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Smarandache-Ruled Surface According to quasi-Frame in Euclidean 3-Space E^3

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Abstract. In differential geometry of curves and surfaces, ruled surfaces represent one of the most important topics in surface theory, which is widely used in many fields of engineering and manufacturing. By choosing a curve (base curve) and a line along that curve (ruling), we can define the ruled surface. A lot of researchers in many papers have introduced ruled surfaces of the moving frames of its base curve. The quasi frame has many advantages over the Frenet-Serret frame and Bishop frame. In this article, we will study Smarandache-ruled surfaces according to the quasi-frame in Euclidean 3-space. In our article, we define three special kinds of Smarandache-Ruled surfaces T_qN_q, T_qB_q, and N_qB_q Smarandache ruled surfaces according to the quasi-frame. We determine the first and second fundamental forms, and the Gaussian curvature, mean curvature, and second Gaussian curvature of these surfaces are investigated. The necessary and sufficient conditions for such surfaces to be developable are introduced. Moreover, we obtain the distribution parameter of such Smarandache-ruled surfaces. Also, the conditions for such surfaces to be minimal are investigated. Finally, the ruled surface characteristic properties related to the normal curvatures and geodesic curvature for special curves on these surfaces are obtained. Also, some examples of these surfaces are given.

Key Words and Phrases: Smarandache curves, Ruled Surface, Frenet Frame, Bishop Frame, Quasi-Frame, Gaussian curvature, Mean curvature.

2020 Mathematics Subject Classifications: 53A04, 53A05

1. Introduction

In differential geometry, a ruled surface is a special type of surface; it is defined by choosing a curve and a line along that curve. These surfaces were found and investigated by Gaspard Monge, who established the partial differential equation that satisfies all ruled surfaces. A (differential) one-parameter family of (straight) lines $\{\alpha(t), w(t)\}$ is a correspondence that assigns to each $t \in I$, where I belongs to R a point $\alpha(t) \in R^3$ and a vector $w(t) \in R^3$, $w(t) \neq 0$, the line L_t which passes through $\alpha(t)$ and parallel to $w(t)$ is called the line of the family at t, the parametrized surface $\sigma(t,v) = \alpha(t) + vw(t), t \in I, v \in R$ is called the ruled surface generated by the family $\{\alpha(t), w(t)\}$ where $\{\alpha(t), w(t)\}$ is a one-parameter family of lines. The lines L_t are called the rulings, and the curve $\alpha(t)$ is called a directrix of the surface $\sigma(t,v)$ [1]. From the past to today, many properties of ruled surfaces have been investigated in Euclidean and non- Euclidean spaces; for example, one can see [2, 3, 4]. Applications of ruled surfaces are of the nature that they are used in civil engineering, computer programming, architecture, and solid modeling, for example [8, 9]. Smarandache-ruled surface according to the Frenet-Serret frame of a curve in E^3 is introduced by Quarab in [10]. The author is concerned with TN-Smarandache-ruled surface, TB- Smarandache-ruled surface, NB- Smarandache-ruled surface, and gives the necessary and sufficient conditions for these ruled surfaces to be developable surfaces and minimal surfaces. Smarandache-ruled surface according to a Darboux frame in E^3, also investigated by Ourab [11]. The author presents a new approach to constructing special ruled surfaces and also introduces the definitions of a Smarandache-ruled surface according to the Darboux frame of a curve lying on an arbitrary regular surface in E^3. Smarandache-ruled surface according to the Bishop frame in E^3 introduced by Senyurt, Canli, and Hilal in reference [12]. The authors investigated the fundamental forms and the corresponding curvatures and illustrated some example see[5]. Senyurt, Canli, and Con introduce some special Smarandache-ruled surfaces by Frenet frame in E^3[13]. The authors introduce some new special ruled surfaces with the base TNB- Smarandache curve, where the unit vector of the generator is taken as one of the other Frenet vectors and their linear combinations. Characteristic properties of type-2 Smarandache-ruled surface according to the type-2 Bishop frame in E^3 are introduced in [14]. Al-Dayelwas and Solouma define and investigate a special kind of ruled surface called-2 Smarandache-ruled surface related to the type-2 Bishop frame in E^3, NC- Smarandacehe-ruled surface and NW- Smarandache-ruled surface according to a curve-alternative moving frame in E^3 are examined in reference [15]. Ouarab defined NC- Smarandache-ruled surface and NW- Smarandache-ruled surface according to the alternative moving frame; the necessary and sufficient conditions for those special surfaces are introduced. Frenet-Serret frame is not continuously defined for a C^1-continuous space curve. So, for solving these problems, Coquillart and Mustafa et al.[17, 16] introduced a new frame (a quasi-frame). A study of ruled surfaces in Minkowski space reveals intriguing distinctions because of the structure of Minkowski space, but it also reveals similarities with Euclidean space; the readers can see the references [18, 19, 20, 21]. Also, some special ruled surfaces according to the Flc frame of a given polynomial curve are introduced by Süleyman et al. in reference [22].

Utilizing the quasi-frame of Smarandache curves in Euclidean 3-space, M. Khalifa Saad and A. Abdel-Baky [4] examine skew ruled surfaces. Additionally, the Author demonstrates the connection between Serret-Frenet and quasi-frames and uses the quasi-frame to provide a parametric description of a directional ruled surface. TqNqBq-Smarandache ruled surfaces are particular types of Smarandache ruled surfaces defined by a quasi-frame. M. Elzawy and A. Aloufi[23] examined these surfaces' mean curvature, Gaussian curvature, and first and second fundamental forms. The authors provide the necessary and sufficient conditions for such surfaces to be developable. Additionally, it is determined that these surfaces must be minimal surfaces. Lastly, these surfaces' normal and geodesic curvatures are examined.

In this study, we shall examine the Smarandache-Ruled surface in Euclidean 3-space using the quasi-frame. We define three distinct types of Smarandache-Ruled surfaces in our article: T_qN_q, T_qB_q, and N_qB_q Smarandache ruled surfaces by using a quasi-frame. The first and second fundamental forms, as well as the Gaussian, mean, and second Gaussian curvatures of these surfaces, are ascertained and examined. We provide the necessary and sufficient conditions for such a surface to be developable. We also derive the Smarandache-ruled surfaces distribution parameter. Additionally, conditions that must be met for such surfaces to be minimal surfaces are examined. Lastly, for certain curves on these surfaces, the distinctive features of the ruled surfaces relating to the geodesic and normal curvatures are determined.

2. Preliminaries

Let $\alpha = \alpha(s)$ be a unit speed curve in Euclidean 3-space E^3, $\alpha : I \longrightarrow R^3$, the orthonormal frame $\{t(s), n(s), b(s)\}$ is called the Frenet-Serret Frame of the curve α, where $t(s)$ is called the unit tangent vector field of $\alpha(s)$, $n(s)$ is the principal normal vector field of $\alpha(s)$, and $b(s) = t(s) \times n(s)$ is the binormal vector field of $\alpha(s)$. Throughout the paper, $\alpha'(s)$ will denote the derivative of $\alpha(s)$ with respect to the arc length parameter s, i.e $t(s) = \alpha'(s)$, $n(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$, $b(s) = t(s) \times n(s)$. In this article $\kappa(s)$ and $\tau(s)$ denote the curvature and the torsion of the curve $\alpha(s)$, and are given by $\kappa(s) = \| \alpha''(s) \|$,
$\tau(s) = - < b'(s), n(s) >$. The arc-length derivative of the Frenet-Serret frame is given by the following formula.

$$\begin{pmatrix} t'(s) \\ n'(s) \\ b'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} \tag{1}$$

