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Existence and Regularity in Fractional Elliptic Shape Optimization Problems

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Abstract. This paper explores a fractional elliptic shape optimization problem, focusing on both the existence and regularity of optimal solutions. We start by proving the existence of weak solutions to the underlying fractional elliptic equation, which models nonlocal spatial interactions through the fractional Laplacian. Building on this, we establish the existence of an optimal shape using the framework of Γ-convergence, a powerful variational technique particularly well-suited to overcoming challenges such as non-convexity and lack of compactness that often arise in shape optimization. This approach provides a rigorous analytical foundation capable of capturing the interplay between the nonlocal nature of the fractional operator and the geometric complexity of admissible domains. In the final part of the study, we conduct a thorough regularity analysis of the solutions, presenting results on boundedness, Hölder continuity, and boundary behavior. Taken together, these results significantly enhance our understanding of the interaction between fractional differential operators and geometric optimization. They not only clarify the analytical mechanisms behind nonlocal effects but also establish a solid foundation for future theoretical progress and the development of computational strategies in nonlocal shape optimization.

Key Words and Phrases: Fractional laplacian, Optimal Shape, Regularity.

2010 Mathematics Subject Classifications: : 35R11, 49Q10, 35B65

1. Introduction

Shape optimization governed by partial differential equations (PDEs) is a vibrant field with applications in physics, engineering, and material science. The emergence of fractional elliptic operators has expanded the modeling capabilities for nonlocal phenomena, such as anomalous diffusion and long-range interactions. However, their nonlocal nature introduces significant analytical and computational challenges. Traditionally, shape optimization problems have been studied

in the context of local differential operators. Recent advances have shifted attention toward nonlocal operators, particularly the fractional Laplacian, due to its ability to model long-range interactions and anomalous diffusion. The seminal work of Caffarelli and Silvestre [2] introduced an extension technique that reformulates the fractional Laplacian as a local operator in a higher-dimensional space. This approach has enabled substantial progress in understanding regularity properties of solutions, as further developed by Ros-Oton and Serra [9, 10, 11]. These studies provide comprehensive insights into boundary behavior and regularity for integro-differential equations. In parallel, shape optimization problems involving nonlocal operators have garnered increasing interest. Dalibard and Gérard-Varet [3] explored foundational aspects of such problems, while Bonder, Ritorto, and Salort [1] proposed a general class of shape optimization problems for nonlocal operators. More recently, Fall and collaborators advanced the theory of fractional Laplacian shape optimization [4], and extended it to fractional p-Laplacian operators [5]. Topological and geometric aspects of nonlocal shape optimization have also been addressed. Fall et al. [6] investigated shape and topological optimization for fractional elliptic boundary problems, building upon the geometric framework established by Henrot and Pierre [8]. Additionally, Warma [15] studied fractional capacities and boundary conditions of Neumann and Robin type, enriching the mathematical structure of nonlocal boundary value problems. Regularity estimates for nonlocal Schrödinger equations, as presented by Fall [7], offer valuable analytical tools for the study of nonlocal variational problems. Collectively, these contributions form a robust foundation for the development of shape optimization theory in the nonlocal setting.
This work addresses a shape optimization problem constrained by a fractional elliptic PDE. The objective is to determine a domain that minimizes a cost functional under the influence of a fractional operator. This work distinguishes itself from the contributions of Fall et al. and Ros-Oton and Serra by extending the framework of shape optimization to fractional operators through a systematic use of Γ-convergence. While previous studies mainly focused on the existence and qualitative properties of solutions to fractional equations, our study demonstrates for the first time the existence of an optimal domain in a shape optimization problem governed by the fractional Laplacian. Moreover, we obtain regularity results that go beyond the classical bounds: we establish boundedness, Hölder continuity, and control of boundary behavior, thereby extending known regularity results for solutions of fractional PDEs. The specific contribution of this article lies in bridging shape optimization theory with nonlocal operators: by combining Γ-convergence with regularity analysis, we provide a new analytical framework that surpasses existing approaches and opens the way to further theoretical and numerical developments in nonlocal shape optimization. The structure of the

paper is as follows:

- **Section 2:** We establish the existence of weak solutions to the governing equation, which forms the foundation of the optimization framework.
- **Section 3:** We prove the existence of an optimal shape using the theory of Γ-convergence. This method allows us to handle the variational structure and nonlocal behavior effectively.
- **Section 4:** We perform a detailed regularity analysis, focusing on:
  - Boundedness of solutions,
  - Hölder regularity,
  - Boundary regularity.

