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On Solvability of Boundary Value Problem for m-th Order Elliptic Equations in Grand Sobolev Spaces

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Abstract. We consider an m-th order elliptic equation with continuous coefficients in the principal part. Let $L_{p)}(\Omega)$, $1 < p < +\infty$, denote the grand Lebesgue space on a bounded n-dimensional domain $\Omega \subset R^n$ and let $W^m_{p)}(\Omega)$ be the corresponding grand Sobolev space, generated by the norm of $L_{p)}(\Omega)$. With respect to the grand Sobolev space $W^m_{p)}(\Omega)$ we define the trace operator Γ_S and trace space $W^m_{p)}(S)$, corresponding to the sufficiently smooth surface $S \subset \overline{\Omega}$. We prove the validity of the continuous embedding $W^m_{p)}(S) \subset L_{p)}(S; d\sigma)$, where $d\sigma$ is an area element of S. It is established that if the boundary $\partial\Omega$ of Ω is sufficiently smooth, then, under certain conditions on the coefficients of the considered elliptic operator,a Schauder-type estimate up to the boundary holds for this operator. Such estimates play a key role in establishing the Fredholmness of boundary value problems (BVP) for elliptic equations.

Key Words and Phrases: Grand Sobolev space, elliptic equations, Schauder type estimates, BVP

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1. Introduction

A priori estimates get own origin from the classic S. N. Bernstein's work and subsequent development belongs to the Y. Schauder. For higher-order elliptic equations such estimates were obtained by S. Agmon, A. Douglis and L. Nirenberg [1]. Using these estimates in [1] it is established the Fredholmness of BVP for elliptic equations. It should be noted that in recent years, for concrete problems in mechanics, mathematical physics and pure mathematics interest in the study of various mathematical questions-such as harmonic analysis, approximation theory, differential equations and etc., in the context of so-called non-standard function

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spaces has increased significantly (see, e.g., [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 30]). Examples of such spaces include Lebesgue space with variable summability index, grand Lebesgue space, Morrey space, Orlicz space and etc. Morrey and grand Lebesgue spaces are nonseparable. For this reason, differential equations in these spaces can be considered under two different statement: A) non separable statement and B) separable statement. In case of A) all data of the equation are taken from the considered nonseparable spaces. In contrast, in case B), one first defines an appropriate separable subspace of considered nonseparable space in which the set of smooth functions is dense, and all data of the equation are taken from spaces generated by this subspace. Note that all differential equations regarding the Morrey spaces have so far been considered mainly in the nonseparable statement (see, e.g., [8, 18, 21, 23, 24, 25]). In contrast, differential equations in grand Lebesgue spaces have generally been studied in the separable statement (see, e.g., [9, 14, 15, 16, 17]), with a primary focus on second-order elliptic equations and their associated boundary value problems.

We consider an m-th order elliptic equation with constant coefficients in the grand Sobolev space $W_{p)}^{m}(\Omega)$ on $\Omega = \Omega' \times (0,T)$, where $\Omega' \subset R^{n-1}$ is a bounded domain. It is considered BVP with boundary conditions on Ω', that satisfy the complementing condition (Shapiro-Lopatinski condition). All results from the well-known work of Agmon, Douglis and Nirenberg [1] concerning classical Sobolev space are extended to this setting. These results play a key role in establishing a Schauder-type estimate up to the boundary for this operator.

2. Preliminaries

2.1. Notations

- Z_+ is the set of non-negative integers; C is the complex plane;
- $x \in R^n \Leftrightarrow x = (x_1; ...; x_n)$, $|x| = \sqrt{x_1^2 + ... + x_n^2}$, $R_+^n = \{x \in R^n : x_n > 0\}$;
- $B_r(x_0) = \{x \in R^n : |x - x_0| < r\}$, $B_r(0) \equiv B_r$, $B_r^+ = \{x \in B_r : x_n \geq 0\}$, $B_r' = \left\{x \in R^{n-1} : |x| < r\right\}$;
- $\Omega_r(x_0) = \Omega \bigcap B_r(x_0)$, $\Omega_r = \Omega_r(0)$;
- $mesM = |M|$ denotes the Lebesgue measure ($n-$dimentional) of a set $M \subset R^n$;
- $\partial\Omega$ is the boundary of the domain $\Omega \subset R^n$, $\overline{\Omega} = \Omega \bigcup \partial\Omega$;

- $\rho(x;M) = \inf\limits_{y\in M}|x-y|$, $d_\Omega = diam\Omega = \sup\limits_{x;y\in\Omega}|x-y|$, $f/_M$ denotes the restriction of f to M;
- $C_0^\infty(\Omega)$ is the set of infinitely differentiable functions with compact support in Ω;
- $C^\infty\left(\overline{\Omega}\right)$ is the set of infinitely differentiable functions on $\overline{\Omega}$;
- $C^m(\Omega)$ is the set of m-times order continuously differentiable functions in Ω;
- $C_0^m(\Omega)$ is the set of m-times continuously differentiable functions with compact support in Ω;
- B-space denotes a Banach space;
- R_T is the range of the operator T, $Ker\,T$ is the kernel of the operator T;
- $[X;\,Y]$ denotes the Banach space of bounded linear operators acting from X to Y, $[X] = [X;\,X]$;
- $\alpha = (\alpha_1;...;\alpha_n) \in Z_+^n$ is a multiindex, $\partial_i = \frac{\partial}{\partial x_i}$, $\partial^\alpha = \partial_1^{\alpha_1}...\,\partial_n^{\alpha_n}$; $\xi = (\xi_1;...\xi_n) \in R^n \;\Rightarrow\; \xi^\alpha = \xi_1^{\alpha_1}...\,\xi_n^{\alpha_n}$;
- S^{n-1} is the unit sphere in R^n, $d\sigma$ is an area element of S^{n-1};
- p' the conjugate to number $p:\ \frac{1}{p'}+\frac{1}{p}=1$;
- $\delta(\cdot)$ is the Dirac delta.

Throughout the paper, by C and c we denote constants that may vary from line to line.

2.2. Elliptic equations

Let L be a uniformly elliptic operator of m-th order

$$L = \sum_{|\alpha|\le m} a_\alpha(x)\,\partial^\alpha,$$

with real coefficients $a_\alpha \in L_\infty(\Omega)$, i.e. $\exists\ \delta > 0$, such that

$$\delta\,|\xi|^m \le \sum_{|\alpha|=m} a_\alpha(x)\,\xi^\alpha \le \delta^{-1}\,|\xi|^m, \quad \text{a.e. } x\in\Omega.$$

Assign to L a "tangential operator"

$$L_{x_0} = \sum_{|\alpha|=m} a_\alpha (x_0)\, \partial^\alpha,$$

at point $x_0 \in \Omega$ and without loss of generality, we assume that the operator L_{x_0} is elliptic at the point x_0 (in what follows, this condition is assumed to hold). Denote by $J_{x_0}(\cdot)$ the fundamental solution of the equation $L_{x_0}\varphi = 0$ with constant coefficients $a_\alpha (x_0)$. The function $J_{x_0}(\cdot)$ is a parametrics for the equation $L\varphi = 0$ with a singularity at the point x_0. Let

$$(S_{x_0}\varphi)(x) = \psi(x) = \int J_{x_0}(x-y)\,\varphi(y)\,dy,$$

and assume

$$T_{x_0} = S_{x_0}(L_{x_0} - L).$$

When $x_0 = 0 \in \Omega$ the case L_{x_0}, S_{x_0} and T_{x_0} we will denote by L_0, S_0 and T_0, respectively.

Regarding the coefficients of the operator L we accept the following:

Condition (A). *i)* $a_\alpha \in C(\overline{\Omega})$, $\forall\, \alpha \in Z_+^n : |\alpha| = m$; *ii)* $a_\alpha \in L_\infty(\Omega)$, $\forall\, \alpha \in Z_+^n : |\alpha| < m$.

2.3. Integral operators and fundamental solution

In obtaining the main results, we will use the following integral operators from harmonic analysis.

Riesz potential R_γ:

$$(R_\gamma f)(x) = \int \frac{f(y)\,dy}{|x-y|^{n-\gamma}},\ 0 < \gamma < n;$$

Hardy-Littlewood maximal operator M:

$$(Mf)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|\,dy;$$

Calderon-Zygmund singular operator S:

$$(Sf)(x) = p.v. \int k(x;y)\, f(y)\,dy,$$

with a standard kernel $k(\cdot;\cdot)$, satisfying the following conditions. The kernel $k(\cdot;\cdot)$ is defined on the set $\Omega^0 = R^n \times R^n \setminus \{(x;x) : x \in R\}$, such that, for some $\delta > 0$ and $B < +\infty$, it holds

$$|k(x;y)| \le B\,|x-y|^{-n};$$

$$\left|k\left(x;y\right)-k\left(x';y\right)\right|\leq B\left|x-x'\right|^{\delta}\left|x-y\right|^{-n-\delta};$$
$$\left|k\left(x;y\right)-k\left(x;y'\right)\right|\leq B\left|y-y'\right|^{\delta}\left|x-y\right|^{-n-\delta},$$

for all $(x;y)$, $(x';y)$, $(x;y')\in\Omega^0$, such that $|x-x'|\leq\frac{1}{2}|x-y|$ and $|y-y'|\leq\frac{1}{2}|x-y|$.

We will also use the following properties of the fundamental solution $J_0(\cdot)$ (see e.g., the monograph [26]) of the constant-coefficient elliptic operator L_0:

(i) $J_0(\cdot)\in C^{\infty}(R\backslash\{0\})$;

(ii) $\partial^{\alpha}J_0(\lambda x)=|\lambda|^{-n}\partial^{\alpha}J_0(x)$, $\forall\alpha\in Z_+^n:|\alpha|=m$;

(iii) $\int_{S^{n-1}}\partial^{\alpha}J_0(\xi)\,\sigma_{\xi}=0$, $\forall\alpha\in Z_+^n:|\alpha|=m$;

(iv) $|\partial^{\alpha}J_0(x)|\leq C|x|^{m-n-|\alpha|}$, $\forall\alpha\in Z_+^n:|\alpha|\leq m$.