Since the Frenet-Serret frame is undefined at an inflection point(i.e, curvature=0), the Frenet-Serret frame is not continuously defined for a C^1-continuous space curve. So, for solving these problems, Coquillart and Mustafa et al. [17, 16] introduced a New frame (a quasi-frame). For a unit speed curve $\alpha(s)$, the New frame (quasi-frame) along $\alpha(s)$ is given by

$$T_q(s) = t(s),\ N_q(s) = \frac{t(s) \times \zeta}{\| t(s) \times \zeta \|},\ B_q(s) = T_q(s) \times N_q(s), \tag{2}$$

Where ζ is called the projection vector, and $T_q(s)$ is a unit tangent vector field, $N_q(s)$ is the quasi-normal, and $B_q(s)$ is the quasi-binormal. The relation between the Frenet-Serret frame and the quasi-frame is given by the following formula [17, 4].

$$\begin{pmatrix} T_q(s) \\ N_q(s) \\ B_q(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\varphi & sin\varphi \\ 0 & -sin\varphi & cos\varphi \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} \tag{3}$$

Where φ is the angle between the principal normal $n(s)$ and the quasi-normal N_q (see figure 1). $\{T_q(s), N_q(s), B_q(s)\}$ is the quasi-frame, $\{t(s), n(s), b(s)\}$ is the Frenet-Serret frame. If $\varphi = 0$ the quasi-frame reduces to the Frenet-Serret frame, and if $k_3 = 0$ the quasi-frame reduces to the Bishop-frame. The variation equations of the quasi-frame in equation (3) are given by the following formula

$$\begin{pmatrix} T_q'(s) \\ N_q'(s) \\ B_q'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{pmatrix} \begin{pmatrix} T_q(s) \\ N_q(s) \\ B_q(s) \end{pmatrix} \tag{4}$$

The triple k_1, k_2, and k_3 are called the quasi-curvature functions of the curve $\alpha(s)$ and denoted by

$$k_1 =< T_q'(s), N_q(s) >= \kappa \cos\varphi, \tag{5}$$

$$k_2 =< T_q'(s), B_q(s) >= -\kappa \sin\varphi, \tag{6}$$

$$k_3 =< N_q'(s), B_q(s) >= \tau(s) + \varphi'(s). \tag{7}$$

The quasi-frame has many advantages over the Frenet-Serret frame and Bishop frame. For example, the quasi-frame can be defined along a line (κ=0). The quasi frame is singular whenever $\tau(s)$ and ζ are parallel, in that case the projection vector ζ can be chosen as $\zeta = (0, 0, 1)$, $\zeta = (0, 1, 0)$ or $\zeta = (1, 0, 0)$. The parametric equation of the ruled surface $\sigma(s, v)$ is denoted by

$$\sigma(s, v) = \alpha(s) + vX(s).$$

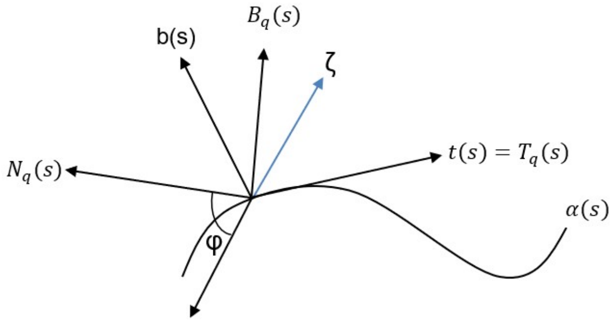

Figure 1: Relation between Frenet and quasi frames

Where $\alpha(s)$ is a curve and $X(s)$ is a generator vector. Let M be a surface in E^3, given with the parametrization $\sigma(s,v)$, the surface M is regular if $\sigma_s \times \sigma_v \neq 0$ for all points in M, where σ_s and σ_v are the partial derivatives of $\sigma(s,v)$ with respect to s and v. The coefficients of the first fundamental form for M are defined by[1]

$$E =< \sigma_s, \sigma_s >, \; F =< \sigma_s, \sigma_v >, \; G =< \sigma_v, \sigma_v > . \tag{8}$$

Where $<,>$ is the Euclidean inner product; the unit normal vector field of the regular surface M is defined by the relation,

$$U = \frac{\sigma_s \times \sigma_v}{\parallel \sigma_s \times \sigma_v \parallel}. \tag{9}$$

The coefficients of the second fundamental form of M are defined by

$$e =< \sigma_{ss}, U >, \; f =< \sigma_{sv}, U >, \; g =< \sigma_{vv}, U > . \tag{10}$$

The Gaussian curvature and the mean curvature of M are given by

$$K = \frac{eg - f^2}{EG - F^2}, \quad and \tag{11}$$

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}. \tag{12}$$

The distribution parameter λ of the ruled surface $\sigma(s,v) = \alpha(s) + vX(s)$ is given by

$$\lambda = \frac{det(\alpha'(s), X(s), X'(s))}{\parallel X'(s) \parallel}. \tag{13}$$

The second Gaussian curvature K_{II} of $\sigma(s,v)$ in E^3 is defined by replacing the components of the first fundamental form E, F, and G by the second fundamental form e, f, and g in Brioschi's formula, respectively. The second Gaussian curvature of a surface M is denoted by

$$K_{II} = \frac{1}{eg-f^2}\left(\begin{vmatrix} -\frac{1}{2}e_{vv}+f_{sv}-\frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s-\frac{1}{2}e_v \\ f_v-\frac{1}{2}g_s & e & f \\ \frac{1}{2}g_v & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_v & \frac{1}{2}g_s \\ \frac{1}{2}e_v & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix}\right). \tag{14}$$

The geodesic curvature, the normal curvature, and the geodesic torsion associated with the curve $\alpha(s)$ on the surface $\sigma(s,v)$ can be computed as follows [14]:

$$k_g =< U \wedge t(s), t'(s) >, \tag{15}$$
$$k_n =< \alpha'', U >, \tag{16}$$
$$\tau_g =< U \wedge U', t'(s) > . \tag{17}$$

For a curve $\alpha(s)$ lying on a surface, the following are important definitions,

(I) $\alpha(s)$ is a geodesic if and only if $k_g \equiv 0$.

(II) $\alpha(s)$ is a asymptotic line if and only if $k_n \equiv 0$.

(III) $\alpha(s)$ is a principal line if and only if $\tau_g \equiv 0$.

(IV) A regular surface if flat(developable) if and only if $K \equiv 0$.

(V) A regular surface is a minimal surface if $H \equiv 0$.

(VI) A surface is called II-flat if the $K_{II} \equiv 0$.

3. Main Results

In this section, we will define three types of Smarandache-ruled surfaces in Euclidean 3-space E^3 according to the quasi-frame $\{T_q, N_q, B_q\}$.

$$\begin{cases} \psi^{T_qN_q}(s,v) = \frac{1}{\sqrt{2}}(T_q(s)+N_q(s)) + vB_q(s) \\ \psi^{T_qB_q}(s,v) = \frac{1}{\sqrt{2}}(T_q(s)+B_q(s)) + vN_q(s) \\ \psi^{N_qB_q}(s,v) = \frac{1}{\sqrt{2}}(N_q(s)+B_q(s)) + vT_q(s) \end{cases}$$

3.1. T_qN_q-Smarandache-Ruled Surface according to quasi frame

In this subsection we will define the T_qN_q-Smarandache-ruled surface according to quasi frame $\psi^{T_qN_q}(s,v)$. We will compute the first and the second fundamental forms of such a surface. We will also introduce the mean curvature and the Gaussian curvature of $\psi^{T_qN_q}(s,v)$.

