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Open-Consequences, Kernel Completions, and Graded Specialization in Intuitionistic Fuzzy Topology

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Abstract. We study open-consequence equivalence in intuitionistic fuzzy topological spaces, identifying intuitionistic fuzzy sets that have the same open supersets. This equivalence is characterized by an open-kernel operator whose fixed points form a canonical kernel completion of the topology and represent the induced quotient. We show that this completion has both frame and coframe structures. Using Heyting implication on Atanassov intuitionistic fuzzy values, we introduce a graded specialization preorder and relate its graded upward sets to the original topology and its kernel completion. We also establish compatibility with intuitionistic fuzzy continuous mappings and prove that κ-compactness is invariant under open-consequence equivalence. These results provide a unified lattice- and order-theoretic framework for intuitionistic fuzzy topology.

Key Words and Phrases: Intuitionistic fuzzy topology, open-superset equivalence, kernel operator, complete distributivity, graded specialization, fuzzy preorder, quotient frame

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1. Introduction

Atanassov's intuitionistic fuzzy sets encode positive membership, negative membership, and residual hesitation subject to a compatibility constraint [1]. Çoker introduced intuitionistic fuzzy topological spaces by replacing ordinary subsets with intuitionistic fuzzy sets while retaining the standard closure requirements for open families [2]. The resulting theory is also naturally viewed as an L-valued topology because intuitionistic fuzzy sets are precisely fuzzy sets valued in the complete lattice L^* of admissible membership–nonmembership pairs [16, 5].

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This lattice perspective is important: it separates results that genuinely use the two-coordinate intuitionistic structure from formal repackagings of general L-fuzzy phenomena; see the broader methodological discussion in [8].

Levine introduced an equivalence relation on subsets of a topological space by identifying sets that possess the same open supersets [11]. The intuitionistic fuzzy and soft analogues were studied in [7, 6]. Frame-theoretic links for intuitionistic fuzzy topologies were considered in [15], while relation- and point-based preorder representations appear in [17, 19, 13]. The construction developed here is different in object and scope: its frame is the fixed-point completion of the whole lattice $\mathrm{IF}(X)$ under open consequences, rather than the original frame of open sets, and its graded preorder is defined on the underlying universe by the canonical residual of L^*. Thus, the present paper does not claim the underlying equivalence relation, frame methods, or intuitionistic fuzzy preorder theory as new. Instead, it asks the structural questions left open at their intersection:

1. Which order generates the equivalence relation?
2. What algebraic object represents the quotient by equivalence?
3. How does the quotient interact with graded specialization preorders?
4. Is the construction functorial under intuitionistic fuzzy continuous mappings?
5. Which properties depend only on the open-consequence profile?

The answers require two ingredients that are largely absent from the earlier treatment. The first is complete distributivity. We show that L^*, and hence $\mathrm{IF}(X) = (L^*)^X$, is completely distributive. This allows a precise use of the unrestricted distributive law to prove that the open-kernel operator preserves arbitrary joins. Its fixed points therefore form a complete sublattice, not merely an abstract closure system. The second ingredient is the canonical Heyting implication of L^*. Previous formulations of intuitionistic fuzzy preorders have been developed on different underlying domains and through varying relation structures, most notably on the base universe and on collections of intuitionistic fuzzy points [17, 19, 13]. Using the residual implication canonically induced by the frame structure, we define a graded specialization relation that applies to arbitrary intuitionistic fuzzy values without requiring pointwise comparability. This formulation situates the construction naturally within the established theory of (L)-valued preorders and fuzzy topological spaces [9, 18].

Relation with general L-valued topology We emphasize that several ingredients used below belong to the general theory of L-valued topology. The lattice-theoretic viewpoint is consistent with [8], while the relationship between L-valued topologies and specialization preorders is developed in [9, 18]; related intuitionistic fuzzy preorder constructions appear in [19]. Accordingly, we do not claim these general constructions as new. The contribution of the present paper lies in their interaction with open-consequence equivalence, particularly the kernel completion, quotient representation, and their functorial and specialization structures. The Atanassov-specific part is the realization of this framework on the two-coordinate lattice L^*, including its explicit residual implication.

The main contributions of the paper are as follows.

(I) The open-kernel operator is proved to preserve arbitrary joins. Its fixed-point family $\mathcal{K}(\tau)$ is the smallest complete sublattice of $\mathrm{IF}(X)$ containing the topology τ, and it is completely distributive. Hence it is simultaneously a frame and a coframe.

(II) The quotient of all intuitionistic fuzzy sets by equality of open-superset profiles is represented canonically by $\mathcal{K}(\tau)$. We distinguish the consequence order, which corresponds to reverse inclusion of kernels, from the information order, which corresponds to ordinary inclusion.

(III) A graded specialization preorder Ω_τ is constructed from the canonical Heyting implication of L^*. The topology, its kernel completion, and the full Alexandrov topology of Ω_τ-upper sets induce the same specialization relation and satisfy

$$\tau \subseteq \mathcal{K}(\tau) \subseteq \mathrm{Up}(\Omega_\tau).$$

The second inclusion can be strict even for a finite Alexandrov intuitionistic fuzzy topology.

(IV) The assignment $\tau \mapsto \mathcal{K}(\tau)$ is contravariantly functorial under inverse images. The corresponding kernel operators satisfy a lax naturality inequality, and a subcategory of kernel-exact maps is identified on which quotient representatives are transported exactly.

(V) Compactness, Lindelöfness are extended to a general κ-compactness framework, which is shown to be invariant under open-consequence equivalence. For finite topologies, every equivalence class has a unique open representative and can be computed in linear time in the open-incidence table.

The paper is organized as follows. Section 2 develops the lattice and implication structure of intuitionistic fuzzy values. Section 3 introduces the consequence preorder and kernel. Section 4 proves the completion and quotient theorems. Section 5 develops graded specialization and the Alexandrov envelope. Section 6 establishes functoriality. Section 7 treats compactness invariants, while Section 8 gives finite algorithms and examples.

2. The lattice of intuitionistic fuzzy values

Let
$$L^* = \{(a,b) \in [0,1]^2 : a + b \leq 1\}$$
with the Atanassov order
$$(a,b) \leq_* (c,d) \quad \Longleftrightarrow \quad a \leq c \text{ and } b \geq d.$$
The least and greatest elements are $0_* = (0,1)$ and $1_* = (1,0)$. For a family $(a_i, b_i)_{i \in I}$,
$$\bigvee_{i \in I} (a_i, b_i) = \left(\sup_{i \in I} a_i, \inf_{i \in I} b_i \right),$$
$$\bigwedge_{i \in I} (a_i, b_i) = \left(\inf_{i \in I} a_i, \sup_{i \in I} b_i \right).$$
These pairs remain in L^*. For example, if $b = \inf_i b_i$, then $a_i \leq 1 - b_i \leq 1 - b$ for all i, so $\sup_i a_i + b \leq 1$.

An intuitionistic fuzzy set [1] on X is a function $A : X \to L^*$. We write
$$A(x) = (\mu_A(x), \nu_A(x))$$
and denote the complete lattice of all such functions by $\mathrm{IF}(X) = (L^*)^X$. Its order and lattice operations are pointwise. The constant bottom and top functions are denoted by $\widetilde{0}$ and $\widetilde{1}$.

Although the following proposition may be implicit in the existing lattice-theoretic literature, we have not found an explicit statement of it. We include a proof for completeness and for later use.

The lattice L^* is well known to be order-isomorphic to the lattice of closed subintervals of $[0,1]$; see, for example, [5, 8]. Indeed, the ordered-pair representation
$$I_{\leq} = \{(u,v) \in [0,1]^2 : u \leq v\}$$
may be identified with the interval lattice
$$\mathfrak{I}([0,1]) = \{[u,v] : 0 \leq u \leq v \leq 1\}$$
through $(u,v) \mapsto [u,v]$, with the coordinatewise endpoint order. We recall this representation and use it to establish explicitly the complete distributivity property needed later in the paper.

Proposition 1. *The lattice L^* is completely distributive. Consequently, for every set X, the function lattice*

$$\mathrm{IF}(X) = (L^*)^X$$

is a completely distributive complete lattice. In particular, both L^* and $\mathrm{IF}(X)$ are frames and coframes.

Proof. Consider the set

$$\mathbb{I}_{\leq} = \{(u, v) \in [0, 1]^2 : u \leq v\},$$

equipped with the coordinatewise order. Define

$$\phi : L^* \longrightarrow \mathbb{I}_{\leq}, \qquad \phi(a, b) = (a, 1 - b).$$

Since $(a + b \leq 1)$ if and only if $(a \leq 1 - b)$, the map ϕ is well defined. Its inverse is

$$\phi^{-1}(u, v) = (u, 1 - v),$$

so ϕ is bijective.

