T^{1/2}.$$

Let $T_4 (0 < T_4 \leq T)$ be a number such that $c_{13} T_4^{1/2} < u_0$. Then

$$|M_1[u(f_k)]_x| \leq \|F\|_D^{(1,0)} + u_0,$$

i.e. in the domain $\overline{D} = [0, \gamma(t)] \times [0, T_5]$ $(T_5 = \max(T_1, T_2, T_3, T_4))$ we have $M_1[B'] \subset B'$.

Let's show that for every $u_k(x,t) \in B'$, $k=\overline{1,m}$ the relations $M_2[f(u_k)] \le f_0$, $M_2[E'] \subset E'$ hold.

We have

$$|M_2[f(u_k)]| \le \left[\left|h_k'(t)\right| + |u_{kx}(d,t)| + |u_{kx}(e,t)|\right] \setminus \int_e^d |g_k(x,t,u)|\,dx.$$

$$\left|h_k'(t)\right| \le h^*, \quad t \in [0,T]$$

$$|u_{kx}(d,t)| \le |F_{kx}(d,t)| + |\upsilon_{kx}(d,t)| + |y_{kx}(d,t)| \le$$
$$\le \|F\|_D^{(1,0)} + c_{23} \left\|\tilde{\Phi}\right\|_Q^{(0)} T^{1/2} + c_{24} \left\|\tilde{\Phi}\right\|_Q^{(0)} T^{1/2}.$$

As

$$\left\|\tilde{\Phi}\right\|_Q^{(0)} \le \max_D |f_k(t)|\,|g_k(x,t,u)| + \|F\|_D^{(2,1)} = f_0 g^* + \|F\|_D^{(2,1)},$$

we have

$$|u_{kx}(d,t)| \le c_{25} \|F\|_D^{(2,1)} T^{1/2} + c_{26} f_0 g^* T^{1/2}, \ t \in [0,T], \ k=\overline{1,m}.$$

Similarly,

$$|u_{kx}(e,t)| \le c_{27} \|F\|_D^{(2,1)} T^{1/2} + c_{28} f_0 g^* T^{1/2}, \ t \in [0,T], \ k=\overline{1,m}.$$

So we have

$$|M_2[f(u_k)]| \le \left\{\left(h^* + c_{29} \|F\|_D^{(2,1)} T^{1/2}\right) + c_{30} f_0 g^* T^{1/2}\right\} / (d-e) g_*.$$

Let $T_6 (0 < T_6 \le T)$ be a number such that $(d-e)g_* - c_{30} g^* T_6^{1/2} > 0$. Then, for $f_0 = \left(h^* + c_{29} \|F\|_D^{(2,1)} T_6^{1/2}\right) / \left[(d-e)g_* - c_{30} g^* T_6^{1/2}\right]$ we have $M_2\left[f(u_k)\right] \le f_0$ and $M_2[E'] \subset E'$.

Thus, $M[A'] \subset A'$, $(x,t) \in \overline{D} \times [0,T_7]$ $(T_7 = \min(T_5, T_6))$.

Now let's show that for every $(u^1, f^1), (u^2, f^2) \in A'$ the relation

$$\left\|M[(u^1,f^1)] - M[(u^2,f^2)]\right\| = \left\|M_1[u^1(f_k^1)] - M_1[u^2(f_k^2)]\right\| +$$

$$+ \left\|M_2[f^1(u_k^1)] - M_2[f^2(u_k^2)]\right\| \le c\left[\left\|u^1 - u^2\right\|_D^{(1,0)} + \left\|f^1 - f^2\right\|_T^{(0)}\right]$$

holds and $c < 1$.