Definition 1. *For a regular curve $\alpha(s)$ in E^3 with quasi frame $\{T_q(s), N_q(s), B_q(s)\}$, the T_qN_q-Smarandache-ruled surface according to quasi frame is defined as follows:*

$$\psi^{T_qN_q}(s,v) = \frac{1}{\sqrt{2}}(T_q(s) + N_q(s)) + vB_q(s). \tag{18}$$

Theorem 1. *Let $\psi^{T_qN_q}(s,v)$ be T_qN_q-Smarandache-Ruled Surface defined by equation* (18)*. Then the mean curvature $H^{T_qN_q}$ of $\psi^{T_qN_q}(s,v)$ will be given by*

$$\begin{aligned} H^{T_qN_q} =& \frac{1}{2(v^2(k_2^2+k_3^2)+\sqrt{2}vk_1(k_2-k_3)+k_1^2)\sqrt{(k_1-v\sqrt{2}k_3)^2+(k_1+v\sqrt{2}k_2)^2}} \\ & [\sqrt{2}v^2(k_3k_2' - k_1k_3^2 - k_3'k_2 - k_1k_2^2) + v(2k_1^2k_3 - 2k_1^2k_2 - k_2'k_1 + k_3k_1' + k_2k_1' - k_3'k_1) \\ & + \sqrt{2}(k_1k_2k_3 - k_1^3) + \frac{\sqrt{2}k_1}{2}(k_2^2+k_3^2)] \end{aligned}$$

and the Gaussian curvature $K^{T_qN_q}$ of $\psi^{T_qN_q}(s,v)$ will be given by

$$K^{T_qN_q} = \frac{-k_1^2(k_2+k_3)^2}{((k_1-v\sqrt{2}k_3)^2+(k_1+v\sqrt{2}k_2)^2)(v^2(k_2^2+k_3^2)+\sqrt{2}vk_1(k_2-k_3)+k_1^2)}.$$

Proof. Consider the $\psi^{T_qN_q}(s,v)$ be the T_qN_q-Smarandache-Ruled Surface according to quasi frame $\{T_q(s), N_q(s), B_q(s)\}$, then by using equation(18).

$$\begin{aligned} \psi_s^{T_qN_q} &= (\frac{-k_1}{\sqrt{2}} - vk_2)T_q + (\frac{k_1}{\sqrt{2}} - vk_3)N_q + (\frac{k_2}{\sqrt{2}} + \frac{k_3}{\sqrt{2}})B_q, \\ \psi_v^{T_qN_q} &= B_q, \\ \psi_s^{T_qN_q} \times \psi_v^{T_qN_q} &= \frac{1}{\sqrt{2}}[(k_1 - v\sqrt{2}k_3)T_q + (k_1 + v\sqrt{2}k_2)N_q], \\ \| \psi_s^{T_qN_q} \times \psi_v^{T_qN_q} \| &= \frac{1}{\sqrt{2}}\sqrt{(k_1-v\sqrt{2}k_3)^2+(k_1+v\sqrt{2}k_2)^2}. \end{aligned}$$

The unit normal of $\psi^{T_qN_q}(s,v)$ is given by

$$U^{T_qN_q} = \frac{\psi_s^{T_qN_q} \times \psi_v^{T_qN_q}}{\| \psi_s^{T_qN_q} \times \psi_v^{T_qN_q} \|} = \frac{(k_1-v\sqrt{2}k_3)T_q + (k_1+v\sqrt{2}k_2)N_q}{\sqrt{(k_1-v\sqrt{2}k_3)^2+(k_1+v\sqrt{2}k_2)^2}}.$$

The components of the first fundamental form of the T_qN_q-Smarandache ruled surface $\psi^{T_qN_q}(s,v)$ according to the quasi-frame will be given by the following relation:

$$\begin{aligned} E^{T_qN_q} &= (v^2+\frac{1}{2})(k_2^2+k_3^2)+\sqrt{2}k_1v(k_2-k_3)+k_2k_3+k_1^2,\\ F^{T_qN_q} &= \frac{1}{\sqrt{2}}(k_2+k_3),\\ G^{T_qN_q} &= 1. \end{aligned}$$

The second derivatives $\psi_{ss}^{T_qN_q}$, $\psi_{sv}^{T_qN_q}$ and $\psi_{vv}^{T_qN_q}$ are

$$\begin{aligned} \psi_{ss}^{T_qN_q} &= [-\frac{k_1'}{\sqrt{2}}-vk_2'-\frac{k_1^2}{\sqrt{2}}+vk_3k_1-\frac{k_2^2}{\sqrt{2}}-\frac{k_2k_3}{\sqrt{2}}]T_q+[-\frac{k_1^2}{\sqrt{2}}-vk_2k_1+\frac{k_1'}{\sqrt{2}}\\ &\quad -vk_3'-\frac{k_2k_3}{\sqrt{2}}-\frac{k_3^2}{\sqrt{2}}]N_q+[-\frac{k_1k_2}{\sqrt{2}}-vk_2^2+\frac{k_3k_1}{\sqrt{2}}-vk_3^2+\frac{k_2'}{\sqrt{2}}+\frac{k_3'}{\sqrt{2}}]B_q,\\ \psi_{sv}^{T_qN_q} &= -k_2T_q-k_3N_q,\\ \psi_{vv}^{T_qN_q} &= 0. \end{aligned}$$

The components of the second fundamental form of the T_qN_q-Smarandache ruled surface $\psi^{T_qN_q}(s,v)$ according to the quasi-frame will be given by the following relations:

$$\begin{aligned} e^{T_qN_q} &= \frac{v^2\gamma_1(s)+v\gamma_2(s)+\gamma_3(s)}{\mu_1(s,v)},\\ f^{T_qN_q} &= \frac{-k_1(k_2+k_3)}{\mu_1(s,v)},\\ g^{T_qN_q} &= 0. \end{aligned}$$

Where

$$\begin{aligned} \mu_1(s,v) &= \sqrt{(k_1-v\sqrt{2}k_3)^2+(k_1+v\sqrt{2}k_2)^2},\\ \gamma_1(s) &= \sqrt{2}(k_3k_2'-k_1k_3^2-k_1k_2^2-k_3'k_2),\\ \gamma_2(s) &= 2k_1^2k_3-2k_1^2k_2-k_2'k_1+k_3k_1'+k_2k_1'-k_3'k_1,\\ \gamma_3(s) &= -\frac{k_1k_2^2}{\sqrt{2}}-\frac{k_1k_3^2}{\sqrt{2}}-\sqrt{2}k_1^3-\sqrt{2}k_1k_2k_3. \end{aligned}$$

The Gaussian curvature and the mean curvature of T_qN_q-Smarandache ruled surface $\psi^{T_qN_q}(s,v)$ be

$$K^{T_qN_q} = \frac{-k_1^2(k_2+k_3)^2}{((k_1-v\sqrt{2}k_3)^2+(k_1+v\sqrt{2}k_2)^2)(v^2(k_2^2+k_3^2)+\sqrt{2}vk_1(k_2-k_3)+k_1^2)}, \tag{19}$$

$$\begin{aligned} H^{T_qN_q} =& \frac{1}{2(v^2(k_2^2+k_3^2)+\sqrt{2}vk_1(k_2-k_3)+k_1^2)\sqrt{(k_1-v\sqrt{2}k_3)^2+(k_1+v\sqrt{2}k_2)^2}} \\ & [\sqrt{2}v^2(k_3k_2'-k_1k_3^2-k_3'k_2-k_1k_2^2)+v(2k_1^2k_3-2k_1^2k_2-k_2'k_1+k_3k_1'+k_2k_1'-k_3'k_1) \\ & +\sqrt{2}(k_1k_2k_3-k_1^3)+\frac{\sqrt{2}k_1}{2}(k_2^2+k_3^2)] \end{aligned}$$