Moreover, for $(a, b), (c, d) \in L^*$,

$$(a, b) \leq_* (c, d) \iff a \leq c \quad \text{and} \quad b \geq d,$$

which is equivalent to $a \leq c \quad \text{and} \quad 1 - b \leq 1 - d$. Thus,

$$(a, b) \leq_* (c, d) \Longleftrightarrow \phi(a, b) \leq \phi(c, d).$$

Hence, ϕ is an order isomorphism.

Arbitrary joins and meets in $\mathbb{I}_{\leq}$ are computed coordinatewise. So, if

$$\{(u_i, v_i) : i \in I\} \subseteq \mathbb{I}_{\leq},$$

then

$$\bigvee_{i \in I} (u_i, v_i) = \left(\sup_{i \in I} u_i, \sup_{i \in I} v_i \right)$$

and

$$\bigwedge_{i \in I} (u_i, v_i) = \left(\inf_{i \in I} u_i, \inf_{i \in I} v_i \right).$$

The above join and meet are still in $\mathbb{I}_{\leq}$, because $u_i \leq v_i$ for every i implies

$$\sup_i u_i \leq \sup_i v_i \quad \text{and} \quad \inf_i u_i \leq \inf_i v_i.$$

The unit interval $[0,1]$, with its usual order, is a completely distributive complete chain. Therefore, for every doubly indexed family

$$\{(u_{ij}, v_{ij}) : i \in I,\ j \in J_i\} \subseteq \mathbb{I}_{\leq},$$

Applying the unrestricted distributive law to each coordinate:

$$\begin{aligned}
\bigwedge_{i\in I} \bigvee_{j\in J_i} (u_{ij}, v_{ij}) &= \left(\inf_{i\in I} \sup_{j\in J_i} u_{ij},\ \inf_{i\in I} \sup_{j\in J_i} v_{ij} \right) \\
&= \left(\sup_{f\in\prod_{i\in I} J_i} \inf_{i\in I} u_{i,f(i)},\ \sup_{f\in\prod_{i\in I} J_i} \inf_{i\in I} v_{i,f(i)} \right) \\
&= \bigvee_{f\in\prod_{i\in I} J_i} \bigwedge_{i\in I} (u_{i,f(i)}, v_{i,f(i)}).
\end{aligned}$$

Hence, $\mathbb{I}_{\leq}$ is completely distributive. Since L^* is completely lattice-isomorphic to $\mathbb{I}_{\leq}$, it follows that L^* is completely distributive.

Finally,

$$\mathrm{IF}(X) = (L^*)^X$$

is ordered pointwise, and all joins and meets are computed pointwise. Let

$$\{A_{ij} : i \in I,\ j \in J_i\} \subseteq \mathrm{IF}(X).$$

For every $x \in X$, complete distributivity of L^* yields

$$\left(\bigwedge_{i\in I} \bigvee_{j\in J_i} A_{ij} \right)(x) = \bigvee_{f\in\prod_{i\in I} J_i} \left(\bigwedge_{i\in I} A_{i,f(i)} \right)(x).$$

Since this equality holds for every $x \in X$, we conclude that

$$\bigwedge_{i\in I} \bigvee_{j\in J_i} A_{ij} = \bigvee_{f\in\prod_{i\in I} J_i} \bigwedge_{i\in I} A_{i,f(i)}.$$

Therefore, $\mathrm{IF}(X)$ is also completely distributive. See [14, 4] for the unrestricted distributive law.

Because L^* is a frame, it carries a unique Heyting implication $p \Rightarrow q$, characterized by

$$r \wedge p \leq_* q \quad \Longleftrightarrow \quad r \leq_* (p \Rightarrow q). \tag{1}$$

The following explicit formula will be useful. Let $s \Rightarrow_G t$ denote the Gödel implication on $[0,1]$:

$$s \Rightarrow_G t = \begin{cases} 1, & s \leq t, \\ t, & s > t. \end{cases}$$

We recall the residual implication generated by the lattice meet on L^*.

The following implication formula is not new; it was explicitly derived in [3]. We recall it here, together with an alternative derivation through the interval-lattice representation, because the resulting Heyting residual is used throughout the paper.

Proposition 2. *Let $p, q \in L^*$, where $p = (a, b)$ and $q = (c, d)$. Define*

$$r_1 = a \Rightarrow_G c, \qquad r_2 = (1-b) \Rightarrow_G (1-d),$$

where $\Rightarrow_G$ denotes the Gödel implication on the unit interval:

$$s \Rightarrow_G t = \begin{cases} 1, & s \leq t, \\ t, & s > t. \end{cases}$$

Then the Heyting implication of p relative to q is

$$p \Rightarrow q = \big(\min\{r_1, r_2\}, 1 - r_2\big). \tag{2}$$

More precisely, for every $z \in L^*$,

$$z \wedge p \leq_* q \quad \Longleftrightarrow \quad z \leq_* (p \Rightarrow q).$$

Consequently, the above operation is the unique residual of the lattice meet on L^*. In particular, its definition does not require p and q to be comparable.

Proof. We first recall the residuation property of the Gödel implication on the complete chain $[0,1]$. For $x, s, t \in [0,1]$,

$$\min\{x, s\} \leq t \quad \Longleftrightarrow \quad x \leq s \Rightarrow_G t. \tag{3}$$

Indeed, if $s \leq t$, then $s \Rightarrow_G t = 1$, and both sides of (3) hold for every x. If $s > t$, then $s \Rightarrow_G t = t$, and

$$\min\{x, s\} \leq t \quad \Longleftrightarrow \quad x \leq t.$$

Consider now the complete-lattice isomorphism introduced in Proposition 1,

$$\phi : L^* \longrightarrow \mathbb{I}_{\leq}, \qquad \phi(a, b) = (a, 1-b),$$

where

$$\mathbb{I}_{\leq} = \{(u,v) \in [0,1]^2 : u \leq v\}$$

is ordered coordinatewise. Under this isomorphism,

$$\phi(p) = (a, 1-b), \qquad \phi(q) = (c, 1-d).$$

Let $z = (u,v) \in L^*$. Then $\phi(z) = (u, 1-v)$, and the condition $u + v \leq 1$ is equivalent to $u \leq 1 - v$. Since meets in $\mathbb{I}_{\leq}$ are computed coordinatewise, we have

$$\phi(z) \wedge \phi(p) = \big(\min\{u,a\}, \min\{1-v, 1-b\}\big).$$

Therefore,

$$\begin{aligned}\phi(z) \wedge \phi(p) \leq \phi(q) \quad \iff \quad & \min\{u,a\} \leq c \\ & \text{and } \min\{1-v, 1-b\} \leq 1-d.\end{aligned}$$

Applying the Gödel residuation law to the two coordinates gives

$$u \leq r_1 \quad \text{and} \quad 1 - v \leq r_2.$$

We must now determine the largest element of the interval lattice $\mathbb{I}_{\leq}$ satisfying these two inequalities. In the lattice $[0,1]^2$, the coordinatewise residual would be (r_1, r_2). However, this pair need not belong to $\mathbb{I}_{\leq}$, because it may fail to satisfy $r_1 \leq r_2$.

Define

$$h = \big(\min\{r_1, r_2\}, r_2\big).$$

Clearly, $\min\{r_1, r_2\} \leq r_2$, so $h \in \mathbb{I}_{\leq}$.

We claim that h is the largest element of $\mathbb{I}_{\leq}$ below (r_1, r_2). Clearly, $h \leq (r_1, r_2)$. Now suppose that $(x,y) \in \mathbb{I}_{\leq}$ and $(x,y) \leq (r_1, r_2)$. Then $x \leq r_1,\ y \leq r_2$, and, since $(x,y) \in \mathbb{I}_{\leq}$, we obtain

$$x \leq y \leq r_2.$$

Thus,

$$x \leq \min\{r_1, r_2\} \quad \text{and} \quad y \leq r_2,$$

which means

$$(x,y) \leq h.$$

Hence h is the greatest interval value satisfying the residuation inequalities.