Considering the relation (15) for (u^1, f^1) and $(u^2, f^2) \in A'$, and then subtracting two relations from each other, we obtain

$$\left|M_1[u^1(f_k^1)] - M_1[u^2(f_k^2)]\right| \le \int_0^t |G_x(x,t-\tau)| \left|\rho_{1k}^1(\tau) - \rho_{1k}^2\right| d\tau +$$

$$+\int_0^t |G_x(x-\gamma(\tau),t-\tau)| \left|\rho_{2k}^1(\tau) - \rho_{2k}^2\right| d\tau + \int_0^t \int_{-\infty}^{+\infty} G(x-\xi,t-\tau)\times$$

$$\times \left|\tilde{\Phi}_k^1(\xi,\tau) - \tilde{\Phi}_k^2(\xi,\tau)\right| d\xi d\tau,\ (x,t)\in \overline{D}, \tag{24}$$

where

$$\left|y_k^1(x,t) - y_k^2(x,t)\right| \le$$

$$\le \int_0^t \int_{-\infty}^{+\infty} G(x-\xi,t-\tau) \left[\left|f_k^1(\tau) - f_k^2(\tau)\right| \ \left|g_k(\xi,\tau,u^1)\right| +\right.$$

$$\left. + \left|f_k^2(\tau)\right| \left|g_k(\xi,\tau,u^1) - g_k(\xi,\tau,u^2)\right|\right] d\xi d\tau \le$$

$$\le c_{14} \left[\left\|u^1 - u^2\right\|_D^{(0)} + \left\|f^1 - f^2\right\|_T^{(0)}\right] t, \quad (x,t)\in\overline{D},$$

and taking into account the relations

$$\left|\rho_{1k}^1(t) - \rho_{1k}^2(t)\right| \le \int_0^t |G_x(-\gamma(\tau),t-\tau)| \left|\rho_{1k}^1(\tau) - \rho_{2k}^2(\tau)\right| d\tau +$$

$$+2\left|y_k^1(0,t) - y_k^2(0,t)\right|,$$

$$\left|\rho_{2k}^1(t) - \rho_{2k}^2(t)\right| \le \int_0^t |G_x(\gamma(t)-\gamma(\tau),t-\tau)| \left|\rho_{2k}^1(\tau) - \rho_{2k}^2(\tau)\right| d\tau +$$

$$+2\int_0^t |G_x(\gamma(\tau),t-\tau)| \left|\rho_{1k}^1(\tau) - \rho_{1k}^2(\tau)\right| d\tau + 2\left|y_k^1(\gamma(t),t) - y_k^2(\gamma(t),t)\right|,$$

$$\left|\rho_{1k}^1(t) - \rho_{1k}^2(t)\right| \le c_{15} \left\|\rho_2^1 - \rho_2^2\right\|_T^{(0)} t^{1/2} + c_{16}\left[\left\|u^1 - u^2\right\|_D^{(0)} + \left\|f^1 - f^2\right\|_T^{(0)}\right] t$$

$$\left|\rho_{2k}^1(t) - \rho_{2k}^2(t)\right| \le c_{17} \left[\left\|\rho_1^1 - \rho_1^2\right\|_T^{(0)} + \left\|\rho_2^1 - \rho_2^2\right\|_T^{(0)}\right] t^{1/2} +$$

$$+c_{18}\left[\left\|u^1 - u^2\right\|_D^{(0)} + \left\|f^1 - f^2\right\|_T^{(0)}\right] t,\quad t\in[0,T]$$

$$\left\|\rho_1^1 - \rho_1^2\right\|_T^{(0)} + \left\|\rho_2^1 - \rho_2^2\right\|_T^{(0)} \le$$

$$\le c_{19}\left[\left\|u^1 - u^2\right\|_D^{(0)} + \left\|f^1 - f^2\right\|_T^{(0)}\right] T_8,\ \ (0 < T_8 \le T,\ c_{17} T_8^{1/2} < 1),$$

from (24) we obtain

$$\left\|M_1[u^1(f_k^1)] - M_1[u^2(f_k^2)]\right\|_D^{(0)} \le c_{20}\left[\left\|u^1 - u^2\right\|_D^{(0)} + \left\|f^1 - f^2\right\|_T^{(0)}\right] T.$$

Similar calculations for

$$\left|\left(M_1[u^1(f_k^1)]-M_1[u^2(f_k^2)]\right)_x\right| \leq \left|v_{kx}^1(x,t)-v_{kx}^2(x,t)\right|+\left|y_{kx}^1(x,t)-y_{kx}^2(x,t)\right|$$

yield

$$\max_{\overline{D}}\left|\left(M_1[u^1(f_k^1)]-M_1[u^2(f_k^2)]\right)_x\right| \leq c_{21}\left[\left\|u^1-u^2\right\|_D^{(0)}+\left\|f^1-f^2\right\|_T^{(0)}\right]T^{1/2}.$$

