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Shape Derivative of the Fractional p-Laplacian Operators Via Minmax Method

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Abstract. This article investigates shape optimization problems governed by Fractional p-Laplacian operators of the form $(-\Delta)^s_p$ where $0 < s < 1$ and $p \geq 2$. We begin by establishing the existence of a weak solution to the associated variational problem, which will guarantee the calculation of the shape derivative. Then we derive the shape derivative of the functional using the minmax method, which provides a robust framework for nonlocal sensitivity analysis.

Key Words and Phrases: Shape optimization, Shape derivative, Fractional p-laplacian, minmax.

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1. Introduction and results

This work focuses on the in-depth mathematical study of shape optimization a field that lies at the crossroads of the calculus of variations, geometric analysis, and partial differential equations. Shape optimization concerns the study of optimal configurations of domains with respect to prescribed cost functionals, often constrained by PDEs. The framework relies on smooth perturbations of admissible domains to develop shape calculus tools, enabling the derivation of first- and second-order shape derivatives. The mathematical foundation of shape optimization draws heavily on boundary variation techniques and the theory of free boundary problems. It has broad applications in fluid dynamics, structural mechanics, inverse problems, and control theory. In particular, the shape derivative provides a sensitivity measure of the functional with respect to domain deformations, which is essential for gradient-based optimization algorithms. The present study contributes to this theory by extending shape calculus to nonlocal and nonlinear operators, with a focus on the fractional p-Laplacian. This work is motivated by our previous investigations into the fractional p-Laplacian, with further context available in [22]. In mathematical analysis, the fractional Laplacian is a nonlocal operator that generalizes the classical Laplacian by extending the notion of spatial derivatives to fractional powers. It serves as a fundamental tool in the formulation of fractional partial differential equations (PDEs), which arise in various physical and engineering contexts. In this paper, we focus on the fractional p-Laplacian, a nonlinear and nonlocal generalization of the classical Laplacian that incorporates both fractional-order differentiation and p-growth conditions. This operator is particularly relevant for modeling complex phenomena such as anomalous diffusion, turbulence, and nonlocal transport processes. For a comprehensive overview of its applications and theoretical foundations, we refer the reader to [2] and the references therein. Multiple definitions of the fractional Laplacian exist, including formulations based on Fourier transform techniques [17], singular integral representations [33], and semigroup theory [26]. These distinct approaches reflect the operator's versatility and its applicability across various domains. Notable physical systems where the fractional Laplacian plays a central role include quasi-geostrophic flows, water wave dynamics, molecular transport, and plasma diffusion. Given the diversity of definitions and modeling frameworks, a range of analytical and numerical methods has been developed to address problems involving the fractional Laplacian. A seminal contribution by Caffarelli and Silvestre [13] introduced the extension method, which recasts the fractional Laplacian as a Dirichlet-to-Neumann map for a local PDE in an extended space. This approach has since become a cornerstone in the analysis of nonlocal equations. Further developments by Fall et al. [23] and others have advanced the regularity theory and optimization techniques for both local and nonlocal operators. The fractional p-Laplacian has also been extensively studied in the context of anomalous diffusion. In particular, the works of Metzler and Klafter [24, 25] provide a probabilistic and physical interpretation of fractional dynamics, deriving fractional PDEs from continuous-time random walk models and Levy flight processes. These studies underscore the relevance of fractional operators in capturing nonlocal and nonlinear behaviors in complex systems. Recent developments in the numerical treatment of nonlocal operators have yielded robust schemes for approximating the fractional Laplacian, particularly those based on singular integral representations, as detailed in [2, 5]. These approaches provide a rigorous discretization framework for operators defined via hypersingular kernels, enabling accurate resolution of boundary value problems in fractional Sobolev spaces. From an analytical standpoint, Caffarelli et al. [13, 16] established foundational results concerning the spectral and extension-based characterizations of fractional powers of the Laplacian and general integro-differential operators. Their work elucidates the interplay between nonlocality and regularity, and demonstrates how extension techniques can recover local PDE formulations from nonlocal models, thereby facilitating the derivation of regularity estimates and maximum principles. In the context of boundary regularity, Ros-Oton, Xavier, and Serra [6, 7, 28, 30, 31] introduced refined analytical tools to investigate the fine properties of solutions near the boundary of the domain. Their contributions include sharp Hölder continuity results and boundary Harnack inequalities for solutions to fractional elliptic equations, which are instrumental in understanding the behavior of nonlocal operators in irregular geometries. Building upon these analytical foundations, Fall et al. [23] extended the regularity theory to encompass nonlocal Schrodinger-type equations, establishing existence, uniqueness, and regularity results for weak solutions in fractional Sobolev spaces. Their framework accommodates singular potentials and nontrivial boundary conditions, thereby broadening the applicability of nonlocal models in quantum mechanics and related fields. Parallel to these developments, shape optimization problems governed by fractional operators have emerged as a fertile area of research. Dalibard et al. [14, 15] investigated the existence of optimal domains for energy functionals under the constraint $s = \frac{1}{2}$, employing variational techniques and geometric measure theory. Subsequently, Fall et al. [21] generalized the analysis to the regime $0 < s < 1$, utilizing nonlocal shape calculus and variational methods to derive first-order shape derivatives and establish optimality conditions in the presence of fractional p-Laplacian constraints. Previous investigations, notably in [22], have addressed shape derivative analysis of associated functionals via perturbations induced by vector fields. This foundational approach motivates our interest in exploring the concept of topological derivatives, now framed within the context of recent developments presented in [8, 9, 27]. In this work, we pursue the computation of the topological derivative through the lens of the minmax variational principle. For a comprehensive exposition of this methodology, the reader is referred to [10]. Regarding applied contexts, [22] provides a detailed derivation of the topological derivative for a functional arising in linear thermoplasticity, while [27] presents a practical application involving Helmholtz-type problems. The principal objective of this article is to characterize the shape derivative of the functional

$$F(\Omega_\epsilon) = F(\Omega_\epsilon, u_\epsilon).$$

A detailed analysis of the existing literature, particularly the works cited above, reveals a significant lack of comprehensive studies addressing shape optimization problems governed by nonlocal operators, such as the fractional Laplacian. This observation constitutes the principal motivation for the present investigation. Fractional nonlocal operators, notably the fractional p-Laplacian Δ_p^s, naturally emerge in a wide range of applied mathematical models, including those in continuum mechanics, phase transition dynamics, population dispersal, and evolutionary game theory. In this study, we rigorously examine the following shape optimization problem:

$$\min_{\Omega \in \mathcal{O}} F(\Omega). \tag{1}$$

Let F denote a prescribed cost functional, $\mathcal{O}$ the set of admissible domains, and Ω an open and bounded region within $\mathbb{R}^N$, with $N \geq 2$, satisfying the following conditions

$$F(\Omega) = \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\mid u_\Omega(x) - u_\Omega(y) \mid^{p-2} (u_\Omega(x) - u_\Omega(y))}{\mid x - y \mid^{N+ps}} dxdy, \tag{2}$$

with u_Ω the solution to the following p-Laplacian operator

$$\begin{cases} (-\Delta)_p^s u_\Omega = f \text{ in } \Omega, \\ u_\Omega = 0 \text{ on } \mathbb{R}^N \setminus \Omega, \\ p \geq 2. \end{cases} \tag{3}$$

The decision to impose the lower bound $p \geq 2$ (rather than merely $p > 1$) is motivated by several key technical and analytical considerations:

- **Regularity of the functional space:** When $p \geq 2$, the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ exhibits stronger convexity and regularity properties, which significantly facilitate variational analysis.
- **Differentiability of the energy functional:** In this regime, the associated energy functional is of class $\mathcal{C}^1$ on $W^{s,p}(\mathbb{R}^N)$, a crucial requirement for applying classical variational tools such as the mountain pass theorem. This regularity is also essential in our context for computing shape derivatives.
- **Control of singularities:** For $1 < p < 2$, the nonlinear term

  $$|u(x) - u(y)|^{p-2}(u(x) - u(y))$$

  becomes singular as $u(x) \to u(y)$, which introduces significant analytical challenges, especially in the nonlocal setting.
- **Compactness and convexity properties:** The space $W^{s,p}(\mathbb{R}^N)$ enjoys better compactness and convexity properties when $p \geq 2$, making it easier to construct minimizing sequences.