Applying ϕ^{-1}, we get

$$p \Rightarrow q = \phi^{-1}(h)$$

$$
\begin{aligned}
&= \phi^{-1}(\min\{r_1, r_2\}, r_2) \\
&= (\min\{r_1, r_2\}, 1 - r_2).
\end{aligned}
$$

We can see that this pair belongs to L^*, since

$$\min\{r_1, r_2\} + (1 - r_2) \leq 1.$$

It remains to verify the residuation law. Using that ϕ preserves the order and finite meets, we have

$$
\begin{aligned}
z \wedge p \leq_* q &\iff \phi(z) \wedge \phi(p) \leq \phi(q) \\
&\iff \phi(z) \leq h \\
&\iff \phi(z) \leq \phi(p \Rightarrow q) \\
&\iff z \leq_* (p \Rightarrow q).
\end{aligned}
$$

Therefore, the stated equivalence is the residual of the meet.

Finally, a residual is uniquely determined. If $w \in L^*$ also satisfies

$$z \wedge p \leq_* q \quad \iff \quad z \leq_* w$$

for every $z \in L^*$, then substituting $z = w$ and $z = p \Rightarrow q$ shows

$$w \leq_* (p \Rightarrow q) \quad \text{and} \quad p \Rightarrow q \leq_* w.$$

Hence $w = p \Rightarrow q$.

Corollary 1. *For $p = (a, b)$ and $q = (c, d)$, the canonical implication can equivalently be written as*

$$
p \Rightarrow q = \begin{cases}
(1, 0), & a \leq c \textit{ and } b \geq d, \\
(c, 0), & a > c \textit{ and } b \geq d, \\
(1 - d, d), & a \leq c \textit{ and } b < d, \\
(c, d), & a > c \textit{ and } b < d.
\end{cases}
$$

Proof. The four cases follow by substituting the corresponding values of the Gödel implications r_1 and r_2 into the formula of Proposition 2.

For example, if $a \leq c$ and $b \geq d$, then $r_1 = r_2 = 1$, so

$$p \Rightarrow q = (1, 0) = 1_*.$$

This is precisely the case $p \leq_* q$.

If $a > c$ and $b < d$, then

$$r_1 = c, \qquad r_2 = 1 - d.$$

Since $q = (c, d) \in L^*$, we have $c \leq 1 - d$, and therefore

$$p \Rightarrow q = (c, d) = q.$$

The remaining cases are obtained similarly.

Remark 1. *The representation of intuitionistic fuzzy sets as L^*-valued sets is classical, see [16, 5]. Propositions 1 and 2 state the two properties needed below: unrestricted distributivity and a canonical residual. Although other intuitionistic fuzzy implications have been studied [3], our specialization relation uses the residual determined by the frame structure.*

Definition 1. *An* intuitionistic fuzzy topology *on X is a family $\tau \subseteq \mathrm{IF}(X)$ such that $\widetilde{0}, \widetilde{1} \in \tau$, arbitrary joins of members of τ belong to τ, and finite meets of members of τ belong to τ. The pair (X, τ) is an intuitionistic fuzzy topological space, and members of τ are intuitionistic fuzzy open sets.*

This is the Çoker convention [2]. We do not require every constant L^*-valued set to be open. When all constants are open, the topology is called stratified or in the sense of Lowen [12].

3. Open consequences and kernel equivalence

For $A \in \mathrm{IF}(X)$, define its open-consequence profile [7] by

$$\mathcal{O}_\tau(A) = \{U \in \tau : A \leq U\}.$$

This profile is never empty because $\widetilde{1} \in \mathcal{O}_\tau(A)$.

Definition 2. *For $A, B \in \mathrm{IF}(X)$, write*

$$A \preccurlyeq_\tau B \iff \mathcal{O}_\tau(A) \subseteq \mathcal{O}_\tau(B).$$

Equivalently, every intuitionistic fuzzy open set containing A also contains B. We call $\preccurlyeq_\tau$ the open-consequence preorder. Define

$$A \equiv_\tau B \iff A \preccurlyeq_\tau B \text{ and } B \preccurlyeq_\tau A.$$

Thus $A \equiv_\tau B$ exactly when $\mathcal{O}_\tau(A) = \mathcal{O}_\tau(B)$.

The relation $\equiv_\tau$ is the equivalence relation studied in [7].

Definition 3. *[7] For $A \in \mathrm{IF}(X)$, define the open kernel as*

$$k_\tau(A) = \bigwedge \mathcal{O}_\tau(A) = \bigwedge\{U \in \tau : A \leq U\}. \tag{4}$$

The kernel spectrum *is*

$$\mathcal{K}(\tau) = \{k_\tau(A) : A \in \mathrm{IF}(X)\}.$$

Proposition 3. *The map $k_\tau : \mathrm{IF}(X) \to \mathrm{IF}(X)$ is a closure operator in the order-theoretic sense:*

(i) $A \leq k_\tau(A)$;

(ii) $A \leq B$ *implies* $k_\tau(A) \leq k_\tau(B)$;

(iii) $k_\tau(k_\tau(A)) = k_\tau(A)$.

Moreover, $k_\tau(\widetilde{0}) = \widetilde{0}$.

Proof. (i) The first statement follows because $A \leq U$ for every $U \in \mathcal{O}_\tau(A)$. (ii) If $A \leq B$, then $\mathcal{O}_\tau(B) \subseteq \mathcal{O}_\tau(A)$, and hence $\bigwedge \mathcal{O}_\tau(A) \leq \bigwedge \mathcal{O}_\tau(B)$. (iii) By definition, $A \leq k_\tau(A)$, which implies $\mathcal{O}_\tau(k_\tau(A)) \subseteq \mathcal{O}_\tau(A)$. Conversely, if $U \in \mathcal{O}_\tau(A)$, then $k_\tau(A) \leq U$ by definition, so $U \in \mathcal{O}_\tau(k_\tau(A))$. Thus, $\mathcal{O}_\tau(k_\tau(A)) = \mathcal{O}_\tau(A)$, which proves $k_\tau(k_\tau(A)) = k_\tau(A)$. Since $\widetilde{0}$ is open and is the least intuitionistic fuzzy set, $k_\tau(\widetilde{0}) = \widetilde{0}$.

The closure properties above were established in [7]; they are recalled here in the present order-theoretic notation.

Theorem 1 (Kernel representation of consequence)**.** *For $A, B \in \mathrm{IF}(X)$, the following are equivalent:*

(i) $A \preccurlyeq_\tau B$;

(ii) $B \leq k_\tau(A)$;

(iii) $k_\tau(B) \leq k_\tau(A)$.

Consequently,

$$A \equiv_\tau B \quad \Longleftrightarrow \quad k_\tau(A) = k_\tau(B).$$

Proof. (i) $\implies$ (ii) If $A \preccurlyeq_\tau B$, then $\mathcal{O}_\tau(A) \subseteq \mathcal{O}_\tau(B)$, and hence $B \leq k_\tau(B) \leq k_\tau(A)$. (ii) $\implies$ (i) Let $B \leq k_\tau(A)$ and $U \in \mathcal{O}_\tau(A)$. Thus, $k_\tau(A) \leq U$. By assumption, $B \leq U$. Hence, $U \in \mathcal{O}(B)$, and therefore $A \preccurlyeq_\tau B$. (ii) $\implies$ (iii) If $B \leq k_\tau(A)$, monotonicity and idempotence give

$$k_\tau(B) \leq k_\tau(k_\tau(A)) = k_\tau(A).$$

(iii) $\implies$ (ii) Follows from $B \leq k_\tau(B)$. The equivalence characterization is obtained by applying the result in both directions.

Corollary 2. *[7] Every equivalence class contains a unique kernel-fixed member, namely $k_\tau(A)$. It is the greatest member of $[A]_{\equiv_\tau}$ with respect to the pointwise order.*

Thus the kernel is the canonical maximal representative of all intuitionistic fuzzy sets with the same open-consequence profile.

3.1. Relation with the earlier equivalence theory

The present paper builds on the equivalence theory introduced in [7]. To make the relationship precise, we separate the results inherited from that work from the extensions developed here.

In [7], two intuitionistic fuzzy sets were defined to be equivalent when they have the same intuitionistic fuzzy open supersets. The corresponding open-superset family, called the equivalence cover there, is denoted here by $\mathcal{O}_\tau(A)$. That paper also introduced the kernel

$$k_\tau(A) = \bigwedge \mathcal{O}_\tau(A),$$

proved that

$$A \equiv_\tau B \quad \Longleftrightarrow \quad k_\tau(A) = k_\tau(B),$$

and showed that $k_\tau(A)$ is the greatest member of the equivalence class of A. It also established the basic closure properties of the kernel operator and its preservation of finite unions. These facts are recalled here only as background and are cited accordingly.