So we have

$$\left\|M_1[u^1(f_k^1)]-M_1[u^2(f_k^2)]\right\|_D^{(1,0)} \leq c_{22}\left[\left\|u^1-u^2\right\|_D^{(0)}+\left\|f^1-f^2\right\|_T^{(0)}\right]T_8^{1/2}. \quad (25)$$

Considering the formula (5) for (u^1,f^1) and (u^2,f^2), and then subtracting two relations from each other, we obtain

$$\left|M_2\left[f^1(u_k^1)\right]-M_2[f^2(u_k^2)]\right| \leq \left[\left|u_{kx}^1(d,t)-u_{kx}^2(d,t)\right|+\left|u_{kx}^1(e,t)-u_{kx}^2(e,t)\right|\right]/$$

$$\int\limits_e^d g_k(x,t,u^1)dx+\left\{\left[\left|h_k^{2\prime}(t)\right|+\left|u_{kx}^2(d,t)\right|+\left|u_{kx}^2(e,t)\right|\right]\int\limits_e^d\left|g_k(x,t,u^1)-g_k(x,t,u^2)\right|dx\right\}/$$

$$/\left[\int_e^d g_k(x,t,u^1)dx\int_e^d g_k(x,t,u^2)dx\right].$$

Using the equation (9), we get

$$\left|u_{kx}^1(d,t)-u_{kx}^2(d,t)\right| \leq \int_0^t\left|G_{xx}(d,t-\tau)\right|\left|\rho_{1k}^1-\rho_{1k}^2\right|d\tau+$$

$$+\int_0^t\left|G_{xx}(d-\gamma(\tau),t-\tau)\right|\left|\rho_{2k}^1(\tau)-\rho_{2k}^2(\tau)\right|d\tau+\int_0^t\int_{-\infty}^{+\infty}G(d-\xi,t-\tau)\times$$

$$\times\left[\left|f_k^1(\tau)-f_k^2(\tau)\right|\left|g_k(\xi,\tau,u^1)\right|+\left|f_2(\tau)\right|\left|g_k(\xi,t,u^1)-g_k(\xi,\tau,u^2)\right|\right]d\xi d\tau.$$

From here, by the inequality (13), the condition 2^0 and the relation

$$\left\|\rho_1^1-\rho_1^2\right\|_{T_{11}}^{(0)}+\left\|\rho_2^1-\rho_2^2\right\|_{T_{11}}^{(0)} \leq c_{31}\left[\left\|f^1-f^2\right\|_T^{(0)}+\left\|u^1-u^2\right\|_D^{(0)}\right]T^{1/2},$$

we have

$$\left|u_{kx}^1(d,t)-u_{kx}^2(d,t)\right| \leq c_{32}\left[\left\|f^1-f^2\right\|_T^{(0)}+\left\|u^1-u^2\right\|_D^{(0)}\right]T^{1/2},\ \ t\in[0,T].$$

Similarly,

$$\left|u_{kx}^1(e,t)-u_{kx}^2(e,t)\right| \leq c_{33}\left[\left\|f^1-f^2\right\|_T^{(0)}+\left\|u^1-u^2\right\|_D^{(0)}\right]T^{1/2},\ \ t\in[0,T].$$

So we get

$$\left\|M_2[f^1(u_k^1) - M_2[f_k^2(u_k^2)]\right\|_T^{(0)} \le c_{34}\left[\left\|f^1 - f^2\right\|_T^{(0)} + \left\|u^1 - u^2\right\|_D^{(0)}\right] T^{1/2}. \quad (26)$$

Combining the inequalities (25) and (26), we obtain

$$\left\|M[u^{1‘}, f^1] - M[u^2, f^2]\right\| \le c_{35}T^{1/2}\left[\left\|u^1 - u^2\right\|_D^{(0)} + \left\|f^1 - f^2\right\|_T^{(0)}\right].$$

Let $T_9(0 < T_9 \le T)$ be a number such that $c = c_{35}T^{1/2} < 1$. Then, by the contraction mapping principle [19, p.69] it follows that the problem (5), (9) has a unique solution $(u_k(x,t)), f_k(t) \in B' \times E' = A'$, $k = \overline{1,m}$ in the domain $\overline{D}_* = [0, \gamma(t)] \times [0, T^*]$ $(T^* = \min(T_8, T_9)$.