- **Physical relevance:** Many models arising in mechanics, physics, and biology naturally involve powers $p \geq 2$, corresponding to more regular diffusive behaviors that are better captured within this framework.

This work constitutes a continuation and broadening of the results presented in [14, 21, 22], now applied to the fractional p-Laplacian operator for parameters $0 < s < 1$ and $p \geq 2$. In contrast to previous methodologies, we adopt a variational framework oriented toward shape differentiation via the minmax principle. Initially, we establish the existence of weak solutions by leveraging the foundational results of M. Fall et al. [21]. This analytical foundation enables a rigorous derivation of the shape derivative of the associated functional within the fractional Sobolev spaces, using the minmax approach as a central tool.
Remark When the second term in the first equation of (1.3) is absent, the framework developed in [21, 22] can be extended to build on the shape derivative approach introduced by Dalibard and Gérard-Varet [14]. This extension leverages the vector field method, which works well as long as the governing equation maintains a relatively simple structure. However, once the nonlinear term is introduced, the analysis becomes significantly more complex. Challenges such as nonlocal interactions, nonlinear couplings, and the breakdown of symmetry emerge, making the classical vector field method inadequate. As a result, deriving the shape derivative in the style of Dalibard and Gérard-Varet is no longer feasible, and alternative strategies like minmax techniques or variational perturbation methods must be employed to fully capture the intricacies of the problem. While the classical Hadamard method offers a formal route for computing shape and topological derivatives, it often proves difficult to implement numerically. This is especially true when dealing with singular integrals or boundary variations, which are notoriously hard to approximate accurately particularly in settings involving complex geometries or nonlocal operators like the fractional Laplacian.
In contrast, the minmax (or variational) approach provides a more robust and practical alternative for numerical simulations. By recasting the derivative computation as a saddle-point problem, it enables the derivation of shape and topological derivatives in a weak (distributional) sense. This formulation lends itself to more stable numerical schemes that are less sensitive to geometric irregularities, making it especially effective for problems governed by nonlocal or fractional operators.
The main results of this work are as follows: We begin with the existence result for a weak solution to the system described in equation (3).

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be an open domain of class $\mathcal{C}^2$, with $N > 1$, and let $s \in (0,1)$, $p \in [2, +\infty)$. Then, there exists a unique weak solution $u \in W^{s,p}(\Omega)$ to* *the problem described in equation* (3)*. Furthermore, this solution is the minimizer of the variational problem*

$$\inf_{u\in W^{s,p}(\Omega)} F(u,u),$$

where the functional F is defined by

$$F(u,v) = \langle(-\Delta)_p^s u, v\rangle_{W^{s,p}(\Omega)} - \int_\Omega f(x)v(x)dx.$$

The second main result concerns the shape derivative of the functional (2), which is stated as follows:

Theorem 2. *Let Ω be the solution to the optimization problem* $\min\{F(\Omega),\ \Omega \in \mathcal{O}\}$*. If the function $\mathcal{R}(\epsilon)$ admits a finite limit $\mathcal{R}(u,p)$, then the shape derivative of $F(\Omega)$ in the direction of a smooth vector field $V \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ is given by:*

$$\begin{aligned} DF(\Omega,V) = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}(u(x)-u(y))K_0'(x,y)\,dxdy \\ &+ \frac{1}{C(N,s)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}(u(x)-u(y))(p(x)-p(y))K_0'(x,y)\,dxdy \\ &- \int_\Omega (\nabla f\cdot V(0))p\,dx - \int_\Omega fp\,divV(0)\,dx + \mathcal{R}(u,p) \end{aligned}$$

where p is the solution to the following adjoint equation and φ a test function in $W^{s,p}(\mathbb{R}^N)$:

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (p-1)|u(x)-u(y)|^{p-2}(\varphi'(x)-\varphi'(y))K_0(x,y)\,dxdy \\ &\quad = -\frac{(p-1)}{C(N,s)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}(\varphi'(x)-\varphi'(y))(p(x)-p(y))K_0(x,y)\,dxdy. \end{aligned}$$

This paper is structured as follows: Section 2 presents the functional framework and proves the existence of a weak solution to the fractional p-Laplacian problem. Section 3 introduces the minmax method and develops the theorical tools needed to compute the shape derivative and provides the full derivation of the shape derivative using domain perturbation and adjoint equations. Finally, section 4 concludes the paper and outlines possible extensions.

2. Existence of a weak solution

2.1. On the fractional operators.

We begin by recalling several foundational definitions and results pertinent to shape optimization theory. In this work, we restrict our attention to problems governed by the Laplacian and the nonlinear p-Laplacian operators, with $p \geq 2$. These operators serve as prototypical examples in the study of PDE-constrained optimization, where the objective functional depends on the solution of an elliptic boundary value problem defined over a variable domain. The shape optimization framework typically involves analyzing the sensitivity of a cost functional with respect to smooth perturbations of the domain. This requires the development of shape calculus tools, including the notions of shape derivative and topological derivative, which are central to deriving necessary optimality conditions and implementing gradient-based optimization algorithms. We now introduce the relevant definitions and mathematical structures that will be used throughout the paper.

Definition 1. *Let $p \in [2, +\infty)$ and $s \in (0, 1)$. The fractional p-Laplacian operator is defined for a function u at a point $x \in \mathbb{R}^N$ by:*

$$(-\Delta)^s_p u(x) = 2 \lim_{\epsilon \longrightarrow 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{| u(x) - u(y) |^{p-2} (u(x) - u(y))}{| x - y |^{N+ps}} dy$$

which can also be expressed using the principal value $P.V.$ as:

$$(-\Delta)^s_p u(x) = 2\, P.V. \int_{\mathbb{R}^N} \frac{| u(x) - u(y) |^{p-2} (u(x) - u(y))}{| x - y |^{N+ps}} dy.$$

Here, $P.V.$ denotes the principal value of the integral, ensuring convergence near the singularity at $x = y$.

Remark 1. *For $p \neq 2$, the fractional p-Laplacian operator $(-\Delta)^s_p$ is inherently nonlinear. Its evaluation at a point $x \in \Omega \subset \mathbb{R}^N$ depends not only on the function u, but also on the domain Ω and the ambient dimension N, i.e.,*

$$(-\Delta)^s_p u(x) = (-\Delta)^s_p u(x, \Omega, N).$$

This operator generalizes the classical fractional Laplacian, which corresponds to the linear case $p = 2$. Specifically, for $p = 2$, the fractional Laplacian $(-\Delta)^s$ is defined via the singular integral representation:

$$(-\Delta)^s u(x) = C(N, s)\, P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dy,$$

where $C(N, s)$ is a normalization constant depending on the dimension N and the fractional order $s \in (0, 1)$.

Definition 2. *Let $0 < s < 1$, $p \in [2, +\infty)$, and assume $N \geq sp$. Consider a bounded open set $\Omega \subset \mathbb{R}^N$ with a Lipschitz continuous boundary. For any measurable function u, the Gagliardo seminorm is given by:*

$$[u]_{s,p} = \left(\int_\Omega \int_\Omega \frac{| u(x) - u(y) |^p}{| x - y |^{N+ps}} dxdy \right)^{\frac{1}{p}}$$

1. The fractional Sobolev space $W^{s,p}(\Omega)$ is defined as:

$$W^{s,p}(\Omega) = \{ u \in L^p(\Omega) \mid [u]_{s,p} < +\infty \},$$

and is equipped with the norm:

$$\|u\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx + [u]_{s,p}^p \right)^{\frac{1}{p}}.$$

2. The subspace $W_0^{s,p}(\Omega)$ is defined by:

$$W_0^{s,p}(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^N) \,\middle|\, u = 0 \ \text{almost everywhere in } \mathbb{R}^N \setminus \Omega \right\},$$

and is normed equivalently by setting $\|u\|_{s,p} = [u]_{s,p}$.