The present work begins from these results but develops a different structural direction. First, the equivalence relation is refined to the directed open-consequence preorder $\preccurlyeq_\tau$, and Theorem 1 characterizes this preorder by

$$A \preccurlyeq_\tau B \quad \Longleftrightarrow \quad B \leq k_\tau(A) \quad \Longleftrightarrow \quad k_\tau(B) \leq k_\tau(A).$$

Thus, equality of kernels from [7] appears as the symmetric special case of a directed order-theoretic representation.

Second, Theorem 2 extends the finite-union property of the kernel proved in [7] to arbitrary joins. This leads to the new kernel-completion structure

$$\mathcal{K}(\tau) = \mathrm{Fix}(k_\tau),$$

which is shown to be the smallest complete sublattice of $\mathrm{IF}(X)$ containing τ and to be a completely distributive frame and coframe. The quotient

$$\mathcal{Q}_\tau = \mathrm{IF}(X)/{\equiv_\tau}$$

is then represented as a complete lattice by $\mathcal{K}(\tau)$, with the kernel-information and consequence orders distinguished explicitly.

Finally, [7] already proved invariance of compactness, countable compactness, and the Lindelöf property under equivalence. The covering result of the present paper extends this to κ-compactness for every infinite cardinal κ. The graded specialization construction, the Alexandrov envelope, the functorial kernel spectrum, kernel-exact maps, and the finite computational representation are not part of [7].

4. Kernel completion and quotient frames

The main algebraic characterization is that k_τ preserves arbitrary joins. This property is not automatic for a closure operator; it is a consequence of complete distributivity of $\mathrm{IF}(X)$.

Lemma 1. *Let $(A_i)_{i\in I}$ be a nonempty family in $\mathrm{IF}(X)$. Then*

$$\mathcal{O}_\tau\left(\bigvee_{i\in I} A_i\right) = \left\{\bigvee_{i\in I} f(i) : f \in \prod_{i\in I} \mathcal{O}_\tau(A_i)\right\}. \tag{5}$$

Here

$$\prod_{i\in I} \mathcal{O}_\tau(A_i) = \left\{f : I \longrightarrow \bigcup_{i\in I} \mathcal{O}_\tau(A_i) : f(i) \in \mathcal{O}_\tau(A_i) \text{ for every } i \in I\right\}$$

is the set of choice functions for the family $\big(\mathcal{O}_\tau(A_i)\big)_{i\in I}$.

Proof. For every $i \in I$, the family $\mathcal{O}_\tau(A_i)$ is nonempty because $\widetilde{1} \in \tau$ and $A_i \le \widetilde{1}$. In fact, the constant choice $f(i) = \widetilde{1}$ belongs to $\prod_{i\in I} \mathcal{O}_\tau(A_i)$.

Let

$$f \in \prod_{i\in I} \mathcal{O}_\tau(A_i).$$

Then

$$A_i \leq f(i) \qquad (i \in I).$$

Hence

$$\bigvee_{i\in I} A_i \leq \bigvee_{i\in I} f(i).$$

Since every $f(i)$ is open and τ is closed under arbitrary joins, $\bigvee_{i\in I} f(i) \in \tau$. Therefore

$$\bigvee_{i\in I} f(i) \in \mathcal{O}_\tau\left(\bigvee_{i\in I} A_i\right).$$

Conversely, let

$$U \in \mathcal{O}_\tau\left(\bigvee_{i\in I} A_i\right).$$

Then

$$A_i \leq \bigvee_{i\in I} A_i \leq U \qquad (i \in I),$$

so $U \in \mathcal{O}_\tau(A_i)$ for every $i \in I$. Since I is nonempty, the constant choice

$$f(i) = U \qquad (i \in I)$$

belongs to $\prod_{i\in I} \mathcal{O}_\tau(A_i)$ and satisfies

$$\bigvee_{i\in I} f(i) = U.$$

This proves (5).

Theorem 2. *For every family $(A_i)_{i\in I} \subseteq \mathrm{IF}(X)$, including the empty family,*

$$k_\tau\left(\bigvee_{i\in I} A_i\right) = \bigvee_{i\in I} k_\tau(A_i). \tag{6}$$

For $I = \varnothing$, both joins are understood to be $\widetilde{0}$.

Proof. We first consider the empty-index case. If $I = \varnothing$, then

$$\bigvee_{i\in I} A_i = \widetilde{0} \qquad \text{and} \qquad \bigvee_{i\in I} k_\tau(A_i) = \widetilde{0}.$$

Since $\widetilde{0} \in \tau$ and $\widetilde{0}$ is the least element of $\mathrm{IF}(X)$, every open set contains $\widetilde{0}$. Hence

$$\mathcal{O}_\tau(\widetilde{0}) = \tau.$$

Therefore,

$$k_\tau(\widetilde{0}) = \bigwedge \mathcal{O}_\tau(\widetilde{0}) = \bigwedge \tau = \widetilde{0},$$

because $\widetilde{0} \in \tau$. Thus (6) holds when $I = \varnothing$.

Now suppose that I is nonempty. For every $i \in I$, put

$$J_i = \mathcal{O}_\tau(A_i).$$

Each J_i is nonempty because

$$\widetilde{1} \in \tau \qquad \text{and} \qquad A_i \leq \widetilde{1}.$$

Moreover,

$$\prod_{i\in I} J_i$$

is nonempty, since the constant function

$$f_0(i) = \widetilde{1} \qquad (i \in I)$$

is a choice function.

We now state explicitly the dual unrestricted distributive identity used below. Since $\mathrm{IF}(X)$ is completely distributive by Proposition 1, for every family

$$\{x_{i,j} : i \in I,\ j \in J_i\} \subseteq \mathrm{IF}(X)$$

with each $J_i \neq \varnothing$, we have

$$\bigwedge_{f\in\prod_{i\in I} J_i} \bigvee_{i\in I} x_{i,f(i)} = \bigvee_{i\in I} \bigwedge_{j\in J_i} x_{i,j}. \tag{7}$$

Apply (7) with

$$J_i = \mathcal{O}_\tau(A_i)$$

and

$$x_{i,U} = U \qquad \big(i \in I,\ U \in \mathcal{O}_\tau(A_i)\big).$$

By Lemma 1,

$$\begin{aligned} k_\tau\Big(\bigvee_{i\in I} A_i\Big) &= \bigwedge \mathcal{O}_\tau\Big(\bigvee_{i\in I} A_i\Big) \\ &= \bigwedge_{f\in\prod_{i\in I} \mathcal{O}_\tau(A_i)} \bigvee_{i\in I} f(i) \end{aligned}$$

$$= \bigvee_{i \in I} \bigwedge_{U \in \mathcal{O}_\tau(A_i)} U$$
$$= \bigvee_{i \in I} k_\tau(A_i).$$

The third equality is precisely (7). This proves (6) for every index set I.

Theorem 3 (Kernel-completion theorem)**.** *The following statements hold.*

(i) $\mathcal{K}(\tau) = \mathrm{Fix}(k_\tau)$ *is closed under arbitrary meets and arbitrary joins computed in* $\mathrm{IF}(X)$*.*

(ii) $\mathcal{K}(\tau)$ *is a completely distributive complete sublattice of* $\mathrm{IF}(X)$*; in particular, it is a frame and a coframe.*

(iii) $\mathcal{K}(\tau)$ *is the smallest complete sublattice of* $\mathrm{IF}(X)$ *containing* τ*.*

(iv) *The map* $k_\tau : \mathrm{IF}(X) \to \mathcal{K}(\tau)$ *is left adjoint to the inclusion* $j : \mathcal{K}(\tau) \hookrightarrow \mathrm{IF}(X)$*:*

$$k_\tau(A) \leq C \iff A \leq C \qquad (C \in \mathcal{K}(\tau)). \tag{8}$$

Proof. (i) The fixed points of any closure operator are closed under arbitrary meets. If $(C_i) \subseteq \mathcal{K}(\tau)$, then Theorem 2 gives

$$k_\tau\left(\bigvee_i C_i\right) = \bigvee_i k_\tau(C_i) = \bigvee_i C_i,$$

so the join is fixed as well. (ii) Since all operations in $\mathcal{K}(\tau)$ are inherited from the completely distributive lattice $\mathrm{IF}(X)$, every unrestricted distributive identity remains valid in $\mathcal{K}(\tau)$.