2.4. Grand Sobolev space

Let us define the grand Lebesgue space $L_{p)}(\Omega)$, $1<p<+\infty$, on a bounded domain $\Omega\subset R^n$. This is the Banach space of measurable (in Lebesgue sense) functions f on Ω equipped with the norm

$$\|f\|_{L_{p)}(\Omega)}=\sup_{0<\varepsilon<p-1}\left(\varepsilon\int_{\Omega}|f(x)|^{p-\varepsilon}\,dx\right)^{\frac{1}{p-\varepsilon}}.$$

The space $L_{p)}(\Omega)$ is non separable for which it holds the following continuous embeddings

$$L_p(\Omega)\subset L_{p)}(\Omega)\subset L_{p-\varepsilon_0}(\Omega),$$

for every $\varepsilon_0\in(0,p-1)$.

Let $S\subset\bar{\Omega}$ be an $(n-1)$-dimensional differentiable surface. Define the grand Lebesgue space $L_{p)}(S;d\sigma)$ on the measure space $(S;d\sigma)$ (where $d\sigma$ is an area element of S) equipped with the norm

$$\|f\|_{L_{p)}(S;d\sigma)}=\sup_{0<\varepsilon<p-1}\left(\varepsilon\int_{S}|f|^{p-\varepsilon}\,d\sigma\right)^{\frac{1}{p-\varepsilon}}.$$

The norm of $L_{p)}(\Omega)$ generates the following grand Sobolov space

$$W_{p)}^m(\Omega)=\left\{f\in L_{p)}(\Omega)\ :\ \partial^{\alpha}f\in L_{p)}(\Omega)\ ,\forall\alpha\in Z_+:|\alpha|\leq m\right\},$$

where $\partial^{\alpha}f$ denotes the generalized derivatives in Sobolev sense. Equipped with the norm

$$\|f\|_{W_{p)}^m(\Omega)}=\sum_{\alpha:|\alpha|\leq m}\|\partial^{\alpha}f\|_{L_{p)}(\Omega)},$$

the space $W_{p)}^m(\Omega)$ is a nonseparable B-space.

2.5. Complementing Condition. Poisson's kernels

Let

$$L(x;\partial) = L'(x;\partial) + L''(x;\partial),$$

be an elliptic operator of order m $(m = 2r : r \in N)$, where

$$L'(x;\partial) = \sum_{|\alpha|=m} a_\alpha(x)\, \partial^\alpha,$$

is its principal part and $L'(x;\xi)$, $\xi \in R^n$, is the corresponding characteristic form. Consider the following equation

$$Lu = f,$$

with boundary conditions

$$B_i u = \varphi_i \,,\ i = \overline{1,r},$$

where $f \in L_{p)}(\Omega;X)$ and $\varphi_i \in L_{p)}(\partial\Omega;X)$, $i = \overline{1,r}$; be given functions. Here B_i , $i = \overline{1,r}$, are boundary differential operators of the form

$$B_i u = B_i' u + B_i'' u, i = \overline{1,r},$$

where

$$B_i' u = \sum_{|\beta|=m_i} b_{\beta;i} u, i = \overline{1,r},$$

is the principal part of B_i of order m_i: $0 \le m_1 < ... < m_r \le m-1$. Accept the following complementing condition.

Condition (B). *Let n_ξ denote the outer normal at the point $\xi \in \partial\Omega$ and $\nu_\xi \neq 0$ be a vector parallel to $\partial\Omega$ at point ξ. We require that the polynomials $B_i'(\xi;\nu_\xi + \tau\, n_\xi)$, $i = \overline{1,r}$; considered as polynomials in the variable τ, are linearly independent modulo the polynomial*

$$\prod_{k=1}^{r} \left(\tau - \tau_k^+(\nu_\xi)\right),$$

where $\tau_k^+(\nu_\xi)$ $\left(\tau_k^-(\nu_\xi)\right)$ are the roots of the polynomial $L'(\xi;\nu_\xi + \tau\, n_\xi)$ with positive (respectively, negative) imaginary parts.

Assume

$$M^\pm(\nu;\tau) = \prod_{k=1}^{r} \left(\tau - \tau_k^\pm(\nu)\right) = \sum_{k=0}^{m} a_k^\pm(\nu)\, \tau^{m-k}.$$

Consider the Poisson's kernels $K_i(x;t)$, $i=\overline{1,r}$; defined by expressions (2.1), (2.1)' in the monograph [1] (see, chapter I, §2 , p.25), corresponding to the polynomial $M^{\pm}(\nu;\tau)$. For these kernels the following estimates hold

$$|\partial_x^s K_i| \leq C\left(1+|\log|p||\right)\left(|x|^2+t^2\right)^{\frac{m_i-n-s+1}{2}}, s\geq 0, \tag{1}$$

where $x=(x_1;...;x_{n-1})\in R^{n-1}$, $P=(x;t)\in R^n$ and $|P|=\left(|x|^2+t^2\right)^{\frac{1}{2}}$.

Definition 1. *The boundary of the domain $\Omega\subset R^n$ is said to be uniformly C^m-regular for some $m\in N$ if there exist open sets U_i, $i=\overline{1,r}$; that cover $\partial\Omega$ and coordinate transforms $\Phi_i\in C^m(U_i;B_1(0))$ with inverse $\Phi_i^{-1}\in C^m(B_1(0);U_i)$ such that:*

(i) there exists a $\delta>0$ such that $\{\xi\in\Omega: dist(\xi;\partial\Omega)<\delta\}$ is covered by the sets $\Phi_i^{-1}\left(B_{\frac{1}{2}}(0)\right)$;

(ii) $\Phi_i(U_i\bigcap\Omega)=\{\xi=(\xi_1;...;\xi_n)\in B_1(0):\xi_n>0\}$ for all $i=\overline{1,r}$;

(iii) there exists a constant $M>0$ that serves as an upper bound for the norms of all derivatives of Φ_i and Φ_i^{-1} up to order m for every $i=\overline{1,r}$.

2.6. Extension operator. Trace space $W_{p)}^m(S;d\sigma)$

In [16] the separable subspace $N_{p)}(\Omega)$ of $L_{p)}(\Omega)$ is considered, in which $C_0^\infty(\Omega)$ is dense and generated by this subspace grand Sobolov space $N_{p)}^m(\Omega)$ $\left(N_{p)}^0(\Omega)\equiv N_{p)}(\Omega)\right)$. In that work an extension theorem for functions in $N_{p)}^m(\Omega)$ is established. The proof of this theorem also applies to the more general space $W_{p)}^m(\Omega)$ case. Therefore, the following theorem is valid.

Theorem 1. *Let $\Omega\subset R^n$ be a bounded domain with boundary $\partial\Omega\in C^{(m)}$. Then, for every domain $\Omega'\subset R^n$: $\Omega\subset\subset\Omega'$, there exists the operator $\theta\in\left[W_{p)}^m(\Omega)\,;\,W_{p)}^m(\Omega')\right]$, such that: 1) $(\theta f)/_\Omega=f$, $\forall f\in W_{p)}^m(\Omega)$; 2) $\theta f=f/_\Omega$, $\forall f\in C^\infty(\overline{\Omega})$.*

Let us define the trace operator $\Gamma_S\in\left[W_{p)}^m(\Omega)\,;\,L_{p)}(S;d\sigma)\right]$, corresponding to a surface $S\in C^{(m)}$. Let $\Gamma_S^{(m)}\in[W_1{}^m(\Omega)\,;\,L_1(S)]$ be the trace operator, corresponding to the classical Sobolev space $W_1{}^m(\Omega)$, where $L_p(S)$ is Lebesgue

space on measure space $(S;\, d\sigma)$ with usual Lebesgue norm. Using the continuous embedding $W_{p)}^{m}(\Omega) \subset W_1{}^{m}(\Omega)$, we define $\Gamma_S = \Gamma_S^{(1)}/_{W_{p)}^{1}(\Omega)}$. We have

$$\|\Gamma_S f\|_{L_1(S)} \leq c\,\|f\|_{W_1^m(\Omega)} \leq c\,\|f\|_{W_{p)}^{m}(\Omega)}\,, \forall f \in W_{p)}^{m}(\Omega)\,, \forall m \in N,$$

where c is a constant independent of f. It follows that $\Gamma_S \in \left[W_{p)}^{m}(\Omega)\ ;\ L_1(S)\right]$. Accept

$$W_{p)}^{m;0}(\Omega) = \left\{u \in W_{p)}^{m}(\Omega)\ :\ \ \Gamma_S u = 0\right\}.$$

The following lemma is true.

Lemma 1. *$W_{p)}^{m;0}(\Omega)$ is a subspace of $W_{p)}^{m}(\Omega)$, i.e. $W_{p)}^{m;0}(\Omega)$ is the closure in $W_{p)}^{m}(\Omega)$.*

Proof. Let $\{u_n\} \subset W_{p)}^{m;0}(\Omega)$ be a Cauchy sequence in $W_{p)}^{m}(\Omega)$. Let $\lim\limits_{n} u_n = u \in W_{p)}^{m}(\Omega)$. To show that $u \in W_{p)}^{m;0}(\Omega)$, it is sufficient to prove that $\Gamma_S u = 0$. Indeed (in $L_1(S)$) we have $\Gamma_S u = \lim\limits_{n} \Gamma_S u_n = 0$.