  This analysis is essential for understanding the qualitative behavior of solutions and for ensuring the robustness of numerical methods.
- **Section 5:** We present our conclusions and outline some perspectives for future research.

2. Existence of Weak Solutions

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an admissible domain. We consider the boundary value problem:

$$\begin{cases} (-\Delta)^s u + u^q = f & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

where $0 < s < 1$, $q > 1$, and $f \in L^r(\Omega)$. The associated cost functional is:

$$J(\Omega) = C(N,2) \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy. \tag{2}$$

We consider the fractional semilinear elliptic problem (1), where $\Omega \subset \mathbb{R}^N$ is a bounded open domain, $0 < s < 1$, $q > 1$, and $f \in L^r(\Omega)$ for some $r \geq 1$.
We aim to prove the existence of a weak solution $u \in W_0^{s,2}(\Omega)$, where:

$$W_0^{s,2}(\Omega) := \left\{ u \in W^{s,2}(\mathbb{R}^N) \mid u = 0 \text{ a.e. on } \mathbb{R}^N \setminus \Omega \right\}.$$

Theorem 1. *Let $f \in L^r(\Omega)$, $r \geq 1$, and assume $q < 2_{s^*} := \frac{2N}{N-2s}$. Then there exists a weak solution $u \in W_0^{s,2}(\Omega)$ to problem* (1).

Proof. We use the direct method of the calculus of variations. Define the energy functional:

$$\mathcal{E}(u) := \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy + \frac{1}{q+1}\int_\Omega |u(x)|^{q+1}\, dx - \int_\Omega f(x)u(x)\, dx. \quad (3)$$

Let us consider the energy functional associated with the fractional semilinear problem: $\mathcal{E}(u)$ defined for $u \in W_0^{s,2}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $s \in (0,1)$, and $f \in L^r(\Omega)$ for some $r > 1$. To prove coercivity, we want to show that

$$\mathcal{E}(u) \to +\infty \quad \text{as} \quad \|u\|_{W^{s,2}(\mathbb{R}^N)} \to \infty.$$

Since Ω is bounded and $s \in (0,1)$, the fractional Sobolev embedding theorem ensures that

$$W_0^{s,2}(\Omega) \hookrightarrow L^{q+1}(\Omega),$$

provided $q+1 < 2_{s^*} := \frac{2N}{N-2s}$ if $N > 2s$, or for any $q+1 \in (1,\infty)$ if $N \le 2s$. This embedding is continuous, so there exists a constant $C > 0$ such that

$$\|u\|_{L^{q+1}(\Omega)} \le C\|u\|_{W^{s,2}(\mathbb{R}^N)}.$$

We estimate the linear term using Hölder's inequality:

$$\left|\int_\Omega f(x)u(x)\, dx\right| \le \|f\|_{L^r(\Omega)}\|u\|_{L^{r'}(\Omega)},$$

where r' is the Hölder conjugate of r, i.e. $\frac{1}{r} + \frac{1}{r'} = 1$. Since $r' < 2_{s^*}$, the Sobolev embedding also gives

$$\|u\|_{L^{r'}(\Omega)} \le C\|u\|_{W^{s,2}(\mathbb{R}^N)}.$$

Thus,

$$\left|\int_\Omega f(x)u(x)\, dx\right| \le C\|f\|_{L^r(\Omega)}\|u\|_{W^{s,2}(\mathbb{R}^N)}.$$

Important remark: The linear term is therefore controlled by a constant times the norm of u in $W^{s,2}$, which means it grows at most linearly in $\|u\|_{W^{s,2}}$. Now consider the structure of $\mathcal{E}(u)$. The first term is the squared Gagliardo seminorm, which is equivalent to the full norm on $W_0^{s,2}(\Omega)$. The second term is nonnegative and superlinear in u, while the third term grows at most linearly in $\|u\|_{W^{s,2}}$. Thus, for large values of $\|u\|_{W^{s,2}}$, the quadratic term necessarily dominates the linear term. We obtain

$$\mathcal{E}(u) \ge \frac{1}{2}\|u\|^2_{W^{s,2}(\mathbb{R}^N)} - C\|u\|_{W^{s,2}(\mathbb{R}^N)} + \frac{1}{p+1}\|u\|^{p+1}_{L^{p+q}(\Omega)}.$$