(iii) Every intuitionistic fuzzy open set U is fixed because $k_\tau(U) = U$, so $\tau \subseteq \mathcal{K}(\tau)$. For the minimality part, every $C \in \mathcal{K}(\tau)$ has the form

$$C = k_\tau(C) = \bigwedge \mathcal{O}_\tau(C),$$

an arbitrary meet of intuitionistic fuzzy open sets. Any complete sublattice containing τ must contain every such meet. This proves minimality.

(iv) For (8), if $k_\tau(A) \leq C$, then $A \leq k_\tau(A) \leq C$. Conversely, if $A \leq C$ and C is fixed, monotonicity of the kernel operator gives $k_\tau(A) \leq k_\tau(C) = C$.

Corollary 3. *The following are equivalent:*

(i) τ *is closed under arbitrary meets;*

(ii) $\tau = \mathcal{K}(\tau)$;

(iii) *every open-consequence class has an intuitionistic fuzzy open representative.*

Proof. If τ is closed under arbitrary meets, then every kernel is open and $\mathcal{K}(\tau) \subseteq \tau$. The reverse inclusion always holds. The equivalence of (ii) and (iii) follows from uniqueness of the fixed representative in Corollary 2.

The next example exhibits a proper kernel completion.

Example 1. *Let $X = \{x\}$ and, for $t \in [0,1]$, define the intuitionistic fuzzy set $D_t(x) = (t, 1-t)$. Put*

$$\tau = \{\widetilde{0}\} \cup \{D_t : \tfrac{1}{2} < t \leq 1\}.$$

This is an intuitionistic fuzzy topology: a nonempty join corresponds to the supremum of its parameters, and a finite meet corresponds to their minimum. However, τ is not closed under arbitrary meets because

$$\bigwedge_{n \geq 3} D_{\frac{1}{2}+\frac{1}{n}} = D_{1/2} \notin \tau.$$

In fact,

$$\mathcal{K}(\tau) = \tau \cup \{D_{1/2}\}, \qquad k_\tau(D_{1/2}) = D_{1/2}.$$

Thus the completion can add a genuine limit element even on a one-point universe.

Let

$$\mathcal{Q}_\tau = \mathrm{IF}(X)/{\equiv_\tau}$$

and write $[A]$(interchangeably, $[A]_{\equiv_\tau}$) for the class of A.

Definition 4. *The* kernel-information order *on $\mathcal{Q}_\tau$ is*

$$[A] \sqsubseteq_k [B] \quad \Longleftrightarrow \quad k_\tau(A) \leq k_\tau(B).$$

The consequence order *is*

$$[A] \sqsubseteq_c [B] \quad \Longleftrightarrow \quad A \preccurlyeq_\tau B.$$

Both are well-defined by Theorem 1; moreover, they are opposite orders. The kernel-information order is natural for the lattice structure because

$$[A] \sqsubseteq_k [B] \quad \Longleftrightarrow \quad k_\tau(A) \leq k_\tau(B),$$

so the quotient is directly order-isomorphic to $\mathcal{K}(\tau)$. By Theorem 1,

$$[A] \sqsubseteq_c [B] \quad \Longleftrightarrow \quad k_\tau(B) \leq k_\tau(A),$$

so the consequence order is its opposite.

Theorem 4 (Quotient representation)**.** *The map*

$$\eta_\tau : (\mathcal{Q}_\tau, \sqsubseteq_k) \longrightarrow (\mathcal{K}(\tau), \leq), \qquad \eta_\tau([A]) = k_\tau(A),$$

is an order isomorphism and a complete-lattice isomorphism. Consequently, $(\mathcal{Q}_\tau, \sqsubseteq_k)$ *is a completely distributive frame and coframe. Under the consequence order, the same map is an order isomorphism*

$$(\mathcal{Q}_\tau, \sqsubseteq_c) \cong (\mathcal{K}(\tau), \leq)^{\mathrm{op}}.$$

For every family $([A_i])_{i\in I}$*,*

$$\bigvee_i^k [A_i] = \left[\bigvee_i A_i\right], \tag{9}$$

$$\bigwedge_i^k [A_i] = \left[\bigwedge_i k_\tau(A_i)\right]. \tag{10}$$

Proof. Well-definedness and injectivity are exactly the kernel characterization of equivalence. Surjectivity follows from the definition of $\mathcal{K}(\tau)$. Preservation and reflection of order are built into $\sqsubseteq_k$. The complete-lattice claims follow from Theorem 3. Formula (9) follows from Theorem 2, while (10) follows because arbitrary meets of fixed points are fixed. Finally, Theorem 1 gives

$$[A] \sqsubseteq_c [B] \iff k_\tau(B) \leq k_\tau(A),$$

which is the reverse order.

5. Graded specialization and the Alexandrov envelope

We now connect open-consequence kernels with fuzzy preorder theory. The construction uses the canonical implication from Proposition 2.

Definition 5. *[9, 18, 19] For* $x, y \in X$*, define the graded specialization*

$$\Omega_\tau(x, y) = \bigwedge_{U\in\tau} (U(x) \Rightarrow U(y)). \tag{11}$$

The value $\Omega_\tau(x, y)$ measures the degree to which every open observable that holds at x transfers to y. It is the L^*-valued specialization relation associated with τ.

The construction above is the specialization L-preorder of L-topological theory, specialized to the Atanassov value lattice L^*. Related intuitionistic fuzzy constructions were considered in [19]. Our contribution is not the definition itself, but its formulation using the canonical Heyting residual on the whole of L^*, without pointwise comparability assumptions, and its interaction with the open-kernel completion developed in this paper.

We recall the established notions of an intuitionistic fuzzy preorder and an upper intuitionistic fuzzy set [9, 19]. They are included to fix notation and to formulate the kernel–specialization comparison developed below.

Definition 6. *An L^*-valued relation $R : X \times X \to L^*$ is an L^*-preorder if*

$$R(x,x) = 1_*, \qquad R(x,y) \wedge R(y,z) \leq_* R(x,z)$$

*for all $x, y, z \in X$. An intuitionistic fuzzy set A is R-*upper *if*

$$A(x) \wedge R(x,y) \leq_* A(y) \qquad (x, y \in X). \tag{12}$$

Write $\mathrm{Up}(R)$ *for the family of all R-upper intuitionistic fuzzy sets.*

Proposition 4. *The relation Ω_τ is an L^*-preorder. Every $U \in \tau$ is Ω_τ-upper.*

Proof. For every U, $U(x) \Rightarrow U(x) = 1_*$, so $\Omega_\tau(x,x) = 1_*$. In any Heyting algebra,

$$(a \Rightarrow b) \wedge (b \Rightarrow c) \leq (a \Rightarrow c).$$

Therefore, for each $U \in \tau$,

$$\Omega_\tau(x,y) \wedge \Omega_\tau(y,z) \leq (U(x) \Rightarrow U(y)) \wedge (U(y) \Rightarrow U(z)) \leq U(x) \Rightarrow U(z).$$

Taking the meet over U proves transitivity. Finally,

$$\Omega_\tau(x,y) \leq U(x) \Rightarrow U(y),$$

and residuation gives $U(x) \wedge \Omega_\tau(x,y) \leq U(y)$.

Proposition 5. *For every L^*-preorder R, the family* $\mathrm{Up}(R)$ *is closed under arbitrary joins and arbitrary meets and contains every constant intuitionistic fuzzy set. Hence it is an Alexandrov intuitionistic fuzzy topology.*

Proof. Let (A_i) be R-upper. Since L^* is a frame,

$$\left(\bigvee_i A_i(x)\right) \wedge R(x,y) = \bigvee_i (A_i(x) \wedge R(x,y)) \leq \bigvee_i A_i(y).$$

For meets,

$$\left(\bigwedge_i A_i(x)\right) \wedge R(x,y) \leq A_i(x) \wedge R(x,y) \leq A_i(y)$$

for every i, so the left side is below $\bigwedge_i A_i(y)$. A constant c is upper because $c \wedge R(x,y) \leq c$.

For an L^*-preorder R, we define the relational saturation operator associated with R by

$$c_R(A)(y) = \bigvee_{x \in X} (A(x) \wedge R(x,y)). \tag{13}$$

This operator coincides with the intuitionistic fuzzy upper approximation associated with the inverse relation R^{-1} in [19].