Let $W_{p)}^{m}(S) = \Gamma_S\left(W_{p)}^{m}(\Omega)\right)$ and assume

$$\|\varphi\|_{W_{p)}^{m}(S)} = \inf\left\{\|u\|_{W_{p)}^{m}(\Omega)} : u \in W_{p)}^{m}(\Omega) \quad \&\ \Gamma_S u = \varphi\right\}. \tag{2}$$

Along with this, consider the factor space

$$\mathscr{F}_{p)}^{m}(S) = W_{p)}^{m}(\Omega)\,|_{W_{p)}^{m;0}(\Omega)},$$

equipped with the factor norm

$$\|F\|_{\mathscr{F}_{p)}^{m}(S)} = \inf_{f \in F} \|f\|_{W_{p)}^{m}(\Omega)}\ ,\ \forall F \in \mathscr{F}_{p)}^{m}(S)\ . \tag{3}$$

It is clear that $\mathscr{F}_{p)}^{m}(S)$ with the norm (3) is a B-space. Let $F \in \mathscr{F}_{p)}^{m}(S)$ and $f_1; f_2 \in F$. Consequently, $(f_1 - f_2) \in W_{p)}^{m;0}(\Omega) \Rightarrow \Gamma_S f_1 = \Gamma_S f_2$. Accept $\hat{\Gamma}_S F = \Gamma_S f\ ,\ f \in F$. From the above, it follows that the definition of $\hat{\Gamma}_S F$, is well-defined for all $\forall F \in \mathscr{F}_{p)}^{m}(S)$. Clearly $\hat{\Gamma}_S : \mathscr{F}_{p)}^{m}(S) \to W_{p)}^{m}(S)$. We have

$$\left\|\hat{\Gamma}_S F\right\|_{W_{p)}^{m}(S)} = /\hat{\Gamma}_S F = \varphi/ = \|\varphi\|_{W_{p)}^{m}(S)} = \inf\left\{\|u\|_{W_{p)}^{m}(\Omega)} : \Gamma_S u = \varphi\right\}.$$

It is not hard to see that $F = \left\{u \in W_{p)}^{m}(\Omega) : \Gamma_S u = \varphi\right\}$. Then from the above relation we obtain

$$\left\|\hat{\Gamma}_S F\right\|_{W_{p)}^{m}(S)} = \|F\|_{\mathscr{F}_{p)}^{m}(S)}\,.$$

From this, the following immediately follows.

Statement 2. *The space* $W_{p)}^{m}(S)$ *equipped with the norm* (2) *is a B-space. Moreover, the spaces* $W_{p)}^{m}(S)$ *and* $\mathscr{F}_{p)}^{m}(S)$ *are isometrically isomorphic and* $\hat{\Gamma}_S \in \left[\mathscr{F}_{p)}^{m}(S)\ ;\ W_{p)}^{m}(S)\right]$ *realizes this isomorphism.*

Let $u \in W_{p)}^{m}(\Omega)$ and $\varphi = \Gamma_S u$. We have

$$\|\Gamma_S u\|_{W_{p)}^{m}(S)} = \|\varphi\|_{W_{p)}^{m}(S)} = \inf\left\{\|v\|_{W_{p)}^{m}(\Omega)}\ :\ \Gamma_S v = \varphi\right\} \leq \|u\|_{W_{p)}^{m}(\Omega)}.$$

Thus, we obtain that $\Gamma_S \in \left[W_{p)}^{m}(\Omega)\ ;\ W_{p)}^{m}(S)\right]$. Based on these relations, we call $W_{p)}^{m}(S)$ the trace space and Γ_S the trace operator corresponding to the grand Sobolev space $W_{p)}^{m}(\Omega)$.

It is valid

Statement 3. *Let* $S \subset \bar{\Omega}\ :\ S \in C^{(m)}$ *be some surface . Let* $W_{p)}^{m}(S)$ *be a trace space, defined by* (2) *and* $\Gamma_S \in \left[W_{p)}^{m}(S)\ ;\ L_1(S)\right]$ *be the corresponding trace operator. Then* $\Gamma_S \in \left[W_{p)}^{m}(\Omega)\ ;\ W_{p)}^{m}(S)\right]$ *and the continuous embedding* $W_{p)}^{m}(S) \subset L_{p)}(S;\, d\sigma)$ *holds.*

Proof. Let us prove the continuous embedding $W_{p)}^{m}(S) \subset L_{p)}(S;\, d\sigma)$. Let $x_0 \in S$ be an arbitrary point. Then there exists a sufficiently small neighborhood $S_{x_0} \subset S$ of x_0, which is bijectively projected onto some domain D in a plane of R^n and is represented by the equation

$$x_n = \varphi\left(x'\right) \in C^{(m)}\left(\bar{D}\right)\ ,\ x' = (x_1, ..., x_{n-1}) \in D.$$

Without loss of generality, we may assume that $\bar{\Omega}$ is included in a cube $\Pi_a = \left\{x \in R^n :\ 0 < x_i < a,\ i = \overline{1,n}\right\}$. Let $f \in W_{p)}^{m}(\Omega)$ be an arbitrary function. By Theorem 1, f can be extended to a function $F \in W_{p)}^{m;0}(\Pi_a)\ :\ F/_{\Omega} = f$. Consequently, we also have $F \in W_1^{m;0}(\Pi_a)$. For every smooth surface $S \subset \bar{\Pi}_a$ under the trace $F/_S$ on S we mean $F/_S = \Gamma_S F$, where $\Gamma_S \in [W_1^m(\Pi_a)\ ;\ L_1(S;\, d\sigma)]$ is the corresponding trace operator. Then it is evident that $F/_{x_i=0} = 0\,,\ \forall\, i = \overline{1,n}$, and for $x = (x';\, \varphi(x')) \in S_{x_0}$ we have

$$f(x)/_{S_{x_0}} = f\left(x';\, \varphi\left(x'\right)\right) = \int_0^{\varphi(x')} \frac{\partial f(x';\xi_n)}{\partial \xi_n} d\xi_n, \tag{4}$$

a.e. $x' \in D$. The validity of the above formula follows from Theorem 2.7.1 of the monograph [27] (see, p. 41). Let $\varepsilon \in (0, p-1)$ be an arbitrary number and

accept $p_\varepsilon = p - \varepsilon$ & $p'_\varepsilon :\ \frac{1}{p_\varepsilon} + \frac{1}{p'_\varepsilon} = 1$. Applying Holder's inequality to (4), we obtain

$$\left|f/_{S_{x_0}}\right|^{p_\varepsilon} \leq \left|\varphi\left(x'\right)\right|^{\frac{p_\varepsilon}{p'_\varepsilon}} \int_0^{\varphi(x')} \left|\frac{\partial f\left(x';\xi_n\right)}{\partial \xi_n}\right|^{p_\varepsilon} d\xi_n \leq$$

$$\leq a^{\frac{p_\varepsilon}{p'_\varepsilon}} \int_0^{\varphi(x')} \left|\frac{\partial f\left(x';\xi_n\right)}{\partial \xi_n}\right|^{p_\varepsilon} d\xi_n \leq c \int_0^{\varphi(x')} \left|\frac{\partial f\left(x';\xi_n\right)}{\partial \xi_n}\right|^{p_\varepsilon} d\xi_n,$$

where $c = \max\{a;1\}$. Multiplying both sides by $\varepsilon\sqrt{1+\varphi_{x_1}^2+...+\varphi_{x_{n-1}}^2}$ of this inequality and integrating over D, we obtain

$$\left\|\varepsilon^{\frac{1}{p_\varepsilon}} f/_{S_0}\right\|^{p_\varepsilon}_{L_{p_\varepsilon}\left(S_{x_0}\right)} \leq c \left\|\varepsilon^{\frac{1}{p_\varepsilon}} \frac{\partial f}{\partial x_n}\right\|^{p_\varepsilon}_{L_{p_\varepsilon}(\Omega)}. \tag{5}$$

Since the surface S can be covered by a finite number of surfaces of type S_{x_0}, then summing the corresponding inequalities (from (5)) we establish

$$\left\|\varepsilon^{\frac{1}{p_\varepsilon}} f/_{S}\right\|^{p_\varepsilon}_{L_{p_\varepsilon}(S)} \leq c \left\|\varepsilon^{\frac{1}{p_\varepsilon}} \frac{\partial f}{\partial x_n}\right\|^{p_\varepsilon}_{L_{p_\varepsilon}(\Omega)},$$

where c is independent of f and E. Consequently, it is true

$$\left\|\varepsilon^{\frac{1}{p_\varepsilon}} f/_{S}\right\|_{L_{p_\varepsilon}(S)} \leq c \sum_{|\alpha|=1} \left\|\varepsilon^{\frac{1}{p_\varepsilon}} \partial^\alpha f\right\|_{L_{p_\varepsilon}(\Omega)} \leq c \left\|\varepsilon^{\frac{1}{p_\varepsilon}} f\right\|_{W^m_{p_\varepsilon}(\Omega)}.$$

Taking $\sup\limits_{0<\varepsilon<p-1}$ first on the right-hand side and then the same sup on the left-hand side, from this inequality we obtain

$$\|\Gamma_S f\|_{L_{p)}(S;\,d\sigma)} \leq c\,\|f\|_{W^m_{p)}(\Omega)}\ ,\ \forall\, f \in W^m_{p)}(\Omega). \tag{6}$$

From this estimate it directly follows that $\Gamma_S f \in L_{p)}(S)$ and $\Gamma_S \in \left[W^m_{p)}(\Omega);\ L_{p)}(S;\,d\sigma)\right]$.

Now, let $\varphi \in W^m_{p)}(S)$ and $\Gamma_S u = \varphi$ for some $u \in W^m_{p)}(\Omega)$. Then, by (6) we have

$$\|\varphi\|_{L_{p)}(S;\,d\sigma)} \leq c\,\|u\|_{W^m_{p)}(\Omega)}.$$

Taking the inf over $\left\{u \in W^m_{p)}(\Omega) :\ \Gamma_S u = \varphi\right\}$ on the right-hand side we obtain

$$\|\varphi\|_{L_{p)}(S;\,d\sigma)} \leq c\,\|\varphi\|_{W^m_{p)}(S)}\ ,\ \forall\, \varphi \in W^m_{p)}(S). \tag{7}$$

Introduce also the trace space $N^m_{p)}(S) = \Gamma_S\left(N^m_{p)}(\Omega)\right)$. Since, $\Gamma_S \in \left[W^m_{p)}(\Omega);\ W^m_{p)}(S)\right]$ and $N^m_{p)}(\Omega)$ is a subspace of $W^m_{p)}(\Omega)$, then it is evident that $N^m_{p)}(S)$ is a subspace of $W^m_{p)}(S)$ and as a result $\Gamma_S \in \left[N^m_{p)}(\Omega)\ ;\ N^m_{p)}(S)\right]$. Moreover, since $C^\infty\left(\bar{\Omega}\right)$ is dense in $N^m_{p)}(\Omega)$, then it is obvious that $\Gamma_S\left(C^\infty\left(\bar{\Omega}\right)\right)$ is also dense in $N^m_{p)}(S)$, i.e. for every $\varphi \in N^m_{p)}(S)$, there exists $\{u_n\} \subset C^\infty\left(\bar{\Omega}\right)$, such that $\Gamma_S u_n \to \varphi\,,\ n \to \infty$, in $W^m_{p)}(S)$. Thus, the following corollary is valid.