Theorem 3. *Let $p \in [2, +\infty)$ and $s \in (0, 1)$. Then the fractional p-Laplacian operator*

$$(-\Delta)_p^s : W_0^{s,p}(\Omega) \longrightarrow (W_0^{s,p}(\Omega))'$$

defined by

$$u_\Omega \mapsto (-\Delta)_p^s u_\Omega$$

is well-defined. Moreover, the following properties hold:
1. For all $u, v \in W_0^{s,p}(\Omega)$, the duality pairing is given by:

$$\langle (-\Delta)_p^s u, v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+ps}} \, dx \, dy.$$

2. For all $u, v \in W_0^{s,p}(\Omega)$, the following inequality holds:

$$\langle (-\Delta)_p^s u, v \rangle \leq [u]_{s,p}^{p-1} [v]_{s,p},$$

where $[\cdot]_{s,p}$ denotes the Gagliardo seminorm in $W^{s,p}(\Omega)$.

Proof. 1. As $u \in W_0^{s,p}(\Omega)$ the integral in the definition of $(-\Delta)_p^s$ exists

$$(-\Delta)_p^s u(x) = \int_{\mathbb{R}^N} \frac{| u(x) - u(y) |^{p-2} \, (u(x) - u(y))}{| x - y |^{N+ps}} dy$$

then, $\forall u, v \in W_0^{s,p}(\Omega)$ we have by Fubini's theorem, we get :

$$\langle(-\Delta)_p^s u, v\rangle = 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid u(x)-u(y)\mid^{p-2}(u(x)-u(y))}{\mid x-y\mid^{N+ps}} v(x)dydx.$$

In fact, the duality pairing of the fractional p-Laplacian with a test function v can be expressed as:

$$\langle(-\Delta)_p^s u, v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy,$$

which can be equivalently rewritten as:

$$= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\,v(x)\,dy\,dx + \int_{\mathbb{R}^{2N}} \frac{-|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\,v(y)\,dy\,dx.$$

By exchanging the variables x and y in the second integral, we obtain:

$$\begin{aligned}\langle(-\Delta)_p^s u, v\rangle &= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\,v(x)\,dy\,dx \\ &+ \int_{\mathbb{R}^{2N}} \frac{-|u(x)-u(y)|^{p-2}(-1)(u(x)-u(y))}{|x-y|^{N+ps}}\,v(x)\,dy\,dx,\end{aligned}$$

which simplifies to:

$$\begin{aligned}\langle(-\Delta)_p^s u, v\rangle &= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\,v(x)\,dy\,dx \\ &+ \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\,v(x)\,dy\,dx.\end{aligned}$$

Thus, the two terms are identical and can be combined, yielding:

$$\langle(-\Delta)_p^s u, v\rangle = 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid u(x)-u(y)\mid^{p-2}(u(x)-u(y))}{\mid x-y\mid^{N+ps}} v(x)dxdy. \quad (4)$$

2. By applying Hölder's inequality to the duality pairing of the fractional p-Laplacian, we obtain:

$$\begin{aligned}\langle(-\Delta)_p^s u, v\rangle &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy \\ &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-1}|v(x)-v(y)|}{|x-y|^{(N+ps)\left(\frac{p-1}{p}\right)}\cdot|x-y|^{(N+ps)\left(\frac{1}{p}\right)}}\,dx\,dy\end{aligned}$$

$$\begin{aligned}\langle(-\Delta)_p^s u, v\rangle &\le \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\mid u(x)-u(y)\mid^p}{\mid x-y\mid^{(N+ps)}}dxdy\right)^{\frac{p-1}{p}}\cdot\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\mid u(x)-v(y)\mid^p}{\mid x-y\mid^{(N+ps)}}dxdy\right)^{\frac{1}{p}}\\ &\le [u]_{s,p}^{p-1}[v]_{s,p}\end{aligned}$$

Therefore, we arrive at the following expression:

$$\langle(-\Delta)_p^s u, v\rangle \le [u]_{s,p}^{p-1}[v]_{s,p}.$$

This theorem will be useful in the following.

Theorem 4. . *Let $s \in (0,1)$, $p \in [2,+\infty)$, and $q \in [1,p]$. Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain such that $W^{s,p}(\Omega)$ is well-defined, and let $T \subset L^p(\Omega)$ be a bounded subset. Assume that*

$$\sup_{u\in T}\left(\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right) < +\infty.$$

Then T is relatively compact in $L^q(\Omega)$.

Proof. The proof follows from the compact embedding results for fractional Sobolev spaces. For details, we refer the reader to [15].

2.2. Existence of solution for the nonlocal Dirichlet problem.

The class of problems considered in this section was previously studied by Caffarelli and Silvestre [13] in the special case where the fractional order is $s = \frac{1}{2}$. Their groundbreaking work introduced an extension technique that transforms the fractional Laplacian into a local operator in a higher-dimensional space, thereby simplifying the analysis of nonlocal equations. In [21, 22], we established the shape derivative using the Hadamard method. In the present work, we aim to extend these results to the full range $0 < s < 1$, adopting a variational framework tailored to the fractional p-Laplacian. The goal of this section is to establish an extension result for the associated boundary value problem, which serves as a crucial foundation for deriving shape and topological derivatives in the nonlocal setting.

$$\begin{cases}(-\Delta)_p^s u_\Omega = f \text{ in } \Omega\\ u_\Omega = 0 \text{ on } \mathbb{R}^N\setminus\Omega\\ p\ge 2.\end{cases}\tag{5}$$

To establish the existence result, we employ the Euler-Lagrange equation associated with (5), which allows us to reformulate the problem in terms of a functional $F(u)$. Lemma 1 establishes that the sequence (u_k) is bounded in $W^{s,p}(\Omega)$; in particular, we have $[u]_{s,p} < +\infty$. This result plays a crucial role in the proof of Theorem 1, as it ensures the necessary compactness and weak convergence properties within the fractional Sobolev space.

Lemma 1. *Let $(u_k)_{k\geq 1} \subset W^{s,p}(\Omega)$ be a minimizing sequence for the functional F, satisfying*

$$\lim_{k\to+\infty} F(u_k, u_k) = \inf_{v\in W^{s,p}(\Omega)} F(v, v) = m.$$

Then the sequence $(u_k)_{k\geq 1}$ is bounded in the space $W^{s,p}(\Omega)$.

Proof. Assume that the sequence $\{u_k\} \subset W^{s,p}(\Omega)$ is minimizing for the functional $F(u_k, u_k)$, and let m denote its infimum. Then, there exists an index $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, the following inequality holds:

$$m \leq F(u_k, u_k) \leq m + \frac{1}{k}, \quad \forall k \geq 1.$$

The functional F is defined by:

$$F(u, v) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+ps}}\, dx\, dy - \int_\Omega f(x)v(x)\, dx.$$

From this, we obtain the estimate:

$$F(u_k, u_k) \geq \ [u_k]^{p-1}_{s,p}[u_k]_{s,p} - \frac{p-1}{p} \parallel f \parallel^{\frac{p}{p-1}}_{L^2(\Omega)} - \frac{1}{p} \parallel u_k \parallel^p_{L^2(\Omega)}.$$

Consequently, we deduce:

$$[u_k]^{p-1}_{s,p}[u_k]_{s,p} \leq F(u_k, u_k) + \frac{p-1}{p} \parallel f \parallel^{\frac{p}{p-1}}_{L^2(\Omega)} + \frac{1}{p} \parallel u_k \parallel^p_{L^2(\Omega)}. \tag{6}$$

Since the domain is bounded, we have $L^p(\Omega) \subset L^2(\Omega)$, which justifies the use of L^2 norms in certain estimates.
Assuming that $f \in L^p(\Omega)$, and given a sequence $\{u_k\} \subset L^p(\Omega)$ satisfying the energy bound

$$F(u_k, u_k) \leq m + \frac{1}{k},$$

for some constant $m > 0$ and all $k \in \mathbb{N}$, we deduce that the right-hand side of inequality (6) remains uniformly bounded as $k \to \infty$. Consequently, the left-hand side of (6) must also be bounded, which implies that the Gagliardo seminorm

$[u_k]_{s,p}$ is uniformly bounded by a constant depending only on f and m. This boundedness plays a crucial role in establishing compactness and weak convergence properties in the fractional Sobolev space $W^{s,p}(\Omega)$.