Theorem 5 (Alexandrov saturation)**.** *For an L^*-preorder R, the operator c_R is extensive, monotone, idempotent, and preserves arbitrary joins. Its fixed points are exactly* $\mathrm{Up}(R)$*. Consequently, $c_R(A)$ is the least R-upper intuitionistic fuzzy set containing A.*

Proof. Reflexivity gives

$$A(y) = A(y) \wedge R(y,y) \leq c_R(A)(y).$$

Monotonicity and join preservation follow directly from (13) and the frame law. For idempotence, transitivity gives

$$\begin{aligned} c_R(c_R(A))(z) &= \bigvee_{y,x} A(x) \wedge R(x,y) \wedge R(y,z) \\ &\leq \bigvee_{x} A(x) \wedge R(x,z) = c_R(A)(z). \end{aligned}$$

The reverse inequality follows from extensivity.

If A is R-upper, then every summand in (13) is below $A(y)$, so $c_R(A) \leq A$; extensivity gives equality. Conversely, if $c_R(A) = A$, then

$$A(x) \wedge R(x,y) \leq c_R(A)(y) = A(y),$$

so A is upper. The leastness assertion is the closure-operator universal property.

The next theorem gives the precise relation among the original topology, the kernel completion, and the Alexandrov topology generated by specialization.

Theorem 6. *Let $\Omega = \Omega_\tau$. Then*

$$\tau \subseteq \mathcal{K}(\tau) \subseteq \mathrm{Up}(\Omega). \tag{14}$$

Moreover, for every $A \in \mathrm{IF}(X)$,

$$c_\Omega(A) \leq k_\tau(A). \tag{15}$$

The three families in (14) *induce the same graded specialization preorder:*

$$\Omega_\tau = \Omega_{\mathcal{K}(\tau)} = \Omega_{\mathrm{Up}(\Omega_\tau)}. \tag{16}$$

Proof. The inclusion $\tau \subseteq \mathcal{K}(\tau)$ was proved in Theorem 3. Every member of τ is Ω-upper by Proposition 4. If $C \in \mathcal{K}(\tau)$, then $C = k_\tau(C)$. Since $c_\Omega(C)$ is below every Ω-upper set containing C, and every open set is Ω-upper, we have

$$c_\Omega(C) \leq k_\tau(C) = C.$$

Extensivity gives $C = c_\Omega(C)$, so $C \in \mathrm{Up}(\Omega)$. The same argument for arbitrary A proves (15).

Because $\tau \subseteq \mathcal{K}(\tau)$, taking a meet over the larger family gives

$$\Omega_{\mathcal{K}(\tau)} \leq \Omega_\tau.$$

Conversely, every $C \in \mathcal{K}(\tau)$ is Ω_τ-upper, so

$$\Omega_\tau(x, y) \leq C(x) \Rightarrow C(y)$$

for all x, y. Taking the meet over C gives the reverse inequality.

Let $R = \Omega_\tau$. Every R-upper set A satisfies

$$R(x, y) \leq A(x) \Rightarrow A(y),$$

so $R \leq \Omega_{\mathrm{Up}(R)}$. For the reverse inequality, fix $x \in X$ and consider the representable set $R_x(z) = R(x, z)$. Transitivity shows that R_x is R-upper. Hence

$$\Omega_{\mathrm{Up}(R)}(x, y) \leq R_x(x) \Rightarrow R_x(y) = 1_* \Rightarrow R(x, y) = R(x, y).$$

Thus $\Omega_{\mathrm{Up}(R)} = R$.

Definition 7. *The intuitionistic fuzzy topology τ is* specialization exact *if*

$$\mathcal{K}(\tau) = \mathrm{Up}(\Omega_\tau).$$

Corollary 4. *The following are equivalent:*

(i) *τ is specialization exact;*

(ii) *$k_\tau = c_{\Omega_\tau}$ on $\mathrm{IF}(X)$;*

(iii) *every Ω_τ-upper intuitionistic fuzzy set is an arbitrary meet of members of τ.*

Proof. The fixed points of k_τ are $\mathcal{K}(\tau)$, and the fixed points of c_{Ω_τ} are $\mathrm{Up}(\Omega_\tau)$. Two closure operators on a complete lattice are equal exactly when they have the same fixed-point family. The equivalence with (iii) follows because $\mathcal{K}(\tau)$ is precisely the family of arbitrary meets of open sets.

The next example shows that there is an Alexandrov topology that is not specialization exact.

Example 2. *Let $X = \{x, y\}$ and define the constant intuitionistic fuzzy set*

$$C(x) = C(y) = \left(\tfrac{1}{2}, 0\right).$$

Then

$$\tau = \{\widetilde{0}, C, \widetilde{1}\}$$

is a finite, hence Alexandrov, intuitionistic fuzzy topology. Thus $\mathcal{K}(\tau) = \tau$. Since every open set is constant,

$$\Omega_\tau(u, v) = 1_* \qquad (u, v \in X).$$

An intuitionistic fuzzy set is upper for the universal preorder if and only if it is constant on X. Therefore

$$\mathrm{Up}(\Omega_\tau) = \{\widetilde{p} : p \in L^*\},$$

which strictly contains τ. Hence closure under arbitrary meets does not by itself imply specialization exactness in the non-stratified Çoker setting.

6. Functoriality and kernel-exact maps

We recall the definition of continuity in intuitionistic fuzzy topological spaces as introduced in [2, 10]. Let (X, τ) and (Y, σ) be intuitionistic fuzzy topological spaces. A map $f : X \to Y$ is intuitionistic fuzzy continuous if

$$f^{\leftarrow}(V) = V \circ f \in \tau \qquad (V \in \sigma).$$

The inverse-image map $f^{\leftarrow} : \mathrm{IF}(Y) \to \mathrm{IF}(X)$ preserves arbitrary meets and arbitrary joins, and thus it is a complete-lattice homomorphism.

Theorem 7 (Kernel-spectrum functor)**.** *If $f : (X, \tau) \to (Y, \sigma)$ is intuitionistic fuzzy continuous, then*

$$\mathcal{K}(f) : \mathcal{K}(\sigma) \longrightarrow \mathcal{K}(\tau), \qquad \mathcal{K}(f)(C) = f^{\leftarrow}(C),$$

is a complete-lattice homomorphism. The assignments

$$(X, \tau) \longmapsto \mathcal{K}(\tau), \qquad f \longmapsto \mathcal{K}(f)$$

define a contravariant functor

$$\mathcal{K} : \mathbf{IFTop}^{\mathrm{op}} \longrightarrow \mathbf{CDLat},$$

where **IFTop** *denotes the category of intuitionistic fuzzy topological spaces and intuitionistic fuzzy continuous maps, and* **CDLat** *denotes the category whose objects are completely distributive complete lattices and whose morphisms preserve arbitrary joins and arbitrary meets.*

Proof. Every $C \in \mathcal{K}(\sigma)$ is an arbitrary meet of members of σ, say $C = \bigwedge_j V_j$. Then

$$f^{\leftarrow}(C) = \bigwedge_j f^{\leftarrow}(V_j).$$

Each $f^{\leftarrow}(V_j)$ lies in τ, so $\bigwedge_j f^{\leftarrow}(V_j) \in \mathcal{K}(\tau)$. Because inverse image preserves all meets and joins, its restriction is a complete-lattice homomorphism. Identity and composition laws follow from the corresponding laws for inverse images.

For every intuitionistic fuzzy topological space (X, τ), consider the identity map

$$\mathrm{id}_X : (X, \tau) \longrightarrow (X, \tau).$$

For every $C \in \mathcal{K}(\tau)$,

$$\mathcal{K}(\mathrm{id}_X)(C) = \mathrm{id}_X^{\leftarrow}(C) = C.$$

Hence,

$$\mathcal{K}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{K}(\tau)}.$$

Now let

$$f : (X, \tau) \longrightarrow (Y, \sigma)$$

and

$$g : (Y, \sigma) \longrightarrow (Z, \rho)$$

be intuitionistic fuzzy continuous maps. For every $C \in \mathcal{K}(\rho)$,

$$\begin{aligned}\mathcal{K}(g \circ f)(C) &= (g \circ f)^{\leftarrow}(C) \\ &= f^{\leftarrow}(g^{\leftarrow}(C)) \\ &= \mathcal{K}(f)(\mathcal{K}(g)(C)).\end{aligned}$$

Therefore,

$$\mathcal{K}(g \circ f) = \mathcal{K}(f) \circ \mathcal{K}(g).$$

The direction of the induced map is reversed:

$$f : (X, \tau) \longrightarrow (Y, \sigma)$$

produces

$$\mathcal{K}(f) : \mathcal{K}(\sigma) \longrightarrow \mathcal{K}(\tau).$$

Thus, the construction is contravariant. The identity and composition equations prove that

$$\mathcal{K} : \mathbf{IFTop}^{\mathrm{op}} \longrightarrow \mathbf{CDLat}$$

is a well-defined functor.