Corollary 1. *Let all conditions of Statement 3 hold and let* $\Gamma_S \in \left[W^m_{p)}(\Omega);\ W^m_{p)}(S)\right]$ *be the trace operator. Define* $N^m_{p)}(S) = \Gamma_S\left(N^m_{p)}(\Omega)\right)$. $N^m_{p)}(S)$ *is a subspace of* $W^m_{p)}(S)$ *and the set* $\Gamma_S\left(C^\infty\left(\bar{\Omega}\right)\right)$ *is dense in* $N^m_{p)}(S)$.

Before moving on to the next presentation let us define some function spaces. By ν we denote outer normal to $\partial\Omega$. Let $\Gamma = \Gamma_{\partial\Omega} \in \left[W^m_{p)}(\Omega)\ ;\ L_{p)}(\partial\Omega; d\sigma)\right]$ be the trace operator and set

$$\Gamma_k u = \Gamma\left(\frac{\partial^k u}{\partial \nu^k}\right)\ ,\ k = \overline{0, m-1}.$$

Let $\hat{s}_r = (s_k)_1^r\ :\ 0 \le s_1 < ... < s_r \le m-1,\ s_k \in Z_+\,,\ \forall\, k = \overline{1,r}$. Consider the product space

$$\left(L_{p)}(\partial\Omega; d\sigma)\right)^r = \underbrace{L_{p)}(\partial\Omega; d\sigma) \times ... \times L_{p)}(\partial\Omega; d\sigma)}_{r}.$$

According to $\hat{s}_r$, define

$$W^m_{p)}(\partial\Omega; \hat{s}_r) = \left\{(\varphi_k)_1^r \in L_{p)}(\partial\Omega; d\sigma)^r\ :\ \exists\, u \in W^m_{p)}(\Omega)\ \Rightarrow\ \Gamma_{s_k} u = \varphi_k\,,\ k = \overline{1,r}\right\}\ .$$

We call $W^m_{p)}(\partial\Omega; \hat{s}_r)$ *the trace space, corresponding to* $\hat{s}_r$.

Completely analogously to the above, we define the separable trace space $N_{p)}(\partial\Omega;\ \hat{s}_r)$, corresponding to $\hat{s}_r$. Let

$$\left(N_{p)}(\partial\Omega)\right)^{\ r} = \underbrace{N_{p)}(\partial\Omega) \times ... \times N_{p)}(\partial\Omega)}_{r}.$$

Accept

$$N^m_{p)}(\partial\Omega;\ \hat{s}_r) = \left\{(\varphi_k)_1^r \in \left(N_{p)}(\partial\Omega)\right)^r\ :\ \exists\, u \in N^m_{p)}(\Omega) \Rightarrow \Gamma_{s_k} u = \varphi_k,\ k = \overline{1,r}\right\}.$$

Analogously to the $W_{p)}^{m}(\partial\Omega)$ case, for $\hat{s}_r = (s_k)_1^r$ assume

$$\|(\varphi_k)_1^r\|_{W_{p)}^{m}(\partial\Omega;\hat{s}_r)} = \sum_{k=1}^{r} \|\varphi_k\|_{W_{p)}^{m}(\partial\Omega;\{s_k\})},$$

where

$$\|\varphi\|_{W_{p)}^{m}(\partial\Omega;\{k\})} = \inf\left\{\|u\|_{W_{p)}^{m-k}(\Omega)} : \forall\, u \in W_{p)}^{m}(\Omega) \Rightarrow \Gamma u = \varphi\right\}.$$

Along with this, we also introduce the trace spaces $W_{p)}^{m}(\partial\Omega;\{k\})$, $\forall\, k = \overline{0, m-1}$. So, it is valid

$$W_{p)}^{m}(\partial\Omega;\hat{s}_r) = \bigcap_{k=1}^{r} W_{p)}^{m}(\partial\Omega;\{s_k\}), \tag{8}$$

and moreover, it holds

$$\|(\varphi_k)_1^r\|_{W_{p)}^{m}(\partial\Omega;\hat{s}_r)} \le r \max_{k=\overline{1,r}} \|\varphi_k\|_{W_{p)}^{m}(\partial\Omega;\{s_k\})}. \tag{9}$$

According to $\hat{s}_r$ define trace operator

$$\hat{\Gamma}_{\hat{s}_r} = (\Gamma_{s_k})_1^r : \ W_{p)}^{m}(\Omega) \to W_{p)}^{m}(\partial\Omega;\hat{s}_r).$$

We have

$$\left\|\hat{\Gamma}_{\hat{s}_r} u\right\|_{W_{p)}^{m}(\partial\Omega;\hat{s}_r)} = \|(\Gamma_{s_k} u)_1^r\|_{W_{p)}^{m}(\partial\Omega;\hat{s}_r)} \le r \max_{k=\overline{1,r}} \|\Gamma_{s_k} u\|_{W_{p)}^{m}(\partial\Omega;\{s_k\})} =$$

$$= r \max_{k=\overline{1,r}} \inf\left\{\|\vartheta\|_{W_{p)}^{m}(\partial\Omega)} : \Gamma_{s_k}\vartheta = \Gamma_{s_k} u\right\} \le r\,\|u\|_{W_{p)}^{m}(\Omega)}.$$

Since $C^{\infty}(\bar{\Omega})$ is dense in $N_{p)}^{m}(\Omega)$, then from here it directly follows that $\hat{\Gamma}_{\hat{s}_r}\left(C^{\infty}(\bar{\Omega})\right)$ is also dense in $N_{p)}^{m}(\partial\Omega;\hat{s}_r)$.

Therefore, it is valid the following

Statement 4. *Let $\Omega \subset R^n : \partial\Omega \in C^{(m)}$ be a bounded domain, $W_{p)}^{m}(\partial\Omega;\hat{s}_r)$ is a trace space defined by (8) and let $\hat{\Gamma}_{\hat{s}_r}$ be the corresponding trace operator. Then it holds $\hat{\Gamma}_{\hat{s}_r} \in \left[W_{p)}^{m}(\Omega) ; W_{p)}^{m}(\partial\Omega;\hat{s}_r)\right]$ and moreover, $\hat{\Gamma}_{\hat{s}_r}\left(C^{\infty}(\bar{\Omega})\right)$ is dense in $N_{p)}^{m}(\partial\Omega;\hat{s}_r)$.*

We also introduce the following subspace of $W_{p)}^{m}(\partial\Omega;\hat{s}_r)$. Denote by $\hat{N}_{p)}^{m}(\partial\Omega;\hat{s}_r)$ the closure of $(C_0^{\infty}(\partial\Omega))^{r} = \underbrace{C_0^{\infty}(\partial\Omega)\times ...\times C_0^{\infty}(\partial\Omega)}_{r}$ in $W_{p)}^{m}(\partial\Omega\,;\,\hat{s}_r)$. Since, $(C_0^{\infty}(\partial\Omega))^{r} \subset N_{p)}^{m}(\partial\Omega;\,\hat{s}_r)$, then it is evident that $\hat{N}_{p}^{m}(\partial\Omega\,;\,\hat{s}_r)$ is also a subspace of $N_{p)}^{m}(\partial\Omega\,;\,\hat{s}_r)$.

Define also the following subspace of $W_{p)}^{m}(\Omega)$:

$$\overset{0}{W}{}_{p)}^{m}(\Omega)=\left\{u\in W_{p)}^{m}(\Omega)\,:\,\Gamma(\partial^{\alpha}u)=0\,,\ \forall\alpha\in Z_{+}^{n}\,:\,|\alpha|\leq m-1\right\}.$$

In obtaining the main results we will also use the following grand version of the classical Minkowski inequality.

Statement 5. *Let $\Omega_1\subset R^n$ & $\Omega_2\subset R^m$: $|\Omega_2|<+\infty$, be some domains and $f:\Omega_1\times\Omega_2\to C$ be a measurable function such that $f(x;\cdot)\in L_{p)}(\Omega_2)$, for a.e. $x\in\Omega_1$. Then it holds $(1\leq q<+\infty)$*

$$\left\|\left(\int_{\Omega_1}|f(x;\cdot)|^{q}\,dx\right)^{\frac{1}{q}}\right\|_{L_{p)}(\Omega_2)}\leq\left(\int_{\Omega_1}\|f(x;\cdot)\|_{L_{p)}(\Omega_2)}^{q}\,dx\right)^{\frac{1}{q}}.$$

At the end of this section, we present a result from the monograph [4] that will be used in the derivation of the main results.