Corollary 1. *The second Gaussian curvature of T_qN_q-Smarandache-Ruled Surface is*

$$K_{II}^{T_qN_q} = \frac{1}{(f^{T_qN_q})^2}(\frac{1}{2}e_{vv}^{T_qN_q}-f_{sv}^{T_qN_q})+\frac{f_v^{T_qN_q}}{(f^{T_qN_q})^3}(f_s^{T_qN_q}-\frac{1}{2}e_v^{T_qN_q})$$

where

$$\begin{aligned} f^{T_qN_q} =& \frac{-k_1(k_2+k_3)}{\mu_1(s,v)} \\ f_v^{T_qN_q} =& \frac{k_1(k_2+k_3)\mu_{1v}}{\mu_1^2(s,v)} \\ f_s^{T_qN_q} =& \frac{1}{\mu_1^2}[\mu_1(-k_1(k_2'+k_3')-k_1'(k_2+k_3))+k_1(k_2+k_3)\mu_{1s}] \\ f_{sv}^{T_qN_q} =& \frac{1}{\mu_1^4}[\mu_1^2(\mu_{1v}(-k_1(k_2'+k_3')-k_1'(k_2+k_3)+k_1(k_2+k_3)\mu_{1sv})+2\mu_1\mu_{1v} \\ & (\mu_1(k_1(k_2'+k_3')+k_1'(k_2+k_3))-k_1(k_2+k_3)\mu_{1s})] \\ e_v^{T_qN_q} =& \frac{1}{\mu_1^2}[-\gamma_1\mu_{1v}v^2+(2\gamma_1\mu_1-\gamma_2\mu_{1v})v+\gamma_2\mu_1-\gamma_3\mu_{1v}] \\ e_{vv}^{T_qN_q} =& \frac{1}{\mu_1^4}[(-\gamma_1\mu_{1vv}\mu_1^2+2\gamma_1\mu_1\mu_{1v}^2)v^2+(-\gamma_2\mu_{1vv}\mu_1^2-4\gamma_1\mu_{1v}\mu_1^2+2\gamma_2\mu_{1v}^2\mu_1)v \\ & +2\gamma_1\mu_1^3-\gamma_3\mu_{1vv}\mu_1^2-2\gamma_2\mu_1^2\mu_{1v}+2\gamma_3\mu_{1v}^2\mu_1]. \end{aligned}$$

Corollary 2. *T_qN_q-Smarandache-Ruled Surface is developable if and only if $k_1 = 0$ or $k_2 = -k_3$.*

Proof. Let $\psi^{T_qN_q}$ be a T_qN_q-Smarandache-Ruled Surface; then the Gaussian curvature is given by equation (19). The Gaussian curvature equal zero if $k_1^2(k_2+k_3)^2=0$ i.e when $k_1=0$ or $k_2=-k_3$.

Corollary 3. *T_qN_q-Smarandache-Ruled Surface is a minimal surface if and only if the quasi curvatures satisfy the following differential equation*

$$\sqrt{2}v^2(k_3k_2'-k_1k_3^2-k_3'k_2-k_1k_2^2)+v(2k_1^2k_3-2k_1^2k_2-k_2'k_1+k_3k_1'+k_2k_1'-k_3'k_1)$$
$$+\sqrt{2}(k_1k_2k_3-k_1^3)+\frac{\sqrt{2}k_1}{2}(k_2^2+k_3^2)=0.$$

Corollary 4. *The geodesic curvature, and the normal curvature for the T_qN_q-Smarandache curve is given by*

$$k_g=\frac{-k_2(k_1+v\sqrt{2}k_2)}{\sqrt{(k_1-v\sqrt{2}k_3)^2(k_1+v\sqrt{2}k_2)^2}},$$

$$k_n=\frac{k_1^2\sqrt{2}v(k_3-k_2)+vk_1'\sqrt{2}(k_3+k_2)-2k_1^3-2k_1k_2k_3-k_1(k_2^2+k_3^2))}{\sqrt{2}\sqrt{(k_1-v\sqrt{2}k_3)^2(k_1+v\sqrt{2}k_2)^2}}.$$

Corollary 5. *The distribution parameter for the T_qN_q Smarandache ruled surface is given by*

$$\lambda^{T_qN_q}=\frac{-k_1(k_2+k_3)}{\sqrt{2}\sqrt{k_2^2+k_3^2}}.$$

3.2. N_qB_q-Smarandache-Ruled Surface according to quasi frame

In this subsection, we will introduce the N_qB_q-Smarandache-ruled surface according to quasi frame $\{T_q(s),N_q(s),B_q(s)\}$, the first and second fundamental form of $\psi^{N_qB_q}(s,v)$ will be obtained. Also, the mean curvature and the Gaussian curvature of $\psi^{N_qB_q}(s,v)$ will be obtained.

Definition 2. *For a regular curve $\alpha=\alpha(s)$ in E^3 with quasi frame $\{T_q(s),N_q(s),B_q(s)\}$, the N_qB_q-Smarandache-Ruled Surface according to quasi frame $\psi^{N_qB_q}(s,v)$ will be given by*

$$\psi^{N_qB_q}(s,v)=\frac{1}{\sqrt{2}}(N_q(s)+B_q(s))+vT_q(s). \tag{20}$$

Theorem 2. *Let $\psi^{N_qB_q}(s,v)$ be N_qB_q-Smarandache-Ruled Surface defined by equation (20). Then the mean curvature $H^{N_qB_q}$ of $\psi^{N_qB_q}(s,v)$ will be introduced by the following equation*

$$H^{N_qB_q} = \frac{1}{2(v^2(k_1^2+k_2^2)+v\sqrt{2}k_3(k_2-k_1)+k_3^2)\sqrt{(k_3+\sqrt{2}vk_2)^2+(k_3-\sqrt{2}vk_1)^2}}$$
$$[\sqrt{2}v^2(-k_2'k_1-k_1^2k_3-k_2^2k_3+k_2k_1')+v(k_1'k_3-2k_2k_3^2-k_2k_3'+2k_1k_3^2-k_1k_3'+k_2'k_3)$$
$$+\sqrt{2}(k_1k_2k_3-k_3^3)+\frac{\sqrt{2}k_3}{2}(k_1^2+k_2^2)].$$

and the Gaussian curvature $K^{N_qB_q}$ of $\psi^{N_qB_q}(s,v)$ will be introduced by the following equation

$$K^{N_qB_q} = \frac{-k_3^2(k_1+k_2)^2}{((k_3+\sqrt{2}vk_2)^2+(k_3-\sqrt{2}vk_1)^2)(v^2(k_1^2+k_2^2)+v\sqrt{2}k_3(k_2-k_1)+k_3^2)}.$$

Proof. Consider the $\psi^{N_qB_q}(s,v)$ be the N_qB_q-Smarandache-Ruled Surface, then by using equation(20).

$$\psi_s^{N_qB_q} = (\frac{-k_1}{\sqrt{2}}-\frac{k_2}{\sqrt{2}})T_q+(\frac{-k_3}{\sqrt{2}}+vk_1)N_q+(\frac{k_3}{\sqrt{2}}+vk_2)B_q,$$
$$\psi_v^{N_qB_q} = T_q,$$
$$\psi_s^{N_qB_q}\times\psi_v^{N_qB_q} = \frac{1}{\sqrt{2}}[(k_3+\sqrt{2}vk_2)N_q+(k_3-\sqrt{2}vk_1)B_q],$$
$$\parallel \psi_s^{N_qB_q}\times\psi_v^{N_qB_q} \parallel = \frac{1}{\sqrt{2}}\sqrt{(k_3+\sqrt{2}vk_2)^2+(k_3-\sqrt{2}vk_1)^2}.$$