We define the energy functional $F : W_0^{s,p}(\Omega) \to \mathbb{R}$ by:

$$F(u) := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy - \langle f, u \rangle,$$

with

$$\langle f, u \rangle = \int_\Omega f(x) u(x) dx.$$

Theorem 5. *The functional F is coercive, strictly convex, and weakly lower semi-continuous on $W_0^{s,p}(\Omega)$.*

Proof.

1. **Coercivity:** As $\|u\|_{W^{s,p}} \to \infty$, the energy term dominates and $F(u) \to \infty$. Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain, with $0 < s < 1$, $p \geq 2$, and let $f \in (W_0^{s,p}(\Omega))^*$. We define the energy functional:

$$F(u) := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy - \langle f, u \rangle,$$

for all $u \in W_0^{s,p}(\Omega)$.
We aim to prove that F is coercive, i.e.,

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} \to \infty \quad \Rightarrow \quad F(u) \to \infty.$$

Recall that the norm in $W_0^{s,p}(\Omega)$ is given by:

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \right)^{1/p}.$$

Let us denote this seminorm by $[u]_{W^{s,p}}$. Then:

$$F(u) = [u]_{W^{s,p}}^p - \langle f, u \rangle.$$

Using Hölders inequality for the dual pairing:

$$|\langle f, u \rangle| \leq \|f\|_{(W_0^{s,p})^*} \cdot \|u\|_{W^{s,p}}.$$

Combining the above, we obtain:

$$F(u) \geq \|u\|_{W^{s,p}}^p - \|f\|_{(W_0^{s,p})^*} \cdot \|u\|_{W^{s,p}}.$$

Let $\epsilon := \|u\|_{W^{s,p}}$. Then:

$$F(u) \geq \epsilon^p - \|f\| \cdot \epsilon.$$

Since $p > 1$, the term ϵ^p dominates the linear term $\|f\| \cdot \epsilon$ as $\epsilon \to \infty$. Therefore:

$$\lim_{\epsilon \to \infty} (\epsilon^p - \|f\| \cdot \epsilon) = \infty.$$

Hence,

$$\lim_{\|u\|_{W^{s,p}} \to \infty} F(u) = \infty.$$

The functional F is coercive on $W_0^{s,p}(\Omega)$, which is a key step in applying the direct method in the calculus of variations to prove the existence of a weak solution. Let $u, v \in W_0^{s,p}(\Omega)$, with $u \neq v$, and let $\lambda \in (0,1)$. Define:

$$w := \lambda u + (1 - \lambda)v.$$

We examine the seminorm:

$$[w]^p_{W^{s,p}} = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w(x) - w(y)|^p}{|x - y|^{N+ps}} \, dx \, dy.$$

Note that:

$$w(x) - w(y) = \lambda(u(x) - u(y)) + (1 - \lambda)(v(x) - v(y)).$$

Since the function $\phi(\epsilon) = |\epsilon^p|$ is strictly convex for $p > 1$, we have:

$$|\lambda a + (1 - \lambda)b|^p < \lambda |a|^p + (1 - \lambda)|b|^p \quad \text{for} a \neq b.$$

Applying this pointwise to the integrand yields:

$$|w(x) - w(y)|^p < \lambda |u(x) - u(y)|^p + (1 - \lambda)|v(x) - v(y)|^p,$$

for a set of positive measure (since $u \neq v$). Integrating both sides:

$$[w]^p_{W^{s,p}} < \lambda [u]^p_{W^{s,p}} + (1 - \lambda)[v]^p_{W^{s,p}}.$$

2. Linearity of the Dual Term

The dual pairing $\langle f, u \rangle$ is linear in u, so:

$$\langle f, w \rangle = \lambda \langle f, u \rangle + (1 - \lambda)\langle f, v \rangle.$$

Combining both results:

$$F(w) = [w]^p_{W^{s,p}} - \langle f, w \rangle < \lambda F(u) + (1 - \lambda)F(v),$$

whenever $u \neq v$. Hence, F is strictly convex on $W_0^{s,p}(\Omega)$.
Let $u_n \rightharpoonup u$ weakly in $W_0^{s,p}(\Omega)$. We aim to show:

$$\liminf_{n \to \infty} F(u_n) \geq F(u).$$

(a) The Gagliardo seminorm:

$$[u]^p_{W^{s,p}} := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy$$

is convex and weakly lower semi-continuous. Therefore:

$$\liminf_{n\to\infty} [u_n]^p_{W^{s,p}} \geq [u]^p_{W^{s,p}}.$$

(b) The linear term $\langle f, u\rangle$ is continuous with respect to weak convergence:

$$\langle f, u_n\rangle \to \langle f, u\rangle.$$

(c) Combining both:

$$\liminf_{n\to\infty} F(u_n) = \liminf_{n\to\infty} \left([u_n]^p_{W^{s,p}} - \langle f, u_n\rangle\right) \geq [u]^p_{W^{s,p}} - \langle f, u\rangle = F(u).$$

Thus, F is weakly lower semi-continuous on $W^{s,p}_0(\Omega)$.

Proof. **of theorem 1**
Let $v \in W^{s,p}_0(\Omega)$, multiplying the first equation of (5) by $v \in W^{s,p}_0(\Omega)$ and by integrating on Ω we have :

$$\int_\Omega (-\Delta)^s_p u\, v dx = \int_\Omega f(x)v(x)dx. \tag{7}$$

By using definition of the scalar product in $W^{s,p}(\Omega)$, we have :

$$\int_{\mathbb{R}^{2N}} (-\Delta)^s_p u\, v = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid u(x)-u(y)\mid^{p-2}(u(x)-u(y))(v(x)-v(y))}{\mid x-y\mid^{N+ps}} dxdy.$$

hus, equation (7) can be rewritten as:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy = \int_\Omega f(x)v(x)\,dx.$$

In the subsequent analysis, we define the functional F by:

$$F(u,v) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy - \int_\Omega f(x)v(x)\,dx.$$

in others words,

$$F(u,v) = \langle (-\Delta)^s_p u, v\rangle - \int_\Omega f(x)v(x)dx. \tag{8}$$

We aim to show that the functional F is bounded from below, i.e.,

$$F(u,u) > -\infty,$$

for all admissible $u \in W^{s,p}(\Omega)$. Assuming $f \in L^p(\Omega)$, and that F involves terms of the form

$$F(u,u) = [u]_{s,p}^p - \int_\Omega f(x)u(x)\,dx,$$

we apply Hölderss inequality to estimate the linear term:

$$\left|\int_\Omega f(x)u(x)\,dx\right| \leq \|f\|_{L^p(\Omega)}\|u\|_{L^{p'}(\Omega)},$$

where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p. Since $W^{s,p}(\Omega) \hookrightarrow L^{p'}(\Omega)$ continuously for suitable s and p, it follows that

$$\|u\|_{L^{p'}(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)},$$

for some constant $C > 0$. Therefore, the functional $F(u,u)$ is bounded from below by

$$F(u,u) \geq [u]_{s,p}^p - C\|f\|_{L^p(\Omega)}\|u\|_{W^{s,p}(\Omega)},$$

which implies that $F(u,u) > -\infty$ for all $u \in W^{s,p}(\Omega)$, completing the argument.

$$| F(u,v) | \leq \left(\int_{\mathbb{R}^{2N}} \frac{| u(x)-u(y) |^p}{| x-y |^{N+ps}}dxdy\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^{2N}} \frac{| (v(x)-v(y) |^p}{| x-y |^{N+ps}}dxdy\right)^{\frac{1}{p}} + \int_\Omega |f(x)v(x)|dx,$$

giving directly

$$| F(u,v) | \leq [u]_{s,p}^{p-1}[v]_{s,p} + \| f \|_{L^2(\Omega)} \| v \|_{L^2(\Omega)} .$$

Since $u,\ v \in W_0^{s,p}(\Omega)$, we derive that the functional $F(u,v)$ is bounded.
Since $\Omega \subset \mathbb{R}^{\mathbb{N}}$, Theorem (4) ensures that the Gagliardo seminorm $[\cdot]_{s,p}$ is precompact in $L^p(\Omega)$. Given that the sequence $(u_k)_{k\geq 1}$ is bounded in $W^{s,p}(\Omega)$, by the Banach-Alaoglu Theorem and the Rellich-Kondrachov compactness Theorem adapted to fractional Sobolev spaces, there exists a subsequence $(u_{k_l})_{l\geq 1}$ and a function $u \in W^{s,p}(\Omega)$ such that:
$u_{k_l} \rightharpoonup u$ weakly in $W^{s,p}(\Omega)$, $u_{k_l} \to u$ strongly in $L^p(\Omega)$, $u_{k_l} \rightharpoonup u$ weakly in $L^p(\Omega)$, as $l \to +\infty$.