Theorem 6. *[4] Let T be a sublinear operator and $T\in[L_p(\Omega)]\bigcap[L_{p-\varepsilon_0}(\Omega)]$, for some $p\in(1,+\infty)$ & $\varepsilon_0\in(0,p-1)$. Then $T\in\left[L_{p)}(\Omega)\right]$.*

3. Poisson type integrals

Consider the following elliptic operator with the constant coefficients

$$L_0u=\sum_{|\alpha|=m}a_{\alpha}^{0}\partial^{\alpha}u,$$

and the boundary operators

$$b_i^0u=\sum_{|\alpha|=m_i}b_{\alpha;i}^{0}\partial^{\alpha}u,i=\overline{1,r}\,;\ \ 0\leq m_1<...<m_r\leq m-1\,,$$

with the constant coefficients $\left\{b_{\alpha;i}^{0}\right\}$. Let $\Omega'\subset R^{n-1}$ be some bounded domain and $T\in(0,+\infty)$ be some fixed number. Assume $\Omega=\Omega'\times(0,T)\subset R^n$ and for every point $P=(x;t)\in\Omega:\ x\in\Omega'\ \&\,t\in(0,T)$, accept $|P|=\sqrt{|x|^2+t^2}$, $|x|=$

$\sqrt{x_1^2+...+x_{n-1}^2}$, where $x=(x_1,...,x_{n-1})$. Let $(\varphi_i)_1^r \in \left(L_{p)}(\Omega';dx)\right)^r$ be given functions and consider the following BVP

$$\left.\begin{array}{l} L_0u=0\,, \\ b_i^0u=\varphi_i\,,\ i=\overline{1,r}\,. \end{array}\right\} \tag{10}$$

Formally, as a solution of the BVP (10) we mean the function $u\in N_{p)}^m(\Omega)$, for which $b_i^0u=\varphi_i\,,\ i=\overline{1,r}$, and it holds $(L_0u)(P)=0$, a.e. $P\in\Omega$.

Let's prove the following main

Theorem 7. *Let $(\varphi_k)_1^r \subset W_{p)}^m(\Omega';\hat{s}_r)$, $1<p<+\infty$, and assume that regarding the system of operators $\left\{L_0;\, b_i^0,\, i=\overline{1,r}\right\}$ the Condition (B) holds and $\{K_i(\cdot;\cdot)\}_{i=\overline{1,r}}$ be corresponding to BVP (10) Poisson's kernels. Then the function*

$$u(x;t)=\sum_{i=1}^r\int_{\Omega'}K_i(x-y;t)\ \varphi_i(y)\,dy=\sum_{i=1}^r\left(K_i*\varphi_i\right)(x;t)\,, \tag{11}$$

is a solution of the BVP (10) in $W_{p)}^m(\Omega)$.

We will divide the proof of this theorem into the following stages.

Step I. Let $f\in L_{p)}(a,b)\ ,\ 1<p<+\infty$, where $(a,b)\subset R$ is a finite interval, and consider

$$(Pf)\ (x;y)=V(x;y)=\frac{1}{\pi}\int_a^b\frac{x-t}{(x-t)^2+y^2}f(t)\,dt\,,\ y>0. \tag{12}$$

It is valid the following

Proposition 1. *There exists a constant M_p independent of y and f, such that*

$$\|V(\cdot;y)\|_{L_{p)}(a,b)}\le M_p\|f\|_{L_{p)}(a,b)}\,,\forall\, y>0\,,\ \forall\, f\in L_{p)}(a,b)\,. \tag{13}$$

First, let us preface the proof of this proposition the following

Lemma 2. *Let $\sigma\in(0,p-1)$ be an arbitrary number. Then there exists a constant M_σ, independent of $\varepsilon\in(0,\sigma]$ and $f\in L_{p-\varepsilon}(a,b)$, such that*

$$\|(Pf)(\cdot;y)\|_{L_{p-\varepsilon}(a,b)}\le M_\sigma\|f\|_{L_{p-\varepsilon}(a,b)}\,,\forall y>0,\ \forall f\in L_{p-\varepsilon}(a,b)\,.$$

Proof. Let $\sigma\in(0,p-1)$ be any fixed number. By the results of [1] there exists positive constants $M_k,\ k=1,2$; independent of y and f, such that

$$\|(Pf)(\cdot;y)\|_{L_{p-\sigma}(a,b)}\le M_1\|f\|_{L_{p-\sigma}(a,b)}\,,$$

$$\|(Pf)(\cdot;y)\|_{L_p(a,b)} \le M_2 \|f\|_{L_p(a,b)}, \forall\, y>0,\ \forall\, f \in L_p(a,b).$$

From this, by the Riesz-Thorin interpolation theorem, we have

$$\|(Pf)(\cdot;y)\|_{L_{p-\varepsilon}(a,b)} \le M_1^{1-t_\varepsilon} M_2^{t_\varepsilon} \|f\|_{L_{p-\varepsilon}(a,b)},$$

where $t_\varepsilon \in (0,1) : \frac{1}{p-\varepsilon} = \frac{1-t_\varepsilon}{p-\sigma} + \frac{t_\varepsilon}{p}$. It is evident that

$$M_\sigma = \sup_{0<\varepsilon\le\sigma} M_1^{1-t_\varepsilon} M_2^{t_\varepsilon} < +\infty.$$

Proof of Proposition 1. Take $\sigma \in (0, p-1)$ and fix it. Applying Holder's inequality for $\varepsilon \in (\sigma, p-1)$ we have

$$\|(Pf)(\cdot;y)\|_{L_{p-\varepsilon}(a,b)} \le d^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}} \|(Pf)(\cdot;y)\|_{L_{p-\sigma}(a,b)},$$

where $d = b-a$. Taking into account the relations

$$0 < \frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)} < \frac{p-1-\sigma}{p-\sigma},\ \forall \varepsilon \in (\sigma, p-1)\ \ \&\ \ (p-1)\,\sigma^{-\frac{1}{p-\sigma}} > 1,$$

and applying Lemma 2 from the above we obtain

$$\|(Pf)(\cdot;y)\|_{L_{p)}(a,b)} = \max\left\{ \sup_{0<\varepsilon\le\sigma} \varepsilon^{\frac{1}{p-\varepsilon}} \|(Pf)(\cdot;y)\|_{L_{p-\varepsilon}(a,b)}\ ;\right.$$

$$\left. \sup_{\sigma<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|(Pf)(\cdot;y)\|_{L_{p-\varepsilon}(a,b)} \right\} \le \max\left\{ M_\sigma \sup_{0<\varepsilon\le\sigma} \varepsilon^{\frac{1}{p-\varepsilon}} \|f\|_{L_{p-\varepsilon}(a,b)}\ ;\right.$$

$$\left. \sup_{\sigma<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \sigma^{-\frac{1}{p-\sigma}} \sigma^{\frac{1}{p-\sigma}} d^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}} \|(Pf)(\cdot;y)\|_{L_{p-\sigma}(a,b)} \right\} \le$$

$$\le \max\left\{ M_\sigma \sup_{0<\varepsilon\le\sigma} \varepsilon^{\frac{1}{p-\varepsilon}} \|f\|_{L_{p-\varepsilon}(a,b)}\ ;\right.$$

$$\left. \sup_{\sigma<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \sigma^{-\frac{1}{p-\sigma}} (1+d)^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}} \sup_{0<\varepsilon\le\sigma} \varepsilon^{\frac{1}{p-\varepsilon}} \|(Pf)(\cdot;y)\|_{L_{p-\varepsilon}(a,b)} \right\}.$$

Since, $\sup\limits_{0<\sigma;\varepsilon<p-1} (1+d)^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}} < +\infty$, from here it directly follows that

$$\|(Pf)(\cdot;y)\|_{L_{p)}(a,b)} \le c_\sigma \|f\|_{L_{p)}(a,b)}, \forall\, y>0,\ \forall\, f \in L_{p)}(a,b),$$

where c_σ is a constant independent of y and f.

Proposition 1 is proved.

Step II. In the following we assume that the kernel $K(x;t)$ satisfies the following conditions.

(i) $K(\cdot;\cdot)$ is homogeneous of degree $(-n+1)$ and has the form

$$K(x;t)=\frac{\omega\left(\frac{x}{|P|};\frac{t}{|P|}\right)}{\left(|x|^2+t^2\right)^{\frac{n-1}{2}}}, \tag{14}$$

where $P=(x;t)\in R^n; B_r^+=\left\{P\in R_+^n:|P|=r\right\}; B_r'=\left\{x\in R^{n-1}:|x|<r\right\}$; $P'=(x;0)\in R^{n-1}$;

(ii) $\omega(\cdot;\cdot)$ is continuous on B_1^+ and there exists constants $c>0\,\&\,\alpha'\in(0,1]$, such that

$$\left.\begin{array}{l}|\omega(P')-\omega(Q)|\le c\,|P'Q|^{\alpha'}\ ,\\ \max\limits_{B_1^+}|\omega|\le c\ ,\end{array}\right\}$$

for $\forall\, P'=(x;0)\in B_1^+ \& Q=(y;t)\in B_1^+$, where $|P'Q|$ denotes the geodetic distance between points P' and Q on B_1^+;

(iii)

$$\int_{|x|=1}\omega(x;0)\,d\sigma_x=0, \tag{15}$$

where $d\sigma_x$ is the area element of sphere $\{x\in R^{n-1}:|x|=1\}$.

Let's prove the following

Proposition 2. *Let* $u\in \overset{0}{W}{}_{p)}^{m}(\Omega)\ ,\ 1<p<+\infty$. *Then*

$$\|u\|_{W_{p)}^m(\Omega)}\le c\,\|L_0u\|_{L_{p)}(\Omega)}, \tag{16}$$

where c *is a constant independent of* u.

Proof. Let Γ_0 be a fundamental solution of $L_0u=0$. Then it is known that (see, e.g. the work [1], p.64 and p.128) $\partial^\alpha\Gamma_0$, for $\forall\,\alpha\in Z_+^n\ :\ |\alpha|=m$, is a Calderon-Zygmund type kernel and as a result the singular integral $\partial^m\Gamma_0 * f$ is bounded in $L_{p)}(\Omega)$ (see, e.g. the work [4]), where $\partial^m\Gamma_0=\sum_{|\alpha|=m}\partial^\alpha\Gamma_0$. Let $u\in \overset{0}{W}{}_{p)}^{m}(\Omega)\Rightarrow u\in \overset{0}{W}{}_{p-\varepsilon_0}^{m}(\Omega)\ ,\ \forall\varepsilon_0\in(0,p-1)$. According to classical results (see, e.g. [1], p.128), we have

$$\partial^m u=\partial^m\Gamma_0 * L_0u+const\,u,\ \text{a.e. in }\Omega.$$

From here the validity of (16) follows directly. Since, in $\overset{\circ}{W}{}^{m}_{p)}(\Omega)$ the norms $\|u\|_{W^m_{p)}}(\Omega)$ and $\sum_{|\alpha|=m}\|\partial^\alpha u\|_{L_{p)}(\Omega)}$ are equivalent.