Theorem 1 is proved.

3. Stability of Generalized Solution

Obtaining an estimate which establishes the stability of solution is vital for the concept of well-posedness in the sense of Tikhonov [17].

Define the following well-posedness set:

$$K = \left\{(f,u) | f(t) \in C[0,T], u(x,t) \in C^{1,0}(\overline{D}) \right.,$$
$$\left. |f(t)|, |u(x,t)|, |u_x(x,t)| \le \sigma, (x,t) \in (\overline{D}), \sigma = const > 0\right\}.$$

Assume the problem (9),(5) has solutions $\left\{f_k^i(t), u_k^i(x,t), k = \overline{1,m},\ i = 1,2\right\}$ for two sets of initial data $\left\{g_k^i(x,t,u^i), \varphi_k^i(x), \psi_{1k}^i(t), \psi_{2k}^i(t), h_k^i(t), k = \overline{1,m}, i = 1,2\right\}$. Let's denote these two problems by II_1 and II_2, respectively.

Theorem 2. Let 1) the functions $\left\{g_k^i(\cdot), \varphi_k^i(\cdot), \psi_{1k}^i(\cdot), \psi_{2k}^i(\cdot), h_k^i(\cdot), k = \overline{1,m},\ i = 1,2\right\}$ and $\gamma(t)$ satisfy the conditions 1^0-5^0; 2) problems II_1 and II_2 have solutions $\left\{f_k^i(t), u_k^i(x,t), k = \overline{1,m},\ i = 1,2\right\}$ in the sense of Definition 2 and these solutions belong to the set K.

Then there exists $\tilde{T}(0 < \tilde{T} \le T)$ such that the following stability estimate is true for the problem (9), (5) in the domain $\overline{D} = [0, \gamma(t)][0, \tilde{T}]$:

$$\left\|u^1 - u^2\right\|_D^{(0)} + \left\|f^1 - f^2\right\|_{\tilde{T}}^{(0)} \le \nu\left[\left\|g^1 - g^2\right\|_D^{(0)}\right.$$

$$\left. + \left\|\varphi^1 - \varphi^2\right\|_{[0,\gamma(0)]}^{(2)} + \left\|\psi_1^1 - \psi_1^2\right\|_{\tilde{T}}^{(1)} + \left\|\psi_2^1 - \psi_2^2\right\|_{\tilde{T}}^{(1)} + \left\|h^1 - h^2\right\|_{\tilde{T}}^{(1)}\right], \quad (27)$$

where $\nu = const > 0$ depends on problem data and the set K.

Proof. During the proof, the positive constants depending on problem data and the set K will be denoted by $\nu_i = 1, 2, ...$

Also denote

$$z_k(x,t) = u_k^1(x,t) - u_k^2(x,t), \lambda_k(t) = f_k^1(t) - f_k^2(t),$$

$$\delta_{1k}(x,t,p) = g_k^1(x,t,p) - g_k^2(x,t,p),\ \delta_{2k}(t) = h_k^1(t) - h_k^2(t),$$

$$\delta_{3k}(x,t) = F_k^1(x,t) - F_k^2(x,t),\ \delta_{4k}(x,t) = \tilde{\Phi}_k^1(x,t) - \tilde{\Phi}_k^2(x,t),$$

$$\delta_{5k}(t) = \rho_{1k}^1(t) - \rho_{1k}^2(t),\ \delta_{6k}(t) = \rho_{2k}^1(t) - \rho_{2k}^2(t),$$