The unit normal of $\psi^{N_qB_q}(s,v)$ is given by

$$U^{N_qB_q} = \frac{\psi_s^{N_qB_q}\times\psi_v^{N_qB_q}}{\parallel \psi_s^{N_qB_q}\times\psi_v^{N_qB_q} \parallel} = \frac{(k_3+\sqrt{2}vk_2)N_q+(k_3-\sqrt{2}vk_1)B_q}{\sqrt{(k_3+\sqrt{2}vk_2)^2+(k_3-\sqrt{2}vk_1)^2}}.$$

The components of the first fundamental form of N_qB_q-Smarandache ruled surface $\psi^{N_qB_q}(s,v)$ according to quasi frame

$$E^{N_qB_q} = (v^2+\frac{1}{2})(k_1^2+k_2^2)+v\sqrt{2}k_3(k_2-k_1)+k_1k_2+k_3^2,$$
$$F^{N_qB_q} = \frac{-1}{\sqrt{2}}(k_1+k_2),$$

$$G^{N_qB_q} = 1.$$

The second derivatives $\psi_{ss}^{T_qN_q}$, $\psi_{sv}^{T_qN_q}$ and $\psi_{vv}^{T_qN_q}$ are

$$\begin{aligned}
\psi_{ss}^{N_qB_q} &= [-\frac{k_1'}{\sqrt{2}} - \frac{k_2'}{\sqrt{2}} + \frac{k_1k_3}{\sqrt{2}} - vk_1^2 - \frac{k_2k_3}{\sqrt{2}} - vk_2^2]T_q + [-\frac{k_1^2}{\sqrt{2}} - vk_2k_3 - \frac{k_3'}{\sqrt{2}} \\
&+ vk_1' - \frac{k_2k_1}{\sqrt{2}} - \frac{k_3^2}{\sqrt{2}}]N_q + [\frac{k_3'}{\sqrt{2}} + vk_2' - \frac{k_3^2}{\sqrt{2}} + vk_1k_3 - \frac{k_1k_2}{\sqrt{2}} - \frac{k_2^2}{\sqrt{2}}]B_q, \\
\psi_{sv}^{N_qB_q} &= k_1N_q + k_2B_q, \\
\psi_{vv}^{N_qB_q} &= 0.
\end{aligned}$$

The components of the second fundamental form of N_qB_q-Smarandache ruled surface $\psi^{N_qB_q}(s,v)$ according to quasi frame

$$\begin{aligned}
e^{N_qB_q} &= \frac{v^2\beta_1(s) + v\beta_2 + \beta_3}{\mu_2(s,v)}, \\
f^{N_qB_q} &= \frac{k_3(k_1+k_2)}{\mu_2(s,v)}, \\
g^{N_qB_q} &= 0.
\end{aligned}$$

Where,

$$\begin{aligned}
\mu_2(s,v) &= \sqrt{(k_3 + \sqrt{2}vk_2)^2 + (k_3 - \sqrt{2}vk_1)^2}, \\
\beta_1(s) &= \sqrt{2}(k_2k_1' - k_2'k_1 - k_1^2k_3 - k_2^2k_3), \\
\beta_2(s) &= k_1'k_3 - k_2k_3' - k_1k_3' + k_2'k_3 - 2k_2k_3^2 + 2k_1k_3^2, \\
\beta_3(s) &= -\sqrt{2}(k_1k_2k_3 + k_3^2) - \frac{k_3}{\sqrt{2}}(k_1^2 + k_2^2).
\end{aligned}$$

The Gaussian curvature $K^{N_qB_q}$ and the mean curvature $H^{N_qB_q}$ of N_qB_q-Smarandache ruled surface according to quasi frame,

$$K^{N_qB_q} = \frac{-k_3^2(k_1+k_2)^2}{((k_3 + \sqrt{2}vk_2)^2 + (k_3 - \sqrt{2}vk_1)^2)(v^2(k_1^2 + k_2^2) + v\sqrt{2}k_3(k_2 - k_1) + k_3^2)},$$

$$\begin{aligned}
H^{N_qB_q} &= \frac{1}{2(v^2(k_1^2 + k_2^2) + v\sqrt{2}k_3(k_2 - k_1) + k_3^2)\sqrt{(k_3 + \sqrt{2}vk_2)^2 + (k_3 - \sqrt{2}vk_1)^2}} \\
&[\sqrt{2}v^2(-k_2'k_1 - k_1^2k_3 - k_2^2k_3 + k_2k_1') + v(k_1'k_3 - 2k_2k_3^2 - k_2k_3' + 2k_1k_3^2 - k_1k_3' + k_2'k_3)
\end{aligned}$$

$$+\sqrt{2}(k_1k_2k_3-k_3^3)+\frac{\sqrt{2}k_3}{2}(k_1^2+k_2^2)].$$

Corollary 6. *The second Gaussian curvature of N_qB_q-Smarandache-Ruled Surface is*

$$K_{II}^{N_qB_q}=\frac{1}{(f^{N_qB_q})^2}(\frac{1}{2}e_{vv}^{N_qB_q}-f_{sv}^{N_qB_q})+\frac{f_v^{N_qB_q}}{(f^{N_qB_q})^3}(f_s^{N_qB_q}-\frac{1}{2}e_v^{N_qB_q})$$

where

$$\begin{aligned}
f^{N_qB_q} &= \frac{k_3(k_1+k_2)}{\mu_2(s,v)}\\
f_v^{N_qB_q} &= \frac{-k_3(k_1+k_2)\mu_{2v}}{\mu_2^2(s,v)}\\
f_s^{N_qB_q} &= \frac{1}{\mu_2^2}[\mu_2k_3(k_1'+k_2')+\mu_2k_3'(k_1+k_2))-k_3(k_1+k_2)\mu_{2s}]\\
f_{sv}^{N_qB_q} &= \frac{1}{\mu_2^4}[\mu_2^2(\mu_{2v}(k_3(k_1'+k_2')+k_3'(k_1+k_2))-k_3(k_1+k_2)\mu_{2sv})-2\mu_2\mu_{2v}(\mu_2k_3(k_1'+k_2')\\
&\quad+\mu_2k_3'(k_1+k_2)-k_3(k_1+k_2)\mu_{2s})]\\
e_v^{N_qB_q} &= \frac{1}{\mu_2^2}[-\beta_1\mu_{2v}v^2+(2\beta_1\mu_2-\beta_2\mu_{2v})v+\beta_2\mu_2-\beta_3\mu_{2v}]\\
e_{vv}^{N_qB_q} &= \frac{1}{\mu_2^4}[(-\beta_1\mu_{2vv}\mu_2^2+2\beta_1\mu_2\mu_{2v}^2)v^2+(-\beta_2\mu_{2vv}\mu_2^2-4\beta_1\mu_{2v}\mu_2^2+2\beta_2\mu_{2v}^2\mu_2)v\\
&\quad+2\beta_1\mu_2^3-\beta_3\mu_{2vv}\mu_2^2-2\beta_2\mu_2^2\mu_{2v}+2\beta_3\mu_{2v}^2\mu_2].
\end{aligned}$$

Corollary 7. *N_qB_q-Smarandache-Ruled Surface is developable if and only if $k_3=0$ or $k_1=-k_2$.*

Proof. The proof is clear.