For obtaining Schauder type estimates the following theorem play a key role.

Theorem 8. *Let the kernel $K(\cdot;\cdot)$ have the form (14), where the function $\omega(\cdot)$ satisfies the conditions (i)-(iii). Consider the following Poisson type integral*

$$(Pf)(x;t) = u(x;t) = \int_{\Omega'} K(x-y;t)\, f(y)\, dy. \tag{17}$$

Then there exists a constant $c>0$ such that

$$\|u(\cdot;t)\|_{L_{p)}(\Omega';dx)} \le c\|f\|_{L_{p)}(\Omega';dx)}, \forall t\in(0,T)\ ,\ \forall f\in L_{p)}\left(\Omega';dx\right).$$

This theorem is, in some sense, connected with the following theorem.

Theorem 9. *Let the kernel $k(\cdot)$ have the form ($x=(x_1,...,x_{n-1})$)*

$$k(x) = |x|^{-n+1}\,\omega\left(\frac{x}{|x|}\right), \tag{18}$$

where the function $\omega(\cdot)$ satisfies the condition (iii) and

$$\left(\int_{|x|=1}|\omega(x)|^q\, d\sigma_x\right)^{\frac{1}{q}} = R_q<\infty\,,\ q>1\,.$$

Then for $\forall\, f\in \overset{0}{W}{}^{1}_{p)}(\Omega')$ exists limit in $L_{p)}(\Omega';dx)$:

$$u(x) = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon(x)} k(x-y)\ f(y)\, dy, \tag{19}$$

where $\Omega'_\varepsilon(x) = \Omega'\backslash B'_\varepsilon(x)$. Moreover, it holds

$$\|u\|_{L_{p)}(\Omega';dx)} \le c\|f\|_{L_{p)}(\Omega';dx)},$$

where c independent of f, constant.

The following theorem is a generalization of the above theorems.

Theorem 10. *Let the kernel $K(\cdot;\cdot)$ have the form* (14)*, where the function $\omega(\cdot)$ satisfies the conditions (i)-(iii). Let $f \in L_{p)}(\Omega';dx)$ and consider the Poisson type integral* (17)*. Then :*

(α) *there exists a constant $c>0$, independent of t and f, such that*

$$\|u(\cdot;t)\|_{L_{p)}(\Omega';dx)} \le c\|f\|_{L_{p)}(\Omega';dx)}\ ,\ \forall\, t>0\,;$$

(β) *in $N_{p)}(\Omega')$ exists the limit*

$$u^0(x) = \lim_{t\to 0} u(x;t)\,;$$

(γ) *there exists a constant $c>0$, independent of f, such that*

$$\left\|u^0\right\|_{L_{p)}(\Omega';dx)} \le c\|f\|_{L_{p)}(\Omega';dx)}\,,\forall\, f \in N_{p)}\left(\Omega'\right).$$

Step III (proof of the above theorems). Before beginning the proof of these theorems, let us present the following

Lemma 3. *Let $\omega \in L_1(R)$ and for $f \in L_{p)}(J)$ consider*

$$(Gf)(x) = \int_R \omega(y)\,\tilde{f}(x-y)\,dy\,,\ x\in J,$$

where $J \in R$ is a finite interval and

$$\tilde{f}(x) = \begin{cases} f(x)\ ,\ x\in J\,, \\ 0\,, \qquad x\notin J\,. \end{cases}$$

Then $G \in \left[L_{p)}(J)\right]$.

Proof. Let $\varepsilon \in (0, p-1)$ and $p_\varepsilon = p-\varepsilon$. According to Statement 4 (Minkowski's inequality) we take $q=1$, $\Omega_2 = J$ & $\Omega_1 = R$, and applying this statement we have

$$\|Gf\|_{L_{p_\varepsilon}(J)} \le \int_R |\omega(y)|\ \left\|\tilde{f}(\cdot - y)\right\|_{L_{p_\varepsilon}(J)} dy =$$

$$= \int_R |\omega(y)|\ dy\ \ \|f\|_{L_{p_\varepsilon}(J)} \Rightarrow \|Gf\|_{L_{p)}(J)} \le \|\omega\|_{L_1(R)}\|f\|_{L_{p)}(J)}\,.$$

Consider the existence of the limit (19) for functions $f \in \overset{0}{W}{}^1_{p)}(\Omega')$. We extend the function f to all of R^{n-1} by zero and we will continue to denote the

extended function by f. Let $\varepsilon \in (0, p-1)$ be some number. It is evident that $f \in W^1_{p-\varepsilon}\left(R^{n-1}\right)$. Then it holds

$$f(x-y) - f(x) = \int_0^{|y|} \frac{d}{dt} f\left(x - t\frac{y}{|y|}\right) dt, \quad \text{a.e. } x; y \in R^{n-1}.$$

Applying Minkovski's inequality, we have

$$\|f(\cdot - y) - f(\cdot)\|_{L_{p)}(R^{n-1})} \leq M|y|, \tag{20}$$

where

$$M = \sum_{|\alpha|=1} \|\partial^\alpha f\|_{L_{p)}(\Omega';dx)} \leq \|f\|_{W^1_{p)}(\Omega')}.$$

Let

$$u_\varepsilon(x) = \int_{\Omega'_\varepsilon(x)} k(x-y) f(y) dy.$$

Then taking into account the relation (20) for $u_\varepsilon(\cdot)$ we obtain

$$u_{\varepsilon_2}(x) - u_{\varepsilon_1}(x) = \int_{\varepsilon_2<|y|<\varepsilon_1} \frac{\omega\left(\frac{y}{|y|^{n-1}}\right)}{|y|^{n-1}} \left[f(x-y) - f(x)\right] dy \Rightarrow$$

$$\Rightarrow \|u_{\varepsilon_2} - u_{\varepsilon_1}\|_{L_{p)}(\Omega')} \leq M(\varepsilon_1 - \varepsilon_2) R_1 \to 0, \ \varepsilon_1; \varepsilon_2 \to 0,$$

where

$$R_1 = \int_{|x|=1} |\omega(x)| d\sigma_x.$$

From here it follows that if $f \in \overset{0}{W}_{p)}(\Omega'; dx)$, then the limit (19) exists in $L_{p)}(\Omega')$.

Let us prove that Theorem 9 follows from Theorem 10. To do this, it is sufficient to prove the following

Lemma 4. *Let the kernel $\omega\left(\frac{x}{|x|}\right)$ satisfy the conditions (i)-(iii) and accept*

$$u(x;t) = \int_{\Omega'} \frac{\omega\left(\frac{x-y}{|x-y|}\right)}{\left(|x-y|^2 + t^2\right)^{\frac{n-1}{2}}} f(y)\, dy,$$

$$\vartheta_\varepsilon(x) = \int_{|x-y|>\varepsilon} \omega\left(\frac{x-y}{|x-y|}\right) |x-y|^{-n+1} f(y)\, dy.$$

Then

$$\lim_{\varepsilon \to 0} \left\| \vartheta_\varepsilon(\cdot) - u\left(\cdot; \varepsilon^2\right) \right\|_{L_{p)}(\Omega';dx)} = 0.$$

Proof. We have

$$\vartheta_\varepsilon(x)-u\left(x;\varepsilon^2\right)=\int_{|y|>\varepsilon}\omega\left(\frac{y}{|y|}\right)\left[\frac{1}{|y|^n}-\frac{1}{\left(|y|^2+\varepsilon^4\right)^{\frac{n-1}{2}}}\right]f(x-y)\,dy-$$

$$-\int_{|y|\le\varepsilon}\omega\left(\frac{y}{|y|}\right)\frac{1}{\left(|y|^2+\varepsilon^4\right)^{\frac{n-1}{2}}}f(x-y)\,dy=:J_1^{(\varepsilon)}(x)-J_2^{(\varepsilon)}(x).$$

Let us show that $\lim\limits_{\varepsilon\to0}\left\|J_1^{(\varepsilon)}(\cdot)\right\|_{L_{p)}(\Omega';dx)}=0$. By virtue of Lemma 3 it is sufficient to show that the $L_1(R)$-norm of the kernel

$$G_\varepsilon(y)=\begin{cases}\omega\left(\frac{y}{|y|}\right)\left[\frac{1}{|y|^n}-\frac{1}{\left(|y|^2+\varepsilon^4\right)^{\frac{n-1}{2}}}\right], & |y|>\varepsilon,\\ 0, & |y|\le\varepsilon,\end{cases}$$

tends to zero as $\varepsilon\to0$. This fact is established in [1] (see, p.150).

For $L_{p)}(\Omega';dx)$ norm of integral $J_2^{(\varepsilon)}(\cdot)$ we have

$$\left\|J_2^{(\varepsilon)}(\cdot)\right\|_{L_{p)}(R^{n-1})}\le$$

$$\le\int_{|y|<\varepsilon}\left|w\left(\frac{y}{|y|}\right)\right|\frac{1}{\left(|y|^2+\varepsilon^4\right)^{\frac{n-1}{2}}}\left\|f(\cdot-y)-f(\cdot)\right\|_{L_{p)}(R^{n-1})}dy.$$

Taking into account the inequality (20) from here we obtain

$$\left\|J_2^{(\varepsilon)}(\cdot)\right\|_{L_{p)}(R^{n-1})}\le$$

$$\le M\int_{|y|<\varepsilon}\left|\omega\left(\frac{y}{|y|}\right)\right|\frac{|y|}{\left(|y|^2+\varepsilon^4\right)^{\frac{n-1}{2}}}dy\le MR_1\varepsilon\to0,\ \varepsilon\to0.$$

Let us summarize the obtained results. From Lemma 4 it follows that there exists the $\lim\limits_{t\to0}u(x;t)$ in $L_{p)}(\Omega';dx)$ for the functions $f\in\overset{0}{W}{}^1_{p)}(\Omega')$. Thereby, the proposition (β) of Theorem 10 is proved. Since, $\lim\limits_{\varepsilon\to0}\vartheta_\varepsilon(x)=\lim\limits_{t\to0}u(x;t)$, then Theorem 9 follows from the proposition (γ) of Theorem 10. Moreover, it is obvious

that Theorem 8 follows from the proposition (α) of Theorem 10. Therefore, we proceed to the proof of Theorem 10.