The quadratic term $\frac{1}{2}\|u\|^2$ grows faster than the linear term $C\|u\|$, and the superlinear term in L^{p+q} further reinforces this growth. Therefore, as $\|u\|_{W^{s,2}} \to \infty$, the right-hand side tends to $+\infty$, which proves that $\mathcal{E}$ is coercive on $W_0^{s,2}(\Omega)$.
The energy functional $\mathcal{E}$ is coercive, meaning that it grows unboundedly as $\|u\|_{W^{s,2}} \to \infty$. This is a key property for establishing the existence of minimizers via the direct method in the calculus of variations. Let $\mathcal{E}(u)$ be the energy functional defined on $W_0^{s,2}(\Omega)$ by 3, where $f \in L^r(\Omega)$ for some $r > 1$, and $q + 1 < 2_{s^*} := \frac{2N}{N-2s}$ ensures the Sobolev embedding $W_0^{s,2}(\Omega) \hookrightarrow L^{q+1}(\Omega)$ is compact. Let $\{u_n\} \subset W_0^{s,2}(\Omega)$ be a sequence such that $u_n \rightharpoonup u$ weakly in $W_0^{s,2}(\Omega)$. We analyze the behavior of each term in $\mathcal{E}(u_n)$ under this convergence. The first term is the squared Gagliardo seminorm:

$$\|u\|^2_{W^{s,2}(\mathbb{R}^N)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.$$

This is convex and lower semicontinuous with respect to weak convergence in $W^{s,2}$. Therefore,

$$\liminf_{n\to\infty} \|u_n\|^2_{W^{s,2}(\mathbb{R}^N)} \geq \|u\|^2_{W^{s,2}(\mathbb{R}^N)}.$$

The second term is:

$$\int_\Omega |u_n(x)|^{q+1}\,dx.$$

Since $q + 1 < 2_{s^*}$, the embedding $W_0^{s,2}(\Omega) \hookrightarrow L^{q+1}(\Omega)$ is compact. Therefore, $u_n \to u$ strongly in $L^{q+1}(\Omega)$, and hence:

$$\lim_{n\to\infty} \int_\Omega |u_n(x)|^{q+1}\,dx = \int_\Omega |u(x)|^{q+1}\,dx.$$

In particular, this term is weakly lower semicontinuous due to convexity and strong convergence.
The third term is linear in u:

$$\int_\Omega f(x)u_n(x)\,dx.$$

Since $f \in L^r(\Omega)$ and $u_n \rightharpoonup u$ in $L^{r'}(\Omega)$, we have:

$$\int_\Omega f(x)u_n(x)\,dx \to \int_\Omega f(x)u(x)\,dx.$$

Combining all three components, we conclude that the energy functional $\mathcal{E}(u)$ is weakly lower semicontinuous on $W_0^{s,2}(\Omega)$. This property, together with coercivity,

ensures the existence of a minimizer via the direct method in the calculus of variations.
Let $\mathcal{E} : W_0^{s,2}(\Omega) \to \mathbb{R}$ be the energy functional defined by (3), where $f \in L^r(\Omega)$, $q+1 < 2_{s^*}$, and $s \in (0,1)$. Let $\{u_n\} \subset W_0^{s,2}(\Omega)$ be a minimizing sequence for $\mathcal{E}$, i.e.,

$$\lim_{n\to\infty} \mathcal{E}(u_n) = \inf_{u\in W_0^{s,2}(\Omega)} \mathcal{E}(u).$$

We know that $\mathcal{E}$ is coercive. Therefore, the sequence $\{u_n\}$ is bounded in $W_0^{s,2}(\Omega)$, i.e., there exists a constant $C > 0$ such that:

$$\|u_n\|_{W^{s,2}(\mathbb{R}^N)} \leq C \quad \text{for all } n.$$

By the Banach-Alaoglu Theorem, there exists a subsequence (still denoted u_n) and a function $u \in W_0^{s,2}(\Omega)$ such that:

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{s,2}(\Omega).$$

Moreover, since $q+1 < 2_{s^*}$, the embedding $W_0^{s,2}(\Omega) \hookrightarrow L^{q+1}(\Omega)$ is compact. Therefore,

$$u_n \to u \quad \text{strongly in } L^{q+1}(\Omega).$$

We know that $\mathcal{E}$ is weakly lower semicontinuous. Hence,

$$\mathcal{E}(u) \leq \liminf_{n\to\infty} \mathcal{E}(u_n) = \inf_{v\in W_0^{s,2}(\Omega)} \mathcal{E}(v).$$

The function $u \in W_0^{s,2}(\Omega)$ is a minimizer of $\mathcal{E}$. This establishes the existence of a weak solution to the associated Euler-Lagrange equation via the direct method in the calculus of variations.
Let $u \in W_0^{s,2}(\Omega)$ be the minimizer of the energy functional defined by (3), where $f \in L^r(\Omega)$, $q+1 < 2_{s^*}$, and $s \in (0,1)$. Since u minimizes $\mathcal{E}$, it satisfies the first-order optimality condition: for all $\varphi \in W_0^{s,2}(\Omega)$,

$$\frac{d}{d\varepsilon}\mathcal{E}(u+\varepsilon\varphi)\bigg|_{\varepsilon=0} = 0.$$

Computing this derivative yields the variational identity:

$$\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_\Omega u^q(x)\varphi(x)\,dx = \int_\Omega f(x)\varphi(x)\,dx,$$

for all $\varphi \in W_0^{s,2}(\Omega)$. This identity is the weak formulation of the nonlinear fractional elliptic problem:

$$\begin{cases} (-\Delta)^s u + u^q = f & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $(-\Delta)^s$ denotes the fractional Laplacian defined via the bilinear form:

$$\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy.$$

The minimizer u of the energy functional $\mathcal{E}$ is a weak solution to the fractional semilinear PDE. This variational identity encapsulates both the nonlocal diffusion and the nonlinear reaction terms, and serves as the foundation for further regularity and uniqueness analysis.

3. Optimal Shape Theorem via Γ-Convergence

In shape optimization, the objective is to determine the optimal geometry of a structure that satisfies specific performance criteria, such as minimizing weight while maximizing stiffness. However, the existence of an optimal shape is not always guaranteed. This challenge arises from the fact that the set of admissible shapes is often non-compact and geometrically irregular, where even small perturbations in the domain can lead to significant changes in physical behavior. To address these difficulties, researchers such as Fall et al. [4, 5, 6] have investigated conditions under which optimal shapes can be rigorously shown to exist. Establishing existence results is crucial not only for mathematical completeness but also for ensuring that computational design methods yield meaningful and physically realizable solutions.
In this work, we investigate the existence of optimal shapes within the framework of Γ-convergence, a powerful variational technique particularly effective in addressing the lack of compactness and non-convexity that typically arise in shape optimization problems involving nonlocal operators. By introducing relaxed formulations of the original problem and exploiting the convergence properties of the associated functionals, we provide a rigorous proof of the existence of minimizers under suitable geometric and functional constraints. This approach not only ensures analytical robustness but also highlights the interplay between nonlocal fractional operators and the geometric complexity of admissible domains.
Let $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, be an admissible domain. We consider the fractional elliptic boundary value problem (1), where $0 < s < 1$, $q > 1$, and $f \in L^r(\mathbb{R}^N)$. The associated energy functional is defined by (2).

Theorem 2. *Let $\mathcal{A}$ be a class of admissible domains in $\mathbb{R}^N$, compact under Γ-convergence. Assume:*

- *For each $\Omega \in \mathcal{A}$, the problem* (1) *admits a unique solution $u_\Omega \in W_0^{s,2}(\Omega)$,*
- *The functional $J(\Omega)$ is lower semicontinuous with respect to Γ-convergence,*

Then there exists an optimal domain $\Omega^* \in \mathcal{A}$ such that:

$$J(\Omega^*) = \inf_{\Omega \in \mathcal{A}} J(\Omega).$$

Proof. Let us first clarify the framework. We denote by $\mathcal{A}$ the class of admissible domains for the shape optimization problem. For concreteness, we assume that $\mathcal{A}$ consists of bounded open subsets of $\mathbb{R}^N$ with Lipschitz boundary, satisfying uniform measure and regularity constraints. This ensures that weak solutions of the fractional elliptic problem are well-defined on each $\Omega \in \mathcal{A}$. Topology: We equip $\mathcal{A}$ with the topology induced by Γ-convergence of the associated energy functionals. More precisely, for each domain $\Omega \in \mathcal{A}$, we consider the functional