Corollary 8. *N_qB_q-Smarandache-Ruled Surface is minimal surface if and only if the quasi curvatures satisfy the following differential equation*

$$\sqrt{2}v^2(-k_2'k_1-k_1^2k_3-k_2^2k_3+k_2k_1')+v(k_1'k_3-2k_2k_3^2+2k_1k_3^2-k_1k_3'-k_2k_3'+k_2'k_3)$$
$$+\sqrt{2}(k_1k_2k_3-k_3^3)+\frac{\sqrt{2}k_3}{2}(k_1^2+k_2^2)=0.$$

Corollary 9. *The geodesic curvature, and the normal curvature for the N_qB_q-Smarandache curve is given by*

$$k_g = \frac{k_1(k_3 - vk_1\sqrt{2}) - k_2(k_3 + vk_2\sqrt{2})}{\sqrt{(k_3 + vk_2\sqrt{2})^2 + (k_3 - vk_1\sqrt{2})^2}},$$

$$k_n = \frac{\sqrt{2}k_1v(k_3^2 - k_3') - \sqrt{2}vk_2(k_3^2 + k_3') - 2k_1k_2k_3 - 2k_3^3 - k_2^2k_3 - k_1^2k_3}{\sqrt{2}\sqrt{(k_3 + vk_2\sqrt{2})^2 + (k_3 - vk_1\sqrt{2})^2}}.$$

Corollary 10. *The distribution parameter for the N_qB_q Smarandache ruled surface is given by*

$$\lambda^{N_qB_q} = \frac{k_3(k_1 + k_2)}{\sqrt{2}\sqrt{k_1^2 + k_2^2}}.$$

3.3. T_qB_q-Smarandache-Ruled Surface according to quasi frame

In this subsection, we will introduce the T_qB_q-Smarandache-ruled surface according to quasi frame $\{T_q(s), N_q(s), B_q(s)\}$, the first and second fundamental form of $\psi^{T_qB_q}(s,v)$ will be obtained. Also, the mean curvature and the Gaussian curvature of $\psi^{T_qB_q}(s,v)$ will be given.

Definition 3. *For a regular curve $\alpha = \alpha(s)$ in E^3 with quasi frame $\{T_q(s), N_q(s), B_q(s)\}$, the T_qB_q-Smarandache-Ruled Surface according to quasi frame $\psi^{T_qB_q}(s,v)$ will be defined as follows*

$$\psi^{T_qB_q}(s,v) = \frac{1}{\sqrt{2}}(T_q(s) + B_q(s)) + vN_q(s). \tag{21}$$

Theorem 3. *Let $\psi^{T_qB_q}(s,v)$ be T_qB_q-Smarandache-Ruled Surface defined by equation (21). Then the mean curvature $H^{T_qB_q}$ of $\psi^{T_qB_q}(s,v)$ will be introduced by*

$$H^{T_qB_q} = \frac{1}{2(v^2(k_1^2 + k_3^2)) + \sqrt{2}vk_2(k_1 + k_3) + k_2^2)\sqrt{(k_2 + \sqrt{2}vk_3)^2 + (k_2 + \sqrt{2}vk_1)^2}}$$
$$[\sqrt{2}v^2(k_1'k_3 + k_3^2k_2 + k_1^2k_2 - k_3'k_1) + v(k_1'k_2 + 2k_2^2k_3 + k_2'k_3 + 2k_2^2k_1 - k_3'k_2$$
$$- k_2'k_1) + \sqrt{2}(k_2^3 + k_1k_2k_3) - \frac{\sqrt{2}k_2}{2}(k_1^2 + k_3^2)],$$

and the Gaussian curvature $K^{T_qB_q}$ of $\psi^{T_qB_q}(s,v)$ will be introduced by the following equation

$$K^{T_qB_q} = \frac{-k_2^2(k_1 - k_3)^2}{((k_2 + \sqrt{2}vk_3)^2 + (k_2 + \sqrt{2}vk_1)^2)(v^2(k_1^2 + k_3^2) + \sqrt{2}vk_2(k_1 + k_3) + k_2^2)}.$$

Proof. Consider the $\psi^{T_qB_q}(s,v)$ be the T_qB_q-Smarandache-Ruled Surface according to quasi frame, then by using equation(21).

$$\begin{aligned}
\psi_s^{T_qB_q} &= (\frac{-k_2}{\sqrt{2}} - vk_1)T_q + (\frac{k_1}{\sqrt{2}} - \frac{k_3}{\sqrt{2}})N_q + (\frac{k_2}{\sqrt{2}} + vk_3)B_q,\\
\psi_v^{T_qB_q} &= N_q,\\
\psi_s^{T_qB_q} \times \psi_v^{T_qB_q} &= \frac{1}{\sqrt{2}}[(-k_2 - \sqrt{2}vk_3)T_q + (-k_2 - \sqrt{2}vk_1)B_q],\\
\| \psi_s^{T_qB_q} \times \psi_v^{T_qB_q} \| &= \frac{1}{\sqrt{2}}\sqrt{(k_2 + \sqrt{2}vk_3)^2 + (k_2 + \sqrt{2}vk_1)^2}.
\end{aligned}$$

The unit normal of $\psi^{T_qB_q}(s,v)$ is given by

$$U^{T_qB_q} = \frac{\psi_s^{T_qB_q} \times \psi_v^{T_qB_q}}{\| \psi_s^{T_qB_q} \times \psi_v^{T_qB_q} \|} = \frac{(-k_2 - \sqrt{2}vk_3)T_q + (-k_2 - \sqrt{2}vk_1)B_q}{\sqrt{(k_2 + \sqrt{2}vk_3)^2 + (k_2 + \sqrt{2}vk_1)^2}}.$$

The components of the first fundamental form of T_qB_q-Smarandache ruled surface $\psi^{T_qB_q}(s,v)$ according to quasi frame,

$$\begin{aligned}
E^{T_qB_q} &= (v^2 + \frac{1}{2})(k_1^2 + k_3^2) + \sqrt{2}vk_2(k_1 + k_3) + k_2^2 - k_1k_3,\\
F^{T_qB_q} &= \frac{1}{\sqrt{2}}(k_1 - k_3),\\
G^{T_qB_q} &= 1.
\end{aligned}$$

The second derivatives $\psi_{ss}^{T_qB_q}$, $\psi_{sv}^{T_qB_q}$ and $\psi_{vv}^{T_qB_q}$ by the following equation

$$\begin{aligned}
\psi_{ss}^{T_qB_q} &= (\frac{-k_2'}{\sqrt{2}} - vk_1' - \frac{k_1^2}{\sqrt{2}} + \frac{k_1k_3}{\sqrt{2}} - \frac{k_2^2}{\sqrt{2}} - k_2k_3v)Tq + (-\frac{k_1k_2}{\sqrt{2}} - vk_1^2 + \frac{k_1'}{\sqrt{2}}\\
&\quad - \frac{k_3'}{\sqrt{2}} - \frac{k_2k_3}{\sqrt{2}} - vk_3^2)N_q + (-\frac{k_2^2}{\sqrt{2}} - vk_1k_2 + \frac{k_1k_3}{\sqrt{2}} - \frac{k_3^2}{\sqrt{2}} + \frac{k_2'}{\sqrt{2}} + vk_3')B_q,\\
\psi_{vv}^{T_qB_q} &= 0,\\
\psi_{vs}^{T_qB_q} &= -k_1T_q + k_3B_q.
\end{aligned}$$