The kernel operators themselves are generally only lax natural.

Proposition 6 (Lax kernel naturality)**.** *For every intuitionistic fuzzy continuous $f : (X, \tau) \to (Y, \sigma)$ and every $A \in \mathrm{IF}(Y)$,*

$$k_\tau(f^{\leftarrow}(A)) \leq f^{\leftarrow}(k_\sigma(A)). \tag{17}$$

Moreover,

$$\Omega_\tau(x, x') \leq \Omega_\sigma(f(x), f(x')) \qquad (x, x' \in X). \tag{18}$$

Proof. The set $f^{\leftarrow}(k_\sigma(A))$ is in $\mathcal{K}(\tau)$ by Theorem 7 and contains $f^{\leftarrow}(A)$. Since $k_\tau(f^{\leftarrow}(A))$ is the least kernel-fixed set above $f^{\leftarrow}(A)$, inequality (17) follows.

For every $V \in \sigma$, continuity gives $f^{\leftarrow}(V) \in \tau$. Hence

$$\begin{aligned}\Omega_\tau(x, x') &\leq f^{\leftarrow}(V)(x) \Rightarrow f^{\leftarrow}(V)(x') \\ &= V(f(x)) \Rightarrow V(f(x')).\end{aligned}$$

Taking the meet over $V \in \sigma$ proves (18).

Definition 8. *A continuous map $f : (X, \tau) \to (Y, \sigma)$ is* kernel exact *if equality holds in (17) for every $A \in \mathrm{IF}(Y)$.*

Proposition 7. *Identity maps are kernel exact, and composites of kernel-exact maps are kernel exact. For a kernel-exact map f, the assignment*

$$\overline{f} : \mathrm{IF}(Y)/{\equiv_\sigma} \longrightarrow \mathrm{IF}(X)/{\equiv_\tau}, \qquad \overline{f}([A]) = [f^{\leftarrow}(A)],$$

is well defined and, under the kernel-information orders, corresponds to the complete-lattice homomorphism $\mathcal{K}(f)$.

Proof. The identity statement is immediate. If $f : X \to Y$ and $g : Y \to Z$ are kernel exact, then

$$\begin{aligned}k_X((g \circ f)^{\leftarrow} A) &= k_X(f^{\leftarrow} g^{\leftarrow} A) = f^{\leftarrow} k_Y(g^{\leftarrow} A) \\ &= f^{\leftarrow} g^{\leftarrow} k_Z(A) = (g \circ f)^{\leftarrow} k_Z(A).\end{aligned}$$

If $A \equiv_\sigma B$, then $k_\sigma(A) = k_\sigma(B)$, and exactness gives

$$k_\tau(f^{\leftarrow} A) = f^{\leftarrow} k_\sigma(A) = f^{\leftarrow} k_\sigma(B) = k_\tau(f^{\leftarrow} B).$$

Thus $\overline{f}$ is well defined. Under the isomorphisms of Theorem 4, it sends $k_\sigma(A)$ to $f^{\leftarrow} k_\sigma(A)$, which is exactly $\mathcal{K}(f)$.

Definition 9. *Let* $\mathbf{IFTop}_{\mathrm{ke}}$ *denote the wide subcategory of* $\mathbf{IFTop}$ *whose objects are all intuitionistic fuzzy topological spaces and whose morphisms are the kernel-exact intuitionistic fuzzy continuous maps. By Proposition 7, identity maps are kernel exact and the composition of two kernel-exact maps is kernel exact. Hence* $\mathbf{IFTop}_{\mathrm{ke}}$ *is indeed a subcategory of* $\mathbf{IFTop}$*.*

The next result gives a useful sufficient condition for kernel exactness.

Proposition 8. *Let*

$$f : (X, \tau) \longrightarrow (Y, \sigma)$$

be a surjective intuitionistic fuzzy continuous map. Suppose that

$$\tau = \{f^{\leftarrow}(V) : V \in \sigma\}.$$

Then f *is kernel exact.*

Proof. Let $A \in \mathrm{IF}(Y)$. By Proposition 6,

$$k_\tau(f^{\leftarrow}(A)) \leq f^{\leftarrow}(k_\sigma(A)).$$

It remains to prove the reverse inequality.

Let $U \in \tau$ satisfy

$$f^{\leftarrow}(A) \leq U.$$

By the assumption on τ, there exists $V \in \sigma$ such that

$$U = f^{\leftarrow}(V).$$

Hence

$$f^{\leftarrow}(A) \leq f^{\leftarrow}(V).$$

Since f is surjective, the inverse-image map reflects the pointwise order, and therefore

$$A \leq V.$$

Thus V is an open superset of A, so

$$k_\sigma(A) \leq V.$$

Taking inverse images gives

$$f^{\leftarrow}(k_\sigma(A)) \leq f^{\leftarrow}(V) = U.$$

Since this holds for every $U \in \tau$ containing $f^{\leftarrow}(A)$, we obtain

$$f^{\leftarrow}(k_\sigma(A)) \leq k_\tau(f^{\leftarrow}(A)).$$

Combining the two inequalities yields

$$k_\tau(f^{\leftarrow}(A)) = f^{\leftarrow}(k_\sigma(A)).$$

Hence f is kernel exact.

Example 3 (A nonidentity kernel-exact map)**.** *Let*

$$Y = \{0,1\}, \qquad X = \{a,b,c\},$$

and define

$$f(a) = 0, \qquad f(b) = f(c) = 1.$$

Let $V \in \mathrm{IF}(Y)$ be the crisp intuitionistic fuzzy set given by

$$V(0) = 0^*, \qquad V(1) = 1^*,$$

and let

$$\sigma = \{\widetilde{0}, V, \widetilde{1}\}.$$

Put

$$U = f^{\leftarrow}(V),$$

so that

$$U(a) = 0^*, \qquad U(b) = U(c) = 1^*,$$

and define

$$\tau = \{\widetilde{0}, U, \widetilde{1}\}.$$

Then f is surjective and

$$\tau = \{f^{\leftarrow}(W) : W \in \sigma\}.$$

Therefore, by the preceding proposition, f is kernel exact. Notice that f is neither an identity map nor an injective map.

Example 4 (A continuous map that is not kernel exact)**.** *Let*

$$Y = \{0,1\}$$

and let

$$\sigma = \{\widetilde{0}, V, \widetilde{1}\},$$

where

$$V(0) = 0^*, \qquad V(1) = 1^*.$$

Let

$$X = \{x\}, \qquad \tau = \{\widetilde{0}, \widetilde{1}\},$$

and define

$$f(x) = 1.$$

Since

$$f^{\leftarrow}(V) = \widetilde{1},$$

the map $f : (X, \tau) \to (Y, \sigma)$ *is intuitionistic fuzzy continuous.*

Now let $A \in \mathrm{IF}(Y)$ *be the crisp intuitionistic fuzzy set given by*

$$A(0) = 1^*, \qquad A(1) = 0^*.$$

The only member of σ *containing* A *is* $\widetilde{1}$*. Hence*

$$k_\sigma(A) = \widetilde{1}.$$

On the other hand,

$$f^{\leftarrow}(A) = \widetilde{0},$$

and therefore

$$k_\tau(f^{\leftarrow}(A)) = k_\tau(\widetilde{0}) = \widetilde{0}.$$

However,

$$f^{\leftarrow}(k_\sigma(A)) = f^{\leftarrow}(\widetilde{1}) = \widetilde{1}.$$

Consequently,

$$k_\tau(f^{\leftarrow}(A)) < f^{\leftarrow}(k_\sigma(A)),$$

so f *is continuous but not kernel exact.*

7. Open-cover invariants

The open-consequence profile determines more than the kernel. It determines every covering property whose definition refers only to open supersets of the set under consideration.

Definition 10. *Let* κ *be an infinite cardinal. An intuitionistic fuzzy set* A *is* κ-compact *if, whenever* $(U_i)_{i\in I} \subseteq \tau$ *satisfies*

$$A \le \bigvee_{i\in I} U_i,$$

there is $J \subseteq I$ *with* $|J| < \kappa$ *such that*

$$A \le \bigvee_{j\in J} U_j.$$

Thus ω-compactness is ordinary finite-subcover compactness, while ω_1-compactness is the Lindelöf property.