So, let

$$u(x;t)=\int_{R^{n-1}}\frac{\omega\left(\frac{x-y}{|x-y|}\right)}{\left(|x-y|^2+t^2\right)^{\frac{n-1}{2}}}\,f(y)\,dy,$$

where $f\in L_{p)}\left(R^{n-1}\right)\ :\ \mathrm{supp}f\subset\overline{\Omega'}$. By results of the work [1] (see, Theorem A2.1', p.144) there exists the constant $c>0$, independent of t and f, such that

$$\|u(\cdot;t)\|_{L_p(R^{n-1})}\le c\,\|f\|_{L_p(\Omega';dx)},$$

$$\|u(\cdot;t)\|_{L_{p-\varepsilon_0}(R^{n-1})}\le c\,\|f\|_{L_{p-\varepsilon_0}(\Omega')},\forall\,t\in(0,T)\ ,\ \forall\,f\in L_{p)}\left(\Omega';dx\right),$$

where $\varepsilon_0\in(0,p-1)$ is some number. Then, using the scheme of the proof of Theorem 6 and completely analogously to Proposition 1 we establish that there exists a constant c_1, also independent of t and f, such that

$$\|u(\cdot;t)\|_{L_{p)}(R^{n-1})}\le c_1\,\|f\|_{L_{p)}(\Omega';dx)},\forall\,t\in(0,T)\ ,\ \forall\,f\in L_{p)}\left(\Omega';dx\right).$$

It remains to prove propositions (β) and (γ) of Theorem 10 for functions $f\in N_{p)}(\Omega')$. Since, the set $C_0^\infty(\Omega')$ is dense in $N_{p)}(\Omega')$, then these parts are proved completely analogously to [1] (see, p.145). Therefore, this completes the proof of Theorem 8-10.

Step IV. In this stage we will establish some facts, that play an essential role in the proof of Theorem 7.

Lemma 5. *Let $G(x;t)$ be a measurable function on R^n_+, that satisfies the inequality*

$$|G(P)|\le\frac{\omega\left(\frac{P}{|P|}\right)}{|P|^n}\ ,\ \forall\,P=(x;t)\in R^n_+,$$

where ω is a nonnegative function defined on the hemisphere $\Omega_1^+=\left\{P\in R^n_+:|P|=1\right\}$, and moreover

$$\int_{\Omega_1^+}\omega^r\,d\sigma<c,$$

for some $r>1$ and $c>0$. Consider

$$u(x;t)=\int_{\Omega'}\int_0^{+\infty}G(x-y;\,t+s)\;\vartheta(y;s)\;dy\,ds,$$

where $\vartheta(\cdot;t) \in L_{p)}(\Omega';dx)$, $\forall t>0$, *and it holds* $p \geq \frac{r}{r-1}$. *Then* $u \in L_{p)}(\Omega)$ *and*

$$\|u\|_{L_{p)}(\Omega)} \leq c\,\|\vartheta\|_{L_{p)}(\Omega)},$$

where $\Omega = \Omega' \times (0,T)$, $\Omega' \subset R^{n-1}$, *is a bounded domain and the constant* $c>0$ *depends only on* r, c, p *and* n.

Proof. Define the kernel $K(x;t)$, which is homogenous of degree $(-n)$ in R^n for points $P=(x;t):\ K(P)=\omega\left(\frac{P}{|P|}\right)|P|^{-n}$ for $t>0$, and K is an odd function of $t \in (-\infty,+\infty)$. Let $\vartheta(\cdot;t) \in L_{p)}(\Omega';dx)$ and extend $\vartheta(x;t)$ to all of R^n by zero outside of $\Omega \subset R^n$. Then we have

$$u(x;t) \leq \int_{R^n_+} K(y;\ s+t)\ |\vartheta(x-y;s)|\,dy\,ds.$$

Applying Minkowski's inequality (Statement 5) we obtain

$$\|u(\cdot;t)\|_{L_{p)}(\Omega';dx)} \leq \int_{R^n_+} K(y;\ s+t)\ \|\vartheta(\cdot-y;s)\|_{L_{p)}(\Omega';dx)}\ dy\,ds =$$

$$= \int_{R^n_+} K(y;\ s+t)\ \|\vartheta(\cdot;s)\|_{L_{p)}(\Omega';dx)}\ dy\,ds =$$

$$= \int_{\Omega_1^+} \omega(\xi)\,d\sigma_\xi \int_0^{+\infty} \frac{\|\vartheta(\cdot;s)\|_{L_{p)}(\Omega';dx)}}{s+t}\,ds.$$

Assume

$$V(t) = \begin{cases} \|\vartheta(\cdot;t)\|_{L_{p)}(\Omega';dx)}\ ,\ t>0\,, \\ \\ 0\,, \qquad\qquad t<0\,. \end{cases}$$

Then from the above it follows that

$$\|u(\cdot;t)\|_{L_{p)}(\Omega';dx)} \leq c\int_{-\infty}^{+\infty} \frac{V(s)\ ds}{s+t}\,.$$

From here applying the Theorem 6 we directly obtain

$$\|u\|_{L_{p)}(\Omega)} \leq c\,\|\vartheta\|_{L_{p)}(\Omega)}\,.$$

Using this lemma let's prove the following

Theorem 11. *Let*

$$u(x;t)=\int_{\Omega'} K(x-y;t)\ f(y)\ dy\ ,\ t>0\,,$$

where the kernel $K(\cdot;\cdot)$ *has a form* (14) *,* ∂K; $\partial_t^2 K$, *are continuous in* R_+^n, $\sup\limits_{P\in\Omega_1^+}|K(P)|<+\infty$, *and it holds the condition* (15). *Then there exists the constant* $c>0$, *such that*

$$\sum_{|\alpha|=1}\|\partial^\alpha u\|_{L_{p)}(\Omega)}\le c\,\|f\|_{W^1_{p)}(\Omega';dx)}\ ,\ \forall\, f\in W^1_{p)}\left(\Omega';dx\right).$$

For this reason first, let us prove the following simple version of Theorem 11.

Theorem 12. *Let all the conditions of Theorem 11 hold, possibly except for condition* (15). *Then there exists a constant* $c>0$ *such that, for* $\forall\,\alpha:\ |\alpha|=1$ *it holds*

$$\|\partial_x^\alpha u\|_{L_{p)}(\Omega)}\le c\,\|f\|_{W^1_{p)}(\Omega';dx)}\ ,\ \forall\, f\in W^1_{p)}\left(\Omega';dx\right)\ .$$

Proof. Let $f\in W^1_{p)}(\Omega';dx)$ and $\Gamma_{\Omega'}\in\left[W^1_{p)}(\Omega)\ ;\ W^1_{p)}(\Omega';dx)\right]$ be a corresponding trace operator. So, we have

$$\|f\|_{W^1_{p)}(\Omega';dx)}=\inf\left\{\|\vartheta\|_{W^1_{p)}(\Omega)}\ :\ \vartheta\in W^1_{p)}(\Omega)\ \&\ \Gamma_{\Omega'}\vartheta=f\right\}.$$

Then there exists $\vartheta\in W^1_{p)}(\Omega)$ such that $\vartheta(x;0)=:(\Gamma_{\Omega'}\vartheta)(x;0)=f(x)\,,\ x\in\Omega'$, and

$$\|\vartheta\|_{W^1_{p)}(\Omega)}\le 2\,\|f\|_{W^1_{p)}(\Omega';dx)}\,. \tag{21}$$

Without loss of generality, let us assume that $\|f\|_{W^1_{p)}(\Omega';d\sigma)}\ \|\vartheta\|_{W^1_{p)}(\Omega)}\ne 0$. Take $\varepsilon_0\in(0,p-1)$ such that $\|\vartheta\|_{W^1_{p-\varepsilon_0}(\Omega)}\le 2\,\|\vartheta\|_{W^1_{p)}(\Omega)}$. Therefore

$$\|\vartheta\|_{W^1_{p-\varepsilon_0}(\Omega)}\le 4\ \|f\|_{W^1_{p)}(\Omega';dx)}\le 4\ \|f\|_{W^1_{p-\varepsilon_0}(\Omega';dx)}\,.$$

Assume

$$\tilde f(x)=\begin{cases} f(x)\ ,\ x\in\Omega'\,,\\ 0\quad ,\ x\in R^{n-1}\backslash\Omega'\,.\end{cases}$$

Consequently , we have

$$u(x;t)=\int_{R^{n-1}} K(x-y;t)\ \tilde f(y)\,dy\,,\ t>0.$$

Then by the same way as in classical case (see, [1], p.167, Theorem A3.1) it is established the validity of the relation $\left(\partial_i = \frac{\partial}{x_i}\,,\ i = \overline{1, n-1}\right)$

$$-\partial_i u = \int_0^{+\infty} \int_{\Omega'} \partial_s \vartheta\,(y;s)\ \partial_i K\,(x-y;t+s)\ dy\,ds+$$

$$+\int_0^{+\infty} \int_{\Omega'} \partial_i \vartheta\,(y;s)\ \partial_s\, K\,(x-y;t+s)\ dy\,ds.$$