The components of the second fundamental form of T_qB_q-Smarandache ruled surface $\psi^{T_qB_q}(s,v)$ according to quasi frame,

$$e^{T_qB_q} = \frac{v^2\delta_1(s) + v\delta_2(s) + \delta_3(s)}{\mu_3(s,v)},$$

$$f^{T_qB_q} = \frac{k_2(k_1 - k_3)}{\mu_3(s,v)},$$
$$g^{T_qB_q} = 0.$$

Where

$$\mu_3(s,v) = \sqrt{(k_2 + \sqrt{2}vk_3)^2 + (k_2 + \sqrt{2}vk_1)^2},$$
$$\delta_1(s) = \sqrt{2}(k_1'k_3 + k_3^2k_2 + k_1^2k_2 - k_3'k_1),$$
$$\delta_2(s) = k_1'k_2 + 2k_2^2k_3 + k_2'k_3 + 2k_2^2k_1 - k_3'k_2 - k_2'k_1,$$
$$\delta_3(s) = \sqrt{2}(k_2^3 - k_1k_2k_3) + \frac{k_2}{\sqrt{2}}(k_1^2 + k_3^2).$$

The Gaussian curvature $K^{T_qB_q}$ and the mean curvature $H^{T_qB_q}$ of T_qB_q-Smarandache ruled surface $\psi^{T_qB_q}(s,v)$,

$$K^{T_qB_q} = \frac{-k_2^2(k_1-k_3)^2}{((k_2+\sqrt{2}vk_3)^2 + (k_2+\sqrt{2}vk_1)^2)(v^2(k_1^2+k_3^2) + \sqrt{2}vk_2(k_1+k_3) + k_2^2)},$$

$$H^{T_qB_q} = \frac{1}{2(v^2(k_1^2+k_3^2)) + \sqrt{2}vk_2(k_1+k_3) + k_2^2)\sqrt{(k_2+\sqrt{2}vk_3)^2 + (k_2+\sqrt{2}vk_1)^2}}$$
$$[\sqrt{2}v^2(k_1'k_3 + k_3^2k_2 + k_1^2k_2 - k_3'k_1) + v(k_1'k_2 + 2k_2^2k_3 + k_2'k_3 + 2k_2^2k_1 - k_3'k_2$$
$$- k_2'k_1) + \sqrt{2}(k_2^3 + k_1k_2k_3) - \frac{\sqrt{2}k_2}{2}(k_1^2 + k_3^2)].$$

Corollary 11. *The second Gaussian curvature of T_qB_q-Smarandache-Ruled Surface is*

$$K_{II}^{T_qB_q} = \frac{1}{(f^{T_qB_q})^2}(\frac{1}{2}e_{vv}^{T_qB_q} - f_{sv}^{T_qB_q}) + \frac{f_v^{T_qB_q}}{(f^{T_qB_q})^3}(f_s^{T_qB_q} - \frac{1}{2}e_v^{T_qB_q})$$

where

$$f^{T_qB_q} = \frac{k_2(k_1-k_3)}{\mu_3(s,v)}$$
$$f_v^{T_qB_q} = \frac{-k_2(k_1-k_3)\mu_{3v}}{\mu_3^2(s,v)}$$
$$f_s^{T_qB_q} = \frac{1}{\mu_3^2}[\mu_3(k_2'(k_1-k_3) + k_2(k_1'-k_3')) - \mu_{3s}k_2(k_1-k_3)]$$

$$
\begin{aligned}
f_{sv}^{T_qB_q} =&\frac{1}{\mu_3^4}[\mu_3^2(\mu_{3v}(k_2'(k_1-k_3)+k_2(k_1'-k_3'))-\mu_{3sv}k_2(k_1-k_3))-2\mu_3\mu_{3v}(\mu_3k_2'(k_1-k_3)\\
&+\mu_3k_2(k_1'-k_3')-\mu_{3s}k_2(k_1-k_3))]\\
e_v^{T_qB_q} =&\frac{1}{\mu_3^2}[-\delta_1\mu_{3v}v^2+(2\delta_1\mu_3-\delta_2\mu_{3v})v+\delta_2\mu_3-\delta_3\mu_{3v}]\\
e_{vv}^{T_qB_q} =&\frac{1}{\mu_3^4}[(-\delta_1\mu_{3vv}\mu_3^2+2\delta_1\mu_3\mu_{3v}^2)v^2+(-\delta_2\mu_{3vv}\mu_3^2-4\delta_1\mu_{3v}\mu_3^2+2\delta_2\mu_{3v}^2\mu_3)v\\
&+2\delta_1\mu_3^3-\delta_3\mu_{3vv}\mu_3^2-2\delta_2\mu_3^2\mu_{3v}+2\delta_3\mu_{3v}^2\mu_3].
\end{aligned}
$$

Corollary 12. *T_qB_q-Smarandache-Ruled Surface is developable if and only if $k_2 = 0$ or $k_1 = k_3$.*

Proof. The proof is clear.

Corollary 13. *T_qB_q-Smarandache-Ruled Surface is minimal surface if and only if the quasi curvatures satisfy the following differential equation*

$$
\begin{aligned}
&\sqrt{2}v^2(k_1'k_3+k_3^2k_2+k_1^2k_2-k_3'k_1)+v(k_1'k_2+2k_2^2k_3+k_2'k_3+2k_2^2k_1-k_3'k_2-k_2'k_1)\\
&+\sqrt{2}(k_2^3+k_1k_2k_3)-\frac{\sqrt{2}k_2}{2}(k_1^2+k_3^2)=0.
\end{aligned}
$$

Corollary 14. *The geodesic curvature, and the normal curvature for the T_qB_q-Smarandache curve is given by*

$$
k_g=\frac{-k_1(k_2+vk_1\sqrt{2})}{\sqrt{(k_2+\sqrt{2}vk_3)^2+(k_2+\sqrt{2}vk_1)^2}},
$$

$$
k_n=\frac{\sqrt{2}vk_1(k_2^2-k_2')+\sqrt{2}vk_3(k_2^2+k_2')-2k_1k_2k_3+2k_2^3+k_2(k_1^2+k_3^2)}{\sqrt{2}\sqrt{(k_2+\sqrt{2}vk_3)^2+(k_2+\sqrt{2}vk_1)^2}}.
$$

Corollary 15. *The distribution parameter for the T_qB_q Smarandache ruled surface is given by*

$$
\lambda^{T_qB_q}=\frac{k_2(k_1-k_3)}{\sqrt{2}\sqrt{k_1^2+k_3^2}}.
$$

Example 1. *Consider the unit speed helix curve $\alpha(s)=(\frac{3}{5}sins,\frac{3}{5}coss,\frac{4}{5}s)$.*

$$
\begin{aligned}
T_q&=(\frac{3}{5}coss,-\frac{3}{5}sins,\frac{4}{5}),\\
N_q&=(-sins,-coss,0),
\end{aligned}
$$

$$B_q = (\frac{4}{5}coss, -\frac{4}{5}sins, -\frac{3}{5}).$$

Where $\zeta = (0,0,1)$ *and the quasi curvatures are as follows*

$$k_1 = \frac{3}{5}, \;\; k_2 = 0, \;\; k_3 = -\frac{4}{5}.$$

The T_qN_q*-Smarandache-Ruled Surface (see figure 2) is given by*

$$\psi^{T_qN_q}(s,v) = (\frac{1}{\sqrt{2}}(\frac{3}{5}coss - sins) + \frac{4v}{5}coss, \frac{1}{\sqrt{2}}(-\frac{3}{5}sins - coss) - \frac{4v}{5}sins, \frac{4}{5\sqrt{2}} - \frac{3v}{5}).$$

The first fundamental form of T_qN_q *is obtained*

$$\begin{aligned} E^{T_qN_q} =& \frac{16}{25}v^2 + \frac{12\sqrt{2}}{25}v + \frac{17}{25} \\ F^{T_qN_q} =& \frac{-2\sqrt{2}}{5} \\ G^{T_qN_q} =& 1 \end{aligned}$$