Theorem 8. *If $A \equiv_\tau B$, then A is κ-compact if and only if B is κ-compact, for every infinite cardinal κ.*

Proof. Assume that A is κ-compact and that $B \leq \bigvee_{i \in I} U_i$. The join $U = \bigvee_i U_i$ is open. Since $A \equiv_\tau B$, the containment $B \leq U$ implies $A \leq U$. Choose $J \subseteq I$ with $|J| < \kappa$ and $A \leq \bigvee_{j \in J} U_j$. The latter join is open, so equivalence gives

$$B \leq \bigvee_{j \in J} U_j.$$

The converse is symmetric.

It should be noted that compactness, countable compactness, and the Lindelöf property were already shown in [7, Proposition 16] to be invariant under open-consequence equivalence. Theorem 8 extends this covering invariance to arbitrary infinite cardinals κ. In particular, $\kappa = \omega$ gives ordinary compactness, while $\kappa = \omega_1$ gives the Lindelöf property. Countable compactness is not a special case of the above κ-compactness definition; its invariance is recalled from [7, Proposition 16].

Corollary 5. *Compactness, countable compactness formulated through countable open covers, and the Lindelöf property are invariant under $\equiv_\tau$.*

A simple universal property explains information compression through kernels.

Proposition 9. *Let P be a poset and let $F : \mathrm{IF}(X) \to P$ be a monotone map satisfying*

$$F(A) = F(k_\tau(A)) \qquad (A \in \mathrm{IF}(X)).$$

Then there is a unique monotone map $\widehat{F} : \mathcal{K}(\tau) \to P$ such that

$$F = \widehat{F} \circ k_\tau.$$

In particular, any monotone observable that is invariant on open-consequence classes factors uniquely through the kernel completion.

Proof. Define $\widehat{F}(C) = F(C)$ for $C \in \mathcal{K}(\tau)$. Then

$$(\widehat{F} \circ k_\tau)(A) = F(k_\tau(A)) = F(A).$$

Uniqueness follows because k_τ is surjective onto $\mathcal{K}(\tau)$.

8. Finite computation and examples

Suppose that $X = \{x_1, \ldots, x_n\}$ and that the topology is explicitly listed as

$$\tau = \{U_1, \ldots, U_m\}.$$

Comparing two intuitionistic fuzzy values takes constant time. The kernel of a given A can therefore be computed by the following direct procedure.

Kernel algorithm.

1. Initialize $K := \widetilde{1}$.
2. For $j = 1, \ldots, m$, test whether $A(x_i) \leq_* U_j(x_i)$ for all i.
3. Whenever the test succeeds, replace K by $K \wedge U_j$.
4. Return K.

Two sets are equivalent exactly when their returned kernels agree.
We use a unit-cost lattice-operation model: each comparison, meet, join, and implication evaluation in L^* is counted as one elementary operation. Thus, for a topology $\tau = \{U_1, \ldots, U_m\}$ on $X = \{x_1, \ldots, x_n\}$, computing $k_\tau(A)$ requires $O(mn)$ elementary comparisons and lattice operations, since each of the m open sets is tested at the n points. Hence, the number of operations is linear in the mn entries of the open-incidence table. The storage requirement is $O(n)$. Similarly, computing the specialization matrix requires $O(mn^2)$ implication evaluations. These estimates count lattice operations only. The bit complexity depends on the chosen numerical representation of the membership and nonmembership values and is not considered here.

Proposition 10. *If τ is finite, then*

$$\mathcal{K}(\tau) = \tau.$$

Consequently, $\mathrm{IF}(X)/{\equiv_\tau}$ *has exactly* $|\tau|$ *equivalence classes, each with a unique intuitionistic fuzzy open representative.*

Proof. We first prove the equality $\mathcal{K}(\tau) = \tau$. We know that

$$\mathcal{K}(\tau) = \left\{ \bigwedge \mathcal{U} : \mathcal{U} \subseteq \tau \right\},$$

By definition, $\tau \subseteq \mathcal{K}(\tau)$. We now prove the reverse inclusion. Let $C \in \mathcal{K}(\tau)$. By the definition of the kernel completion, there exists a family $\{U_i\}_{i \in I} \subseteq \tau$ such that $C = \bigwedge_{i\in I} U_i$. Since τ is finite, $C \in \tau$. Thus, $\mathcal{K}(\tau) \subseteq \tau$. To determine the number of open-consequence equivalence classes, consider the canonical map

$$\Phi : \mathrm{IF}(X)/{\equiv_\tau} \longrightarrow \mathcal{K}(\tau), \qquad \Phi([A]_{\equiv_\tau}) = k_\tau(A).$$

This map is well defined because

$$A \equiv_\tau B \iff k_\tau(A) = k_\tau(B).$$

It is injective because if $\Phi([A]_{\equiv_\tau}) = \Phi([B]_{\equiv_\tau})$, then $k_\tau(A) = k_\tau(B)$ and therefore $A \equiv_\tau B$, so $[A]_{\equiv_\tau} = [B]_{\equiv_\tau}$. The map is also surjective. If $C \in \mathcal{K}(\tau)$, then $C = k_\tau(C)$ and hence $\Phi([C]_{\equiv_\tau}) = C$. Thus,

$$\mathrm{IF}(X)/{\equiv_\tau} \cong \mathcal{K}(\tau).$$

Since $\mathcal{K}(\tau) = \tau$, we obtain

$$|\mathrm{IF}(X)/{\equiv_\tau}| = |\mathcal{K}(\tau)| = |\tau|.$$

Finally, every $U \in \tau$ represents a distinct class because two equivalent open sets are equal.

Example 5. *Return to the topology in Example 2:*

$$\tau = \{\widetilde{0}, C, \widetilde{1}\}, \qquad C(x) = C(y) = \left(\tfrac{1}{2}, 0\right).$$

For any $A \in \mathrm{IF}(X)$*,*

$$k_\tau(A) = \begin{cases} \widetilde{0}, & A = \widetilde{0}, \\ C, & \widetilde{0} < A \le C, \\ \widetilde{1}, & A \nleq C. \end{cases}$$

Hence the entire continuum-sized lattice $\mathrm{IF}(X)$ *is compressed into three open-consequence classes. In the information order, these classes form the chain*

$$[\widetilde{0}] \sqsubseteq_k [C] \sqsubseteq_k [\widetilde{1}].$$

In the consequence order, the chain is reversed. The graded specialization relation, however, is universal and its Alexandrov topology contains every constant intuitionistic fuzzy set. This single example displays both the quotient compression and the strict inclusion

$$\mathcal{K}(\tau) \subsetneq \mathrm{Up}(\Omega_\tau).$$

Proposition 11 (Signatures)**.** *For a finite topology $\tau = \{U_1, \dots, U_m\}$, define*

$$\mathrm{sig}_\tau(A) = \{j \in \{1, \dots, m\} : A \leq U_j\}.$$

Then, for $A, B \in \mathrm{IF}(X)$,

$$A \equiv_\tau B \quad \Longleftrightarrow \quad \mathrm{sig}_\tau(A) = \mathrm{sig}_\tau(B),$$

and

$$k_\tau(A) = \bigwedge_{j \in \mathrm{sig}_\tau(A)} U_j.$$

Thus the signature is a lossless binary code for all open-containment queries about A.

Proof. The first assertion is the definition of open-consequence equivalence after indexing the open family. The second is the kernel definition. Since $A \leq U$ if and only if $k_\tau(A) \leq U$, the kernel and signature determine one another.

9. Conclusion and research directions

Open-consequence equivalence in intuitionistic fuzzy topology is best understood through its directed consequence preorder and canonical kernel representative. Complete distributivity of the Atanassov truth-value lattice makes the kernel operator join-preserving, and this turns its image into a completely distributive frame and coframe. The resulting kernel completion is simultaneously the quotient representation, the smallest complete sublattice containing the topology, and a reflective information-compression space.

The graded specialization relation obtained from the canonical Heyting implication provides a second representation layer. The original topology, its kernel completion, and the full Alexandrov envelope have the same specialization preorder, but the two inclusions between them can be strict. This separates topological completion by arbitrary open meets from completion by all graded upper sets, a distinction hidden by a direct transfer of the classical Alexandrov correspondence.

Several directions remain. First, specialization exactness should be characterized by intrinsic axioms on stratified intuitionistic fuzzy topologies and compared systematically with existing representation theorems for L-preorders. Second, kernel-exact maps merit a categorical analysis, including limits, colimits, and adjunctions. Third, the quotient frame can be studied through prime and completely join-prime elements to determine when the graded specialization space can be reconstructed from its observable kernel algebra. Finally, the finite signature representation suggests applications to attribute reduction and rough approximation, where large families of intuitionistic fuzzy states can be compressed without changing any admissible open decision rule.

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