Since, the kernels $\partial_i K\,;\ \partial_t K$, satisfy the conditions of Lemma 5, applying this lemma we obtain

$$\|\partial_i u\|_{L_{p)}(\Omega)} \le c \sum_{|\alpha|=1} \|\partial^\alpha \vartheta\|_{L_{p)}(\Omega)} \le\ c\,\|\vartheta\|_{W^1_{p)}(\Omega)}\ \le c\ \|f\|_{W^1_{p)}(\Omega';dx)}\,,$$

consequently

$$\sum_{|\alpha|=1} \|\partial^\alpha u\|_{L_{p)}(\Omega)} \le\ c\ \|f\|_{W^1_{p)}(\Omega';dx)}\,.$$

Proof of Theorem 11. By virtue of Theorem 12 it is sufficient to prove the estimate

$$\|\partial_t u\|_{L_{p)}(\Omega)} \le c\ \|f\|_{W^1_{p)}(\Omega';dx)}\,.$$

Based on the condition (21) for $\vartheta \in W^1_{p)}(\Omega)$, analogously to [1] (see, p.168) as above we obtain

$$-\partial_t u\,(x;t) = \int_0^{+\infty} \int_{\Omega'} \partial_s \vartheta\,(y;s)\ \partial_t K\,(x-y;t+s)\ dy\,ds+$$

$$+\int_0^{+\infty} \int_{\Omega'} \vartheta\,(y;s)\ \partial_t^2 K\,(x-y;t+s)\ dy\,ds = J_1\,(x;t) + J_2\,(x;t)\,.$$

Since, the kernel $\partial_t K$ satisfies the conditions of the Lemma 5, then as above we obtain

$$\|J_1\|_{L_{p)}(\Omega)} \le c\ \|\partial_s \vartheta\|_{L_{p)}(\Omega)}\ \le c\ \|\vartheta\|_{W^1_{p)}(\Omega)} \le c\ \|f\|_{W^1_{p)}(\Omega';dx)}\,.$$

And now, let's establish the following estimate

$$\|J_2\|_{L_{p)}(\Omega)} \le c \sum_{i=1}^{n-1} \|\partial_i \vartheta\|_{L_{p)}(\Omega)}\,, \tag{22}$$

where

$$J_2\,(x;t) = \int_0^{+\infty} \int_{\Omega'} \vartheta\,(y;s)\ \partial_t^2 K\,(x-y;t+s)\ dy\,ds. \tag{23}$$

It holds (see, [1] , p.169, formula (A3.8))

$$\int_{R^{n-1}} \partial_t^2 K(y;t)\, dy = 0\,,\ t > 0.$$

Then the estimate (22) follows from the following

Lemma 6. *Let $J(P)$ be a kernel, continuous in R^{n-1}, that have homogeneous degree $(-n+1)$ at $t > 0$, and $\sup\limits_{|P|=1} |J(P)| < +\infty$, and satisfies*

$$\int_{R^{n-1}} J(x;t)\, dx = 0\,.$$

Let $\sum_{|\alpha|=1} \|\partial^\alpha \vartheta\|_{L_{p)}(\Omega)} < +\infty$. Then the function

$$I(x;t) = \int_0^{+\infty} \int_{\Omega'} \vartheta(y;s)\ J(x-y;t+s)\ dy\, ds,$$

satisfies the inequality

$$\|I\|_{L_{p)}(\Omega)} \le c \sum_{i=1}^{n-1} \|\partial_i \vartheta\|_{L_{p)}(\Omega)}\,,$$

where the constant c depends only on p and n.

The proof of this lemma is similar to that of Lemma A3.1 in [1] (see p. 170), since it reduces to Lemma 5 and therefore, we omit it. Then the estimate (22) is obtained by applying this lemma to the integral (23) with kernel $J(P) = \partial_t^2 K(P)$. As a result, the proof of Theorem 11 is complete.

Step V (Proof of Theorem 7). Using the above obtained results let's prove the Theorem 7.

It is obvious that $\Omega' \subset \partial\Omega$ is a part of the boundary $\partial\Omega$. In the sequel by $\Gamma_k u = \Gamma\left(\frac{\partial^k u}{\partial \nu^k}\right)$, $k = \overline{0, m-1}$; we mean the trace operator $\Gamma_k \in \left[W_{p)}^m(\Omega)\ ;\ W_{p)}^m(\Omega'; dx)\right]$, corresponding to the boundary part Ω', where ν is outer normal to Ω'. First, consider the case, when $\{\varphi_k\}_1^r \subset C_0^\infty(\Omega')$ and extend it to all of R^{n-1} by zero. Consider the following function

$$u(x;t) = \sum_{i=1}^{r} (K_i * \varphi_i)\ (x;t)\ ,\ (x;t) \in R_+^n. \tag{24}$$

By result of the work [1] (see, chapter I, &2) the kernels $\partial^m K_i\,,\ i = \overline{1, r}$; satisfy the conditions (i)-(iii). By virtue of the Theorem 1 (see, [1] , p.26,) the function

(24) is a solution of the BVP (10) in $C^{\infty}\left(\overline{R_{+}^{n}}\right)$. Consequently $u\left(P\right)$ is a solution of BVP also in $C^{\infty}\left(\bar{\Omega}\right)$, and as a result, in grand Sobolev space $W_{p)}^{m}\left(\Omega\right)$.

And now, consider the case, when $\left(\varphi_{k}\right)_{1}^{r} \in W_{p)}^{m}\left(\Omega';\hat{s}_{r}\right)$. Continue $\left(\varphi_{k}\right)_{1}^{r}$ to R^{n-1} by zero and consider again the function (24). It is evident that $\left(\varphi_{i}\right)_{1}^{r} \in W_{p-\varepsilon_{0}}^{m}\left(\Omega';\hat{s}_{r}\right)$, for every fixed $\varepsilon_{0} \in (0,p-1)$. From the characterization of the trace space $W_{p-\varepsilon_{0}}^{m}\left(R^{n-1};\hat{s}_{r}\right)$ (see, e.g. [28] [29, Theorem Gagliardo]) it directly follows that $\left(\varphi_{i}\right)_{1}^{r} \in W_{p-\varepsilon_{0}}^{m}\left(R^{n-1};\hat{s}_{r}\right)$. Then by results of [1] (see, Theorem 14.1, p.129) the function (24) is a solution of the BVP (10) in $W_{p-\varepsilon_{0}}^{m}\left(R_{+}^{n}\right)$. Consequently, it holds

$$\left(L_{0}u\right)\left(x;t\right)=0, \quad \text{a.e. } \left(x;t\right) \in R_{+}^{n}.$$

Then, it only remains to show that $u \in W_{p)}^{m}\left(\Omega\right)$ and that the boundary conditions are satisfied

$$\left(b_{i}^{0}u\right)_{1}^{r}=\left(\varphi_{i}\right)_{1}^{r},$$

in the sense of $W_{p)}^{m}\left(\Omega';\hat{s}_{r}\right)$.

In what follows, we need some relations concerning the kernels $K_{j}\left(\cdot;\cdot\right)$, $j=\overline{1,r}$. These kernels have the form (see, [1], p.27)

$$K_{j}\left(x;t\right)=\Delta_{x}^{\frac{n-1+q}{2}}K_{j;q}\left(x;t\right),$$

where q is an arbitrary positive integer having the same parity as $\left(n-1\right)$; $\Delta_{x}=\sum_{i=1}^{n}\partial_{i}^{2}$ is a Laplacian and the functions $K_{j;q}\left(x;t\right)$ are defined by expressions (7)', (7)", of the work [1] (see, p.27) . Then according to the Theorem 14.1 of the work [1] (see, p.129) for m-th order derivatives of $u\left(\cdot;\cdot\right)$ we have

$$\partial^{m}u\left(x;t\right)=\sum_{j=1}^{r}I_{j}\left(x;t\right),$$

where

$$I_{j}=\partial^{m}\Delta^{\frac{n-1+q-m+m_{j}}{2}}K_{j;q}*\varphi_{j},$$

if $m-m_{j}$ is even and

$$I_{j}=\sum_{i=1}^{n-1}\partial^{m}\partial_{i}\,\Delta^{\frac{n-1+q}{2}-k-1}K_{j;q}*\Delta^{k}\varphi_{j},$$

if $m-m_{j}=2k+1$ (here $q>m-m_{j}-n+1$).

As is shown in [1], I_j can be represented in the form (e.g. in case, when $m-m_j$ is even)

$$I_j = \sum_{k=1}^{n-1} \partial_k I_{j;k},$$

where

$$I_{j;k} = \partial^m \Delta^{\frac{n-1+q-m+m_j}{2}} K_{j;q} * \partial_k \Delta^{\frac{m-m_j-1}{2}} \varphi_j \ .$$

Since, the kernel

$$K(x;t) = \partial^{n+q+m_j-1} K_{j;q}(x;t),$$

satisfies the conditions of Theorem 11 (see, [1], Theorem 3.4 , p.50), then by this theorem we have

$$\|\partial^m u\|_{L_{p)}(\Omega)} \leq \sum_{j=1}^{r} \|I_j\|_{L_{p)}(\Omega)} \leq \sum_{j=1}^{r} \sum_{k=1}^{n-1} \|\partial_k I_{j;k}\|_{L_{p)}(\Omega)} \leq$$

$$\leq c \sum_{j=1}^{r} \sum_{k=1}^{n-1} \left\| \partial_k \Delta^{\frac{m-m_j-1}{2}} \varphi_j \right\|_{L_{p)}(\Omega';dx)} \leq c \sum_{j=1}^{r} \|\varphi_j\|_{W^m_{p)}(\Omega';\{m_j\})} . \tag{25}$$

On the other hand, from estimates (1) for kernels $K_i(\cdot;\cdot)$ follows

$$|K_i(P)| \leq c \frac{1+|\log|P||}{|P|^{n-m_i-1}}, \ i = \overline{1,r},$$

i.e. it have a weak singularities at $P=0$ in R^+_n and by results of the work [16] we have

$$\|u\|_{L_{p)}(\Omega)} \leq c \sum_{j=1}^{r} \|\varphi_j\|_{L_{p)}(\Omega';dx)} . \tag{26}$$

According to results of the work [15] (intermediate estimates for derivatives) from (25) and (26) follows that $u \in W^m_{p)}(\Omega)$.

The Theorem 7 is proved.

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