It follows that the limit function u inherits the regularity and integrability properties of the sequence. Moreover, due to the weak lower semicontinuity of the Gagliardo seminorm and the convexity of the energy functional F, we obtain:

$$F(u,u) \leq \liminf_{l\to\infty} F(u_{k_l}, u_{k_l}),$$

which confirms that u is a minimizer (or candidate minimizer) of the functional F in the admissible space $W^{s,p}(\Omega)$.

$$F(u_{k_l}, u_{k_l}) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid u_{k_l}(x) - u_{k_l}(y) \mid^{p-2} (u_{k_l}(x) - u_{k_l}(y))(u_{k_l}(x) - u_{k_l}(y))}{\mid x-y \mid^{N+ps}} dxdy$$

$$- \int_\Omega f(x) u_{k_l}(x) dx \leq m + \epsilon, \ \ \forall \epsilon \geq 0.$$

It follows from the above inequality, that:

$$\int_{\mathbb{R}^{2N}} \frac{\mid (u_{k_l}(x) - u_{k_l}(y)) \mid^p}{\mid x-y \mid^{N+ps}} dxdy \leq \int_\Omega f(x) u_{k_l}(x) dx + m + \epsilon, \ \ \forall \epsilon \geq 0.$$

Applying Fatou's Lemma, we obtain the following inequality for the limiting behavior of the energy functional:

$$\int_{\mathbb{R}^{2N}} \liminf_{l\longrightarrow+\infty} \frac{\mid (u_{k_l}(x) - u_{k_l}(y)) \mid^p}{\mid x-y \mid^{N+ps}} dxdy \leq \liminf_{l\longrightarrow+\infty} \int_\Omega f u_{k_l} dx + m + \epsilon, \ \ \forall \epsilon \geq 0.$$

$$\int_{\mathbb{R}^{2N}} \frac{\mid u(x) - u(y) \mid^p}{\mid x-y \mid^{N+ps}} dxdy \leq \liminf_{l\longrightarrow+\infty} \int_\Omega f u_{k_l} dx + m + \epsilon, \ \ \forall \epsilon \geq 0.$$

By the weak convergence $u_{k_l} \rightharpoonup u$ in $L^p(\Omega)$, and the fact that $L^p(\Omega)$ is reflexive for $p > 1$, it follows that for any test function $\phi \in L^{p'}(\Omega)$, where $p' = \frac{p}{p-1}$, we have:

$$\lim_{l\to\infty} \int_\Omega u_{k_l}(x)\phi(x)\, dx = \int_\Omega u(x)\phi(x)\, dx.$$

This confirms that $u \in L^p(\Omega)$ and that the sequence (u_{k_l}) converges weakly to u in the dual pairing of $L^p(\Omega)$ and $L^{p'}(\Omega)$. Moreover, since the embedding $W^{s,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact under suitable conditions on Ω, the strong convergence $u_{k_l} \to u$ in $L^p(\Omega)$ also holds. This dual convergence (weak in $W^{s,p}$, strong in L^p) is crucial for passing to the limit in nonlinear terms and establishing the existence of minimizers for variational problems involving the fractional p-Laplacian. We observe that:

$$\liminf_{l\to+\infty} \int_\Omega f u_{k_l}\, dx = \lim_{l\to+\infty} \int_\Omega f u_{k_l}\, dx = \int_\Omega f u\, dx.$$

Moreover, the following inequality holds:

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \le \int_\Omega fu\,dx + m + \epsilon, \quad \forall \epsilon \ge 0.$$

Consequently, the functional F evaluated at u satisfies:

$$F(u,u) = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \int_\Omega fu\,dx \le m + \epsilon, \quad \forall \epsilon \ge 0.$$

This leads to the conclusion:

$$F(u,u) \le m \quad \Longrightarrow \quad F(u,u) = m.$$

3. Shape derivative via Minmax method

3.1. Some preliminary results

In this subsection, we present the methodology for computing the topological derivative of a cost functional associated with a fractional p-Laplacian problem, using the min-max variational framework. This approach is inspired by the works of Delfour and Zolésio [8] and further developed in [27], where topological sensitivity analysis is formulated in terms of perturbations of the domain by small inclusions.

Definition 3. *A Lagrangian function is a mapping of the form*

$$(t,x,y) \mapsto L(t,x,y) : [0,\tau] \times X \times Y \to \mathbb{R}, \quad \textit{with } \tau > 0,$$

where X is a vector space, Y is a non-empty subset of a vector space, and for fixed $t \in [0,\tau]$ and $x \in X$, the function $y \mapsto L(t,x,y)$ is affine.

We associate to the parameter $t \in [0,\tau]$ the parametrized minimax function:

$$g(t) = \inf_{x\in X} \sup_{y \in Y} L(t,x,y), \qquad \text{where } \tau > 0,$$

and define its right derivative at zero as:

$$dg(0) = \lim_{t\to 0^+} \frac{g(t)-g(0)}{t}.$$

Whenever the relevant limits exist, we adopt the following notation for partial directional derivatives of the Lagrangian L:
Time derivative:

$$d_t L(0,x,y) = \lim_{t \to 0^+} \frac{L(t,x,y) - L(0,x,y)}{t}.$$

Derivative with respect to x in direction $\varphi \in X$:

$$d_x L(t,x,y;\varphi) = \lim_{\theta \to 0^+} \frac{L(t,x+\theta\varphi,y) - L(t,x,y)}{\theta}.$$

Derivative with respect to y in direction $\phi \in Y$:

$$d_y L(t,x,y;\phi) = \lim_{\theta \to 0^+} \frac{L(t,x,y+\theta\phi) - L(t,x,y)}{\theta}.$$

Since the function $y \mapsto L(t,x,y)$ is affine for all $(t,x) \in [0,\tau] \times X$, it follows that:

$$\forall\, y, \psi \in Y, \quad d_y L(t,x,y,\psi) = L(t,x,\psi) - L(t,x,0) = d_y L(t,x,0,\psi).$$

State Equation For each $t \geq 0$, the state equation is:

$$\text{Find } x^t \in X \text{ such that } d_y L(t,x^t,0,\psi) = 0 \quad \forall \psi \in Y.$$

The corresponding set of admissible states is denoted by:

$$E(t) = \left\{ x^t \in X \;\middle|\; d_y L(t,x^t,0,\psi) = 0 \text{ for all } \psi \in Y \right\}.$$

Adjoint Equation For each $t \geq 0$, the adjoint equation is:

$$\text{Find } p^t \in Y \text{ such that } d_x L(t,x^t,p^t,\varphi) = 0 \quad \forall \varphi \in X.$$

The corresponding set of adjoint solutions is given by:

$$Y(t,x^t) = \left\{ p^t \in Y \;\middle|\; d_x L(t,x^t,p^t,\varphi) = 0 \text{ for all } \varphi \in X \right\}.$$

Set of Minimisers Finally, the set of minimisers for the parametrized minimax problem is defined as:

$$X(t) = \left\{ x^t \in X \;\middle|\; g(t) = \inf_{x \in X} \sup_{y \in Y} L(t,x,y) = \sup_{y \in Y} L(t,x^t,y) \right\}.$$

Lemma 2. *The following statements hold:*

- *(i)*

$$\inf_{x\in X}\sup_{y\in Y} L(t,x,y) = \inf_{x\in E(t)} L(t,x,0).$$

- *(ii) The minimax value $g(t) = +\infty$ if and only if $E(t) = \emptyset$. In this case, the set of minimizers satisfies $X(t) = X$.*
- *(iii) If $E(t) \neq \emptyset$, then:*

$$X(t) = \left\{ x^t \in E(t) \;\middle|\; L(t,x^t,0) = \inf_{x\in E(t)} L(t,x,0) \right\} \subset E(t),$$

and the minimax value $g(t) < +\infty$.