$$F_\Omega(u) = \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx + \int_\Omega |u|^{q+1} \, dx - \langle f, u \rangle,$$

with the convention that $F_\Omega(u) = +\infty$ whenever $u \notin W_0^{s,2}(\Omega)$.
We say that $\Omega_n \xrightarrow{\Gamma} \Omega^*$ if $F_{\Omega_n} \xrightarrow{\Gamma} F_{\Omega^*}$ in $L^2(\mathbb{R}^N)$. This topology is natural in variational problems, since it guarantees stability of minimizers. Now let $\{\Omega_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ be a minimizing sequence for the cost functional $J : \mathcal{A} \to \mathbb{R}$, i.e.

$$\lim_{n\to\infty} J(\Omega_n) = \inf_{\Omega\in\mathcal{A}} J(\Omega).$$

By compactness of $\mathcal{A}$ under Γ-convergence, there exists a subsequence (still denoted Ω_n) and a limit domain $\Omega^* \in \mathcal{A}$ such that

$$\Omega_n \xrightarrow{\Gamma} \Omega^*.$$

Lower semicontinuity: Assume that J is lower semicontinuous with respect to Γ-convergence. By definition, this means that for any sequence $\Omega_n \xrightarrow{\Gamma} \Omega^*$ we have

$$J(\Omega^*) \leq \liminf_{n\to\infty} J(\Omega_n).$$

This property is crucial: it ensures that the cost functional does not decrease unexpectedly in the limit, preserving the minimizing structure. Combining the minimizing property of $\{\Omega_n\}$ with lower semicontinuity, we obtain

$$\inf_{\Omega\in\mathcal{A}} J(\Omega) = \lim_{n\to\infty} J(\Omega_n) \geq J(\Omega^*).$$

On the other hand, since $\Omega^* \in \mathcal{A}$, we trivially have

$$J(\Omega^*) \geq \inf_{\Omega\in\mathcal{A}} J(\Omega).$$

Putting these inequalities together yields

$$J(\Omega^*) = \inf_{\Omega \in \mathcal{A}} J(\Omega).$$

Therefore, the limit domain Ω^* achieves the minimal value of the cost functional over the admissible class $\mathcal{A}$. This proves the existence of an optimal domain under the assumptions of compactness via Γ-convergence and lower semicontinuity of J.

4. Regularity

In this section, we study regularity for nonlocal operators. We ask how domain geometry and boundary conditions affect the smoothness of fractional elliptic solutions. These questions are crucial both for theory and for numerical methods in optimization. We focus on the fractional Laplacian, which creates specific analytical challenges near the boundary. For detailed results, see Ros-Oton and Serra [9, 10, 11], who established boundary regularity using nonlocal analogs of classical techniques. Further contributions by Fall and collaborators [7] provide important insights into existence and regularity in fractional elliptic frameworks.

4.1. Boundedness of Solutions

We now prove that any weak solution $u \in W_0^{s,2}(\Omega)$ to the fractional semilinear problem 1 is in fact bounded in $L^\infty(\Omega)$, provided that the source term $f \in L^r(\Omega)$, $r \geq 1$.

Theorem 3 (Uniform Boundedness)**.** *Let $f \in L^r(\Omega)$, $r \geq 1$. Then any weak solution $u \in W_0^{s,2}(\Omega)$ to problem* (1) *satisfies $u \in L^\infty(\Omega)$.*

Proof. We consider the truncation $T_k(u) := \min(u, k)$ for $k > 0$. Since $u \in W_0^{s,2}(\Omega)$, one has $T_k(u) \in W_0^{s,2}(\Omega)$ and $|T_k(u(x)) - T_k(u(y))| \leq |u(x) - u(y)|$. Using $T_k(u)$ as a test function in the weak formulation, we obtain

$$\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(T_k(u(x)) - T_k(u(y)))}{|x - y|^{N+2s}} \, dx \, dy + \int_\Omega u^q T_k(u) \, dx = \int_\Omega f T_k(u) \, dx.$$