The second fundamentals of T_qN_q *are obtained*

$$\begin{aligned} e^{T_qN_q} =& \frac{-(48\sqrt{2}v^2 + 72v + 51\sqrt{2})}{25\sqrt{32v^2 + 24\sqrt{2}v + 18}} \\ f^{T_qN_q} =& \frac{12}{5\sqrt{32v^2 + 24\sqrt{2}v + 18}} \\ g^{T_qN_q} =& 0 \end{aligned}$$

The Gaussian curvature $K^{T_qN_q}$ *and the mean curvature* $H^{T_qN_q}$ *of* T_qN_q*-Smarandache ruled surface are*

$$\begin{aligned} K^{T_qN_q} =& \frac{-144}{25(32v^2 + 24\sqrt{2}v + 18)(16v^2 + 12\sqrt{2}v + 9)}, \\ H^{T_qN_q} =& \frac{-48\sqrt{2}v^2 - 72v - 3\sqrt{2}}{(32v^2 + 24\sqrt{2}v + 18)^{\frac{3}{2}}}. \end{aligned}$$

The T_qB_q*-Smarandache-Ruled Surface (see figure 3) is given by*

$$\psi^{T_qB_q}(s,v) = (\frac{7}{5\sqrt{2}}coss - vsins, -\frac{7}{5\sqrt{2}}sins - vcoss, \frac{1}{5\sqrt{2}}).$$

The first fundamental of T_qB_q *are obtained*

$$E^{T_qB_q} = v^2 + \frac{49}{50}$$

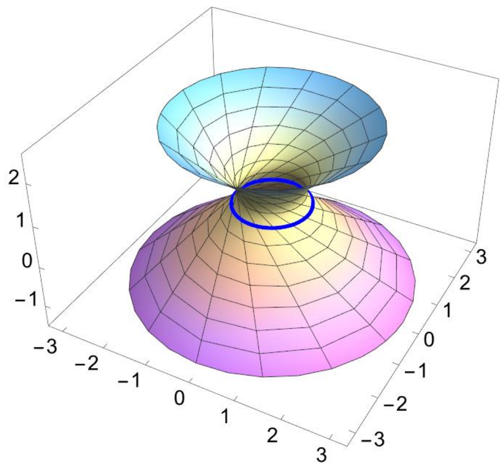

Figure 2: T_qN_q-Smarandache-Ruled Surface

$$F^{T_qB_q} = \frac{7}{5\sqrt{2}}$$
$$G^{T_qB_q} = 1$$

The second fundamentals of T_qB_q are obtained

$$e^{T_qB_q} = 0$$
$$f^{T_qB_q} = 0$$
$$g^{T_qB_q} = 0$$

The Gaussian curvature $K^{T_qB_q}$ and the mean curvature $H^{T_qB_q}$ of T_qB_q-Smarandache ruled surface are

$$K^{T_qB_q} = 0,$$
$$H^{T_qB_q} = \frac{\sqrt{2}}{4v^3}.$$

Since $H^{T_qB_q} = 0$, the T_qB_q-Smarandache ruled surface is a minimal surface.

The N_qB_q-Smarandache-Ruled Surface (see figure 4) is given by

$$\psi^{N_qB_q}(s,v) = (\frac{1}{\sqrt{2}}(\frac{4}{5}coss - sins) + \frac{3v}{5}coss, \frac{1}{\sqrt{2}}(-\frac{4}{5}sins - coss) - \frac{3v}{5}sins, -\frac{3}{5\sqrt{2}} + \frac{4v}{5}).$$

The first fundamental of N_qB_q are obtained

$$E^{N_qB_q} = \frac{9}{25}v^2 + \frac{12\sqrt{2}}{25}v + \frac{41}{50}$$

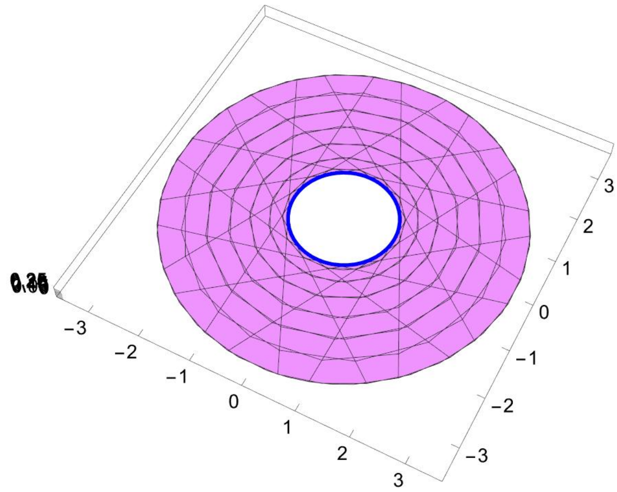

Figure 3: T_qB_q-Smarandache-Ruled Surface

$$F^{N_qB_q} = \frac{-3\sqrt{2}}{10}$$
$$G^{N_qB_q} = 1$$

And the second fundamental of N_qB_q *are obtained*

$$e^{N_qB_q} = \frac{36\sqrt{2}v^2 + 96v - 46\sqrt{2}}{25\sqrt{18v^2 + 24\sqrt{2}v + 32}}$$
$$f^{N_qB_q} = \frac{-12}{5\sqrt{18v^2 + 24\sqrt{2}v + 32}}$$
$$g^{N_qB_q} = 0$$

The Gaussian curvature $K^{N_qB_q}$ *and the mean curvature* $H^{N_qB_q}$ *of* N_qB_q*-Smarandache ruled surface are*

$$K^{N_qB_q} = \frac{-144}{25(18v^2 + 24\sqrt{2}v + 32)(9v^2 + 12\sqrt{2}v + 16)},$$
$$H^{N_qB_q} = \frac{36\sqrt{2}v^2 + 96v - 82\sqrt{2}}{(18v^2 + 24\sqrt{2}v + 32)^{\frac{3}{2}}}.$$

4. Conclusion

In our article, we introduced the definition of Smarandache-Ruled Surface according to a new frame called the quasi frame. T_qN_q, N_qB_q, and T_qB_q Smarandache ruled surfaces are investigated and studied according to the quasi frame

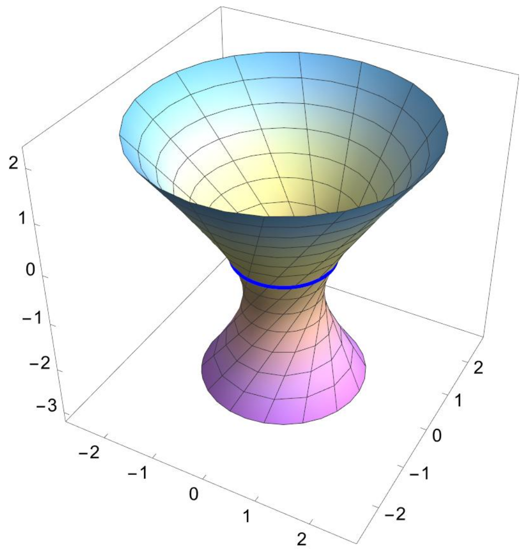

Figure 4: N_qB_q-Smarandache-Ruled Surface

in Euclidean 3-space. The Gaussian curvature, mean curvature, and the second Gaussian curvature for these surfaces are calculated according to the quasi frame in E^3. Also, the necessary and sufficient conditions for such surfaces to be developable surfaces are studied. Some other important geometric properties for these surfaces are introduced. Also, some examples are given.

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