Proof. See [8, 9, 10].

Hypothesis (H0).
Let X be a vector space. We assume the following conditions hold:

- (i) For every $t \in [0,\tau]$, $x^0 \in X(0)$, $x^t \in X(t)$, and $y \in Y$, the mapping

$$\theta \mapsto L(t, x^0 + \theta(x^t - x^0), y) : [0,1] \to \mathbb{R}$$

is absolutely continuous. This implies that for almost every $\theta \in [0,1]$, the derivative exists and is given by

$$\frac{d}{d\theta} L(t, x^0 + \theta(x^t - x^0), y) = d_x L(t, x^0 + \theta(x^t - x^0), y, x^t - x^0),$$

and the function satisfies the integral representation:

$$L(t,x^t,y) = L(t,x^0,y) + \int_0^1 d_x L(t, x^0 + \theta(x^t - x^0), y, x^t - x^0)\, d\theta.$$

- (ii) For all $t \in [0,\tau]$, $x^0 \in X(0)$, $x^t \in X(t)$, $y \in Y$, and $\phi \in X$, the directional derivative

$$d_x L(t, x^0 + \theta(x^t - x^0), y, \phi)$$

exists for almost every $\theta \in [0,1]$, and the function

$$\theta \mapsto d_x L(t, x^0 + \theta(x^t - x^0), y, \phi)$$

belongs to $L^1([0,1])$.

Definition 4. *Given $x^0 \in X(0)$ and $x^t \in X(t)$, the averaged adjoint equation is:*

$$\textit{Find } y^t \in Y \;\; \forall\, \phi \in X, \quad \int_0^1 d_x L(t, x^0 + \theta(x^t - x^0), y, \phi)\, d\theta = 0$$

and the set of solutions is noted $Y(t,x^0,x^t)$.
$Y(0,x^0,x^0)$ clearly reduces to the set of standard adjoint states $Y(0,x^0)$ at $t = 0$.

Theorem 6. *Let*

$$L(t,x,y) : [0,\tau] \times X \times Y \to \mathbb{R}, \quad \tau > 0,$$

be a Lagrangian functional, where X and Y are vector spaces, and for each fixed (t,x), the mapping $y \mapsto L(t,x,y)$ is affine.

Assume that Hypothesis (H0) holds, along with the following conditions:

- (H1) For all $t \in [0,\tau]$, the minimax value $g(t) = \inf_{x\in X} \sup_{y\in Y} L(t,x,y)$ is finite, and the sets of minimizers and adjoint solutions are singletons:

$$X(t) = \{x^t\}, \quad Y(0,x^0) = \{p^0\}.$$

- (H2) The partial derivative with respect to t at $t = 0$ exists:

$$d_t L(0,x^0,p^0) \text{ exists.}$$

- (H3) The following limit exists:

$$R(x^0,p^0) = \lim_{t\to 0^+} \int_0^1 d_x L\left(t, x^0 + \theta(x^t - x^0), p^0, \frac{x^t - x^0}{t}\right) d\theta.$$

Then the right derivative of the minimax function at $t = 0$ exists and is given by:

$$dg(0) = d_t L(0,x^0,p^0) + R(x^0,p^0).$$

Proof. See [8, 9].

Corollary 1. *Let*

$$L(t,x,y) : [0,\tau] \times X \times Y \to \mathbb{R}, \quad \tau > 0,$$

be a Lagrangian functional, where X and Y are vector spaces, and for each fixed (t,x), the mapping $y \mapsto L(t,x,y)$ is affine.

Assume that Hypothesis (H0) holds, along with the following conditions:

- (H1a) For all $t \in [0,\tau]$, the set $X(t)$ is non-empty, the minimax value $g(t)$ is finite, and for each $x \in X(0)$, the set $Y(0,x)$ is non-empty.

- (H2a) For all $x \in X(0)$ and $p \in Y(0,x)$, the partial derivative $d_t L(0,x,p)$ exists.

- (H3a) There exist $x^0 \in X(0)$ and $p^0 \in Y(0,x^0)$ such that the following limit exists:

$$R(x^0,p^0) = \lim_{t\to 0^+} \int_0^1 d_x L\left(t, x^0 + \theta(x^t - x^0), p^0, \frac{x^t - x^0}{t}\right) d\theta.$$

Then the right derivative of the minimax function at $t = 0$ exists, and there exist $x^0 \in X(0)$ and $p^0 \in Y(0,x^0)$ such that:

$$dg(0) = d_t L(0,x^0,p^0) + R(x^0,p^0).$$

In the following, we focus on the principal result concerning the shape derivative of the functional. For further details and theoretical background, the reader is referred to the works of [8, 9, 27].

3.2. The shape derivative of the functional

Shape optimization involves deforming a domain in a controlled and mathematically ideal manner to minimize or maximize a given cost functional. To formalize such deformations, we adopt a differential framework that draws an analogy between domain variations and classical derivatives. This approach is well-established in the literature; see, for example, [1, 10]. The derivative of the cost functional $F(\Omega)$, with respect to domain variations, plays a central role in identifying descent directions and characterizing optimal shapes.
Let $V \in C_c^1(\mathbb{R}^N;\mathbb{R}^N)$ be a smooth vector field with compact support, meaning that V is continuously differentiable and vanishes outside a compact subset of $\mathbb{R}^N$. This vector field represents the direction and magnitude of an infinitesimal deformation applied to the domain.
Using this field, we define a family of transformations of the space, parameterized by $\epsilon \in \mathbb{R}$, as follows:

$$\Phi_\epsilon(x) = x + \epsilon V(x), \quad x \in \mathbb{R}^N.$$

For sufficiently small values of $|\epsilon|$, the map Φ_ϵ is a diffeomorphism from $\mathbb{R}^N$ onto its image. It generates a family of perturbed domains given by $\Omega_\epsilon = \Phi_\epsilon(\Omega)$, which is used in the variational analysis of shape functionals.
This framework allows us to define the shape derivative of J at Ω in the direction V as:

$$DF(\Omega)[V] = \lim_{\epsilon \to 0} \frac{F(\Omega_\epsilon) - F(\Omega)}{\epsilon},$$

provided the limit exists. The computation of this derivative typically involves differentiating the state equation with respect to the domain and applying variational techniques to extract sensitivity information. We now give a provide the proof of Theorem 2 by applying Theorem 6.

Proof. **of theorem 2**
Let us consider the functionnal defined in Ω_ϵ by

$$F(\Omega_\epsilon) = \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\mid u_\Omega(x) - u_\Omega(y) \mid^{p-2} (u_\Omega(x) - u_\Omega(y))}{\mid x - y \mid^{N+ps}} dxdy. \tag{9}$$

where u_{Ω_ϵ} be the solution to the following p-Laplacian operator

$$\begin{cases} (-\Delta)_p^s u = f \text{ in } \Omega_\epsilon, \\ u = 0 \text{ on } \mathbb{R}^N \setminus \Omega_\epsilon, \\ p \geq 2. \end{cases} \tag{10}$$

Let us consider as shape functional F define by

$$F(\Omega) = \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{| u_\Omega(x) - u_\Omega(y) |^{p-2} (u_\Omega(x) - u_\Omega(y))}{| x - y |^{N+ps}} dxdy \tag{11}$$

and $u_\Omega \in W_0^{s,p}(\Omega)$ is solution to the variational problem

$$\int_{\mathbb{R}^{2N}} \frac{| u_\Omega(x) - u_\Omega(y) |^{p-2} (u_\Omega(x) - u_\Omega(y))(v_\Omega(x) - v_\Omega(y))}{| x - y |^{N+ps}} dxdy = \int_\Omega f(x) v_\Omega(x) dx \;\; \forall v \in W^{s,p}(\Omega). \tag{12}$$