$$\delta_{7k}(x,t) = y_k^1(x,t) - y_k^2(x,t).$$

Let's write the equations (9) and (5) for the data $\left\{g_k^i(\cdot), h^i(\cdot), F_k^i(\cdot), \tilde{\Phi}_k^i(\cdot), \rho_{1k}^i(\cdot), \rho_{2k}^i(\cdot), k = \overline{1,m}, i = 1,2\right\}$ and then subtract them from each other. Then we obtain the following equations for the functions $\left\{\lambda_k(t) z_k(x,t),\ k = \overline{1,m}\right\}$:

$$z_k(x,t) = \delta_{3k}(x,t) + \int_0^t G_x(x, t-\tau)\delta_{5k}(\tau)d\tau + \int_0^t G_x(x-\gamma(\tau), t-\tau)\delta_{6k}(\tau)d\tau +$$

$$+ \int_0^t \int_{-\infty}^{+\infty} (x-\xi, t-\tau)\delta_{4k}(\xi,\tau)d\xi, d\tau,\ (x,t) \in D, \tag{28}$$

$$\lambda_k(t) = \left[\delta_{2k}'(t) - z_{kx}(d,t) + z_{kx}(e,t)\right] / \int_e^d g_k^1(x,t,u^1)dx - H_k(t),\ t \in [0,T], \tag{29}$$

where

$$\delta_{5k}(t) = 2\int_0^t G_x(-\gamma(\tau), t-\tau)\delta_{6k}(\tau)d\tau + \delta_{7k}(0,t),$$

$$\delta_{6k}(t) = -2\int_0^t G_x(\gamma(t)-\gamma(\tau), t-\tau)\delta_{6k}(\tau)d\tau -$$

$$-2\int_0^t G_x(\gamma(t), t-\tau)\delta_{5k}(\tau)d\tau - 2\delta_{7k}(\gamma(t),t), \tag{30}$$

$$H_k(t) = \left\{\int_e^d \left[\delta_{1k}(x,t,u^1) + g_k^2(x,t,u^1) - g_k^2(x,t,u^2)\right] dx \times\right.$$

$$\left. \times \left(h_k^{2'}(t) - u_{kx}^2(d,t) + u_{kx}^2(e,t)\right)\right\} / \left[\int_e^d g_k^1(x,t,u^1)dx \cdot \int_e^d g_k^2(x,t,u^2)dx\right]. \tag{31}$$

From (28), we obtain the following relation for the function $z_k(x,t)$ $k = \overline{1,m}$:

$$|z_k(x,t)| \le |\delta_{3k}(x,t)| + \int_0^t |G_x(x,t-\tau)|\,|\delta_{5k}(\tau)|\,d\tau + \int_0^t |G_x(x-\gamma(\tau),t-\tau)| \times$$

$$\times |\delta_{6k}(\tau)|\,d\tau + \int_0^t \int_{-\infty}^{+\infty} G(x-\xi,t-\tau)\,|\delta_{4k}(\xi,\tau)|\,d\tau,\ (x,t)\in D. \qquad (32)$$

By the relations (11), (13), (30) and

$$\delta_{4k}(x,t) = \lambda_k(t) g_k^1(x,t,u^1) + f_k^2(t)\delta_{1k}(x,t,u^2)+$$

$$+f_k^2(t)\left[g_k^1(x,t,u^1) - g_k^1(x,t,u^2)\right] - \delta_{3kt}(x,t) + \delta_{3kxx}(x,t),$$

we have

$$\|\delta_4\|_D^{(0)} \le \nu_1 \left[\|\lambda\|_T^{(0)} + \|\delta_1\|_D^{(0)} + \|z\|_D^{(0)} + \|\delta_3\|_D^{(2,1)}\right], \qquad (33)$$

$$|\delta_{5k}(t)| + |\delta_{6k}(t)| \le \nu_2 \left(\|\delta_5\|_T^{(0)} + \|\delta_6\|_T^{(0)}\right) t^{1/2} + \nu_3 \|\delta_4\|_D^{(0)}\, t.$$

Let $T_{10} (0 < T_{10} \le T)$ be a number such that $\nu_2 T_{10}^{1/2} < 1$. Then,

$$\|\delta_5\|_T^{(0)} + \|\delta_6\|_T^{(0)} \le \nu_4 \left[\|\lambda\|_T^{(0)} + \|\delta_1\|_D^{(0)} + \|z\|_D^{(0)} + \|\delta_3\|_T^{(2,1)}\right] T. \qquad (34)$$