In the context of the perturbed domain, the expression labeled as 12 is reformulated as follows:

$$\int_{\mathbb{R}^{2N}} \frac{| u_\epsilon(x) - u_\epsilon(y) |^{p-2} (u_\epsilon(x) - u_\epsilon(y))(v(x) - v(y))}{| x - y |^{N+ps}} dxdy = \int_\Omega f(x) v(x) dx \tag{13}$$

We return to the initial domain Ω via the transformation Φ_ϵ, exploiting the identity $\Omega_\epsilon = \Phi_\epsilon(\Omega)$. This change of variables allows us to express integrals over the perturbed domain Ω_ϵ in terms of the reference domain Ω. Specifically, for any integrable function $g\colon \Omega_\epsilon \to \mathbb{R}$, we have:

$$\int_{\Omega_\epsilon} g(x)\, dx = \int_\Omega g(\Phi_\epsilon(x))\, |\det D\Phi_\epsilon(x)|\, dx,$$

where $D\Phi_\epsilon(x)$ denotes the Jacobian matrix of the transformation and $|\det D\Phi_\epsilon(x)|$ its determinant. This pullback technique is essential for computing shape derivatives, as it enables us to differentiate the cost functional $J(\Omega_\epsilon)$ with respect to ϵ while remaining in the fixed domain Ω. The derivative of the functional can then be expressed in terms of the deformation field V, the solution u, and the adjoint state, facilitating the derivation of first-order optimality conditions.

$$\int_{\mathbb{R}^{2N}} \frac{| u_\epsilon(x) - u_\epsilon(y) |^{p-2} (u_\epsilon(x) - u_\epsilon(y))(v(x) - v(y))}{| x - y |^{N+ps}} \circ \Phi_\epsilon Jac_{\Phi_\epsilon}(x) Jac_{\Phi_\epsilon}(y)\, dxdy$$

$$= \int_\Omega (fv) \circ \Phi_\epsilon Jac_{\Phi_t}(x)\, dx.$$

It is also possible to express

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid u^\epsilon(x) - u^\epsilon(y) \mid^{p-2} (u^\epsilon(x) - u^\epsilon(y))(v \circ \Phi_(x) - v \circ \Phi_\epsilon(y))}{|\Phi_\epsilon(x) - \Phi_\epsilon(y)|^{N+ps}} Jac_{\Phi(x) Jac_{\Phi_\epsilon}(y)\, dxdy}$$

$$= \int_\Omega f \circ \Phi_\epsilon v \circ \Phi_\epsilon Jac_{\Phi_\epsilon}(x)\, dx.$$

By introducing the change of variables $\phi = v \circ \Phi_\varepsilon$, we obtain:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid u^\epsilon(x) - u^\epsilon(y) \mid^{p-2} (u^(x) - u^\epsilon(y))(\phi(x) - \phi(y))}{|\Phi_\epsilon(x) - \Phi_\epsilon(y)|^{N+ps}} Jac_{\Phi_\epsilon}(x) Jac_{\Phi_\epsilon}(y)\, dxdy$$

$$= \int_\Omega f \circ \Phi_\epsilon \phi Jac_{\Phi_\epsilon}(x)\, dx.$$

By defining $K_\epsilon(x,y) = C(N,s) \dfrac{\mathrm{Jac}_{\Phi_\epsilon}(x)\, \mathrm{Jac}\, \Phi_\epsilon(y)}{|\Phi_\epsilon(x) - \Phi_\epsilon(y)|^{N+ps}}$, the preceding expression transforms into:

$$\frac{1}{C(N,s)} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mid u^\epsilon(x) - u^\epsilon(y) \mid^{p-2} (u^(x) - u^\epsilon(y))(\phi(x) - \phi(y)) K_\epsilon(x,y)\, dxdy$$

$$= \int_\Omega f \circ \Phi_\epsilon \phi Jac_{\Phi_\epsilon}(x)\, dx.$$

The objective functional associated with the perturbed domain Ω is given by

$$F(\Omega_) = C(N,s) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_(x) - u_(y)|^{p-2}(u_(x) - u_(y))}{|x-y|^{N+ps}}\, dxdy. \tag{14}$$

Similarly, we return to the unperturbed domain via the transformation Φ_ϵ. By performing this change of variables, we obtain an equivalent expression for the functional over the original domain Ω, now involving the pullback of the integrand and the Jacobian determinant associated with Φ_ϵ.

$$\begin{aligned} F(\Omega_\epsilon) &= C(N,s) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\epsilon(x) - u_\epsilon(y)|^{p-2}(u_\epsilon(x) - u_\epsilon(y))}{|x-y|^{N+ps}} \circ \Phi_\epsilon Jac_{\Phi_\epsilon}(x) Jac_{\Phi_\epsilon}(y)\, dxdy \\ &= C(N,s) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\epsilon(x) - u_\epsilon(y)|^{p-2}(u_\epsilon(x) - u_\epsilon(y))}{|\Phi(x) - \Phi(y)|^{N+2s}} Jac_{\Phi_\epsilon}(x) Jac_{\Phi_\epsilon}(y)\, dxdy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u_\epsilon(x) - u_\epsilon(y)|^{p-2}(u_\epsilon(x) - u_\epsilon(y))\, K_\epsilon(x,y)\, dxdy. \end{aligned}$$

Leveraging the variational framework and the objective functional defined on the perturbed domain, we construct the corresponding perturbed Lagrangian as follows:

$$
\begin{aligned}
L(\epsilon, \varphi, \phi) = & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\varphi(x) - \varphi(y)|^{p-2} (\varphi(x) - \varphi(y))\, K_(x, y)\, dxdy \\
& + \frac{1}{C(N,s)} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi(x) - \varphi(y)|^{p-2} (\varphi(x) - \varphi(y))\, (\phi(x) - \phi(y)) K_\epsilon(x, y)\, dxdy \\
& - \int_\Omega (f \circ \Phi_\epsilon) \phi(x) Jac_{\Phi_\epsilon}(x)\, dx.
\end{aligned}
$$

The derivative of the Lagrangian with respect to is given by:

$$
\begin{aligned}
d_L(\epsilon, \varphi, \phi) = & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\varphi(x) - \varphi(y)|^{p-2} (\varphi(x) - \varphi(y))\, K'_(x, y)\, dxdy \\
& + \frac{1}{C(N,s)} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\varphi(x) - \varphi(y)|^{p-2} (\varphi(x) - \varphi(y))\, (\phi(x) - \phi(y)) K'_\epsilon(x, y)\, dxdy \\
& - \int_\Omega (\nabla f \cdot V(\epsilon)) \phi Jac_{\Phi_\epsilon}(x)\, dx - \int_\Omega (f \circ \Phi_\epsilon) \phi Jac_{\Phi_\epsilon}(x) div V(\epsilon) \circ \Phi_\epsilon\, dx
\end{aligned}
$$

where

$$
K'_\epsilon(x, y)\big|_{\epsilon=0} = -\left[(N + 2s) \frac{x - y}{|x - y|} \cdot P_V(x, y) - (div V(x) + div V(y)) \right] K_0(x, y)
$$

and $P_V \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ is given by

$$
P_V(x, y) = \frac{V(x) - V(y)}{|x - y|}.
$$

To construct the function $\mathcal{R}()$, we evaluate the derivative of the Lagrangian with respect to φ along a chosen direction φ'. This process yields:"

$$
\begin{aligned}
d_\varphi L(\epsilon, \varphi, \phi; \varphi') = & \int_{\mathbb{R}^{2N}} (p-2)(-1)^{p-2} (\varphi(x) - \varphi(y))^{p-3} (\varphi'(x) - \varphi'(y))(\varphi(x) - \varphi(y)) K_\epsilon(x, y)\, dxdy \\
& + \int_{\mathbb{R}^{2N}} (-1)^{p-2} (\varphi(x) - \varphi(y))^{p-2} (\varphi'(x) - \varphi'(y)) K_\epsilon(x, y)\, dxdy \\
& + \frac{(p-2)(-1)^{p-2}}{C(N,s)} \int_{\mathbb{R}^{2N}} (\varphi(x) - \varphi(y))^{p-3} (\varphi'(x) - \varphi'(y))(\varphi(x) - \varphi(y))(\phi(x)) K_\epsilon(x, y) \\
& - \frac{(p-2)(-1)^{p-2}}{C(N,s)} \int_{\mathbb{R}^{2N}} (\varphi(x) - \varphi(y))^{p-3} (\varphi'(x) - \varphi'(y))(\varphi(x) - \varphi(y))(\phi(y)) K_\epsilon(x, y) \\
& - \frac{(-1)^{p-2}}{C(N,s)} \int_{\mathbb{R}^{2N}} (\varphi(x) - \varphi(y))^{p-2} (\varphi'(x) - \varphi'(y))(\varphi(x) + \varphi(y))(\phi(y)) K_\epsilon(x, y)\, dxdy
\end{aligned}
$$

$$
\begin{aligned}
d_\varphi L(\epsilon,\varphi,\phi;\varphi') &= \int_{\mathbb{R}^{2N}} (p-2)(-1)^{p-2}(\varphi(x)-\varphi(y))^{p-2}(\varphi'(x)-\varphi'(y))K_(x,y)\,dxdy \\
&+ \int_{\mathbb{R}^{2N}} (-1)^{p-2}(\varphi(x)-\varphi(y))^{p-2}(\varphi'(x)-\varphi'(y))K_(x,y)\,dxdy \\
&+ \frac{(p-2)(-1)^{p-2}}{C(N,s)}\int_{\mathbb{R}^{2N}} (\varphi(x)-\varphi(y))^{p-2}(\varphi'(x)-\varphi'(y))(\phi(x))K_(x,y)\,dxdy \\
&- \frac{(p-2)(-1)^{p-2}}{C(N,s)}\int_{\mathbb{R}^{2N}} (\varphi(x)-\varphi(y))^{p-2}(\varphi'(x)-\varphi'(y))(\phi(y))K_\epsilon(x,y)\,dxdy \\
&+ \frac{(-1)^{p-2}}{C(N,s)}\int_{\mathbb{R}^{2N}} (\varphi(x)-\varphi(y))^{p-2}(\varphi'(x)-\varphi'(y))(\phi(y)-\phi(y))K_\epsilon(x,y)\,dxdy \\
&= \int_{\mathbb{R}^{2N}} (p-1)|\varphi(x)-\varphi(y)|^{p-2}(\varphi'(x)-\varphi'(y))K_\epsilon(x,y)\,dxdy \\
&+ \frac{(p-1)}{C(N,s)}\int_{\mathbb{R}^{2N}} |\varphi(x)-\varphi(y)|^{p-2}(\varphi'(x)-\varphi'(y))(\phi(x)-\phi(y))K_\epsilon(x,y)\,dxdy. \\
&= \int_{\mathbb{R}^{2N}} (p-1)|\varphi(x)-\varphi(y)|^{p-2}(\varphi'(x)-\varphi'(y))K_\epsilon(x,y)\,dxdy \\
&+ \frac{(p-1)}{C(N,s)}\int_{\mathbb{R}^{2N}} |\varphi(x)-\varphi(y)|^{p-2}(\varphi'(x)-\varphi'(y))(\phi(x)-\phi(y))K_\epsilon(x,y)\,dxdy.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}(\epsilon) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (p-1)\left|\left(\frac{u_\epsilon(x)+u(x)}{2}\right)-\left(\frac{u_\epsilon(y)+u(y)}{2}\right)\right|^{p-2}\left[\left(\frac{u_\epsilon(x)-u(x)}{\epsilon}\right)\right]K_(x,y)\,dxdy \\
&- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (p-1)\left|\left(\frac{u_(x)+u(x)}{2}\right)-\left(\frac{u_(y)+u(y)}{2}\right)\right|^{p-2}\left[\left(\frac{u_\epsilon(y)-u(y)}{\epsilon}\right)\right]K_\epsilon(x,y)\,dxdy \\
&+ \frac{(p-1)}{C(N,s)}\int_{\mathbb{R}^{2N}} \left|\left(\frac{u_\epsilon(x)+u(x)}{2}\right)-\left(\frac{u_\epsilon(y)+u(y)}{2}\right)\right|^{p-2}\left[\left(\frac{u_\epsilon(x)-u(x)}{\epsilon}\right)\right](p(x)-p(y))\,K_\epsilon(x,y)\,dxdy \\
&- \frac{(p-1)}{C(N,s)}\int_{\mathbb{R}^{2N}} \left|\left(\frac{u_\epsilon(x)+u(x)}{2}\right)-\left(\frac{u_\epsilon(y)+u(y)}{2}\right)\right|^{p-2}\left[\left(\frac{u_\epsilon(y)-u(y)}{\epsilon}\right)\right](p(x)-p(y))\,K_\epsilon(x,y)\,dxdy.
\end{aligned}
$$

Replacing φ' with $\dfrac{u_\epsilon - u}{\epsilon}$ in the adjoint equation for p, we arrive at the following expression:

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (p-1)|u(x)-u(y)|^{p-2}\left[\left(\frac{u_\epsilon(x)-u(x)}{\epsilon}\right)-\left(\frac{u_\epsilon(y)-u(y)}{\epsilon}\right)\right]K_(x,y)\,dxdy \\
&\qquad + \frac{(p-1)}{C(N,s)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}\left(\frac{u_\epsilon(x)-u(x)}{\epsilon}\right)(p(x)-p(y))K_\epsilon(x,y)\,dxdy
\end{aligned}
$$

$$- \frac{(p-1)}{C(N,s)} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2} \left(\frac{u_\epsilon(y)-u(y)}{\epsilon} \right) (p(x)-p(y)) K_\epsilon(x,y)\, dxdy = 0.$$

$$\begin{aligned}
\mathcal{R}() = & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (p-1) \left| \left(\frac{u_\epsilon(x)-u_\epsilon(y)}{2} \right) + \left(\frac{u(x)-u(y)}{2} \right) \right|^{p-2} \left[\left(\frac{u_\epsilon(x)-u(x)}{\epsilon} \right) \right] K_(x,y)\, dxdy \\
& - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (p-1) \left| \left(\frac{u_\epsilon(x)-u_\epsilon(y)}{2} \right) + \left(\frac{u(x)-u(y)}{2} \right) \right|^{p-2} \left[\left(\frac{u_\epsilon(y)-u(y)}{\epsilon} \right) \right] K_(x,y)\, dxdy \\
& + \frac{(p-1)}{C(N,s)} \int_{\mathbb{R}^{2N}} \left| \left(\frac{u_\epsilon(x)-u_\epsilon(y)}{2} \right) + \left(\frac{u(x)-u(y)}{2} \right) \right|^{p-2} \left[\left(\frac{u_\epsilon(x)-u(x)}{\epsilon} \right) \right] (p(x))\, K_\epsilon(x,y) \\
& + \frac{(p-1)}{C(N,s)} \int_{\mathbb{R}^{2N}} \left| \left(\frac{u_\epsilon(x)-u_\epsilon(y)}{2} \right) + \left(\frac{u(x)-u(y)}{2} \right) \right|^{p-2} \left[\left(\frac{u_\epsilon(x)-u(x)}{\epsilon} \right) \right] (-p(y))\, K_\epsilon(x,y) \\
& - \frac{(p-1)}{C(N,s)} \int_{\mathbb{R}^{2N}} \left| \left(\frac{u_\epsilon(x)-u_\epsilon(y)}{2} \right) + \left(\frac{u(x)-u(y)}{2} \right) \right|^{p-2} \left[\left(\frac{u_\epsilon(y)-u(y)}{\epsilon} \right) \right] (p(x))\, K_\epsilon(x,y) \\
& - \frac{(p-1)}{C(N,s)} \int_{\mathbb{R}^{2N}} \left| \left(\frac{u_\epsilon(x)-u_\epsilon(y)}{2} \right) + \left(\frac{u(x)-u(y)}{2} \right) \right|^{p-2} \left[\left(\frac{u_\epsilon(y)-u(y)}{\epsilon} \right) \right] (-p(y))\, K_\epsilon(x,y).
\end{aligned}$$

4. Conclusion and perspectives

We have presented a framework for shape optimization involving fractional p-Laplacian operators. After proving the existence of weak solutions, we derived the shape derivative using the minmax method. Future work will focus on numerical simulations and the extension of this approach to include topological derivatives.

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