Considering (33) and (34) in (32), we obtain

$$|z_k(x,t)| \le \nu_5 \left[\|\delta_1\|_D^{(0)} + \|\delta_3\|_D^{(2,1)}\right] + \nu_6 \left[\|\lambda\|_T^{(0)} + \|z\|_D^{(0)}\right] T^{3/2},\ (x,t)\in D. \quad (35)$$

Let's estimate the function $z_{kx}(x,t)$. From (28) it follows

$$|z_{kx}(x,t)| \le |\delta_{3kx}(x,t)| + \int_0^t |G_{xx}(x,t-\tau)|\,|\delta_{5k}(\tau)|\,d\tau + \int_0^t |G_{xx}(x-\gamma(\tau),t-\tau)| \times$$

$$\times |\delta_{6k}(\tau)|\,d\tau + \int_0^t \int_{-\infty}^{+\infty} G_x(x-\xi,t-\tau)\,|\delta_{4k}(\xi,\tau)|\,d\xi d\tau,\ (x,t)\in D.$$

Similar to the way we derived the inequality (35), we have

$$|z_{kx}(x,t)| \le$$

$$\le \nu_7 \left[\|\delta_1\|_D^{(0)} + \|\delta_3\|_D^{(2,1)}\right] + \nu_8 \left[\|\lambda\|_T^{(0)} + \|z\|_D^{(0)}\right] T^{1/2},\ k=\overline{1,m},\ (x,t)\in D, \quad (36)$$

By (31) and (36),

$$|\lambda_k(t)| \le \nu_9 \left[\|\delta_1\|_D^{(0)} + \|\delta_2\|_T^{(1)} + \|\delta_3\|_T^{(2,1)}\right] + \nu_{10} \left[\|\lambda\|_T^{(0)} + \|z\|_D^{(0)}\right] T^{1/2},$$

$$k=\overline{1,m}\,,\ \ t\in[0,T], \tag{37}$$

From (36), (37) it follows that

$$\|z_k(x,t)\|+\|\lambda_k(t)\|\le\nu_9\left[\|\delta_1\|_D^{(0)}+\|\delta_2\|_T^{(1)}+\|\delta_3\|_T^{(2,1)}\right]+\nu_{10}\left[\|\lambda\|_T^{(0)}+\|z\|_D^{(0)}\right]T^{1/2}.$$

Let $T_{11}(0<T_{11}\le T)$ be a number such that $\nu_{10}T_{11}^{1/2}<1$. Then from the last inequalities we obtain

$$\|\lambda\|_T^{(0)}+\|z\|_D^{(0)}\le\nu_{11}\left[\|\delta_1\|_D^{(0)}+\|\delta_2\|_T^{(1)}+\|\delta_3\|_T^{(2,1)}\right],$$

i.e. the stability estimate (27) is true for the solution of the problem (9), (5) in the sense of Definition 2 in the domain $\overline{\tilde{D}}=[0,\gamma(t)][0,\tilde{T}]$ ($\tilde{T}=\min(T_{10},T_{11})$).

Theorem 2 is proved.

Note that the uniqueness of the generalized solution of problem I follows from the inequality (25) under the conditions $g_k^1(x,t,p)=g_k^2(x,t,p)$, $\varphi_k^1(x)=\varphi_k^2(x)$, $\psi_{1k}^1(t)=\psi_{1k}^2(t)$, $\psi_{2k}^1(t)=\psi_{2k}^2(t)$, $h_k^1(t)=h_k^2(t)$.

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Adalat Ya. Akhundov
Ministry of Science and Education of the Republic of Azerbaijan Institute of Mathematics Baku, Azerbaijan
E-mail: adalatakhund@gmail.com

Arasta Habibova
Baku Engineering University, Depatment mathematics and Computer science, Baku, Azerbaijan
E-mail: ahabibova@beu.edu.az

Aynur H.Hasanova
Ministry of Science and Education of the Republic of Azerbaijan Institute of Mathematics Baku, Azerbaijan
E-mail: aynurhasanova1973@gmail.com

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