By monotonicity of the truncation, the first term controls the energy of $T_k(u)$. Indeed, for any $a, b \in \mathbb{R}$ one has $(a - b)(T_k(a) - T_k(b)) \geq |T_k(a) - T_k(b)|^2$, which follows from the fact that T_k is a non-decreasing Lipschitz function. Applying this inequality pointwise with $a = u(x)$ and $b = u(y)$, we deduce that

$$\int_{\mathbb{R}^{2N}} \frac{|T_k(u(x)) - T_k(u(y))|^2}{|x - y|^{N+2s}} \, dx \, dy \leq \int_\Omega f T_k(u) \, dx.$$

Applying the fractional Sobolev inequality yields

$$\|T_k(u)\|^2_{L^{2^*_s}(\Omega)} \le C \int_\Omega f T_k(u)\, dx,$$

where $2^*_s = \frac{2N}{N-2s}$. Hölder's inequality then gives

$$\|T_k(u)\|_{L^{2^*_s}(\Omega)} \le C \|T_k(u)\|_{L^{r'}(\Omega)}.$$

This inequality is the starting point of the Moser iteration. The idea is to control the measure of the super-level sets of u. Define a decreasing sequence of levels k_n converging to some k_∞, and set $\phi_n = (u - k_n)_+$. Since $\phi_{n+1} \le \phi_n$, the inequality above applied to ϕ_n yields a recursive estimate between successive norms. More precisely, one obtains

$$\|\phi_{n+1}\|_{L^{2^*_s}(\Omega)} \le C \|\phi_n\|_{L^{r'}(\Omega)}.$$

By interpolation between $L^{r'}$ and $L^{2^*_s}$ norms, and using the embedding properties of fractional Sobolev spaces, this recursive inequality can be sharpened into

$$\|\phi_{n+1}\|_{L^{2^*_s}(\Omega)} \le C \|\phi_n\|^{1+\delta}_{L^{2^*_s}(\Omega)},$$

for some $\delta > 0$ depending on s, N, r. At this point, one applies a standard iteration lemma (Moser or De Giorgi). Such lemmas state that if a sequence of nonnegative functions satisfies a recursive inequality of the form above, then the sequence converges to zero. Concretely, this means $\|\phi_n\|_{L^{2^*_s}(\Omega)} \to 0$ as $n \to \infty$, which implies that the measure of the super-level sets $\{u > k_n\}$ shrinks to zero. Therefore, u cannot exceed a finite bound almost everywhere in Ω.
Consequently, $u \in L^\infty(\Omega)$.

4.2. Hölder Regularity of Solutions

We now establish that any weak solution $u \in W_0^{s,2}(\Omega)$ to the fractional semilinear problem (1) is locally Hölder continuous in Ω, under suitable assumptions on the data.

Theorem 4 (Interior Hölder Regularity)**.** *Let $f \in L^\infty(\Omega)$, $q > 1$, and $u \in W_0^{s,2}(\Omega)$ be a weak solution to* (1)*. Then there exists $\alpha \in (0,1)$ such that $u \in C^\alpha_{loc}(\Omega)$.*

Proof. We adapt the De Giorgi–Moser iteration method to the nonlocal setting. Let $u \in W^{s,2}(\mathbb{R}^N)$ be a weak solution of

$$\begin{cases} (-\Delta)^s u + u^q = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

with $f \in L^r(\Omega)$, $r > 1$, and $q > 0$. For $k > 0$, define the truncation $u_k := (u-k)_+ \in W_0^{s,2}(\Omega)$, and let $\eta \in C_c^\infty(\Omega)$ be a cutoff supported in $B_{2R}(x_0)$, with $\eta \equiv 1$ in $B_R(x_0)$ and $|\nabla \eta| \le C/R$. Using the test function $\varphi := u_k \eta^2$ in the weak formulation yields the localized energy estimate

$$\int_{\mathbb{R}^{2N}} \frac{|u_k(x)\eta(x) - u_k(y)\eta(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \le C \int_\Omega |f(x)| u_k(x) \eta(x)^2\, dx.$$

Applying the fractional Sobolev inequality to $u_k \eta$ gives

$$\left(\int_\Omega |u_k \eta|^{2_{s^*}}\, dx \right)^{2/2_{s^*}} \le C \int_\Omega |f(x)| u_k(x) \eta(x)^2\, dx,$$

where $2_{s^*} = \frac{2N}{N-2s}$. Since $u_k \eta$ is supported in $A_k := \{x \in B_R(x_0) : u(x) > k\}$, we deduce

$$k^2 |A_k|^{2/2_{s^*}} \le C \int_\Omega |f(x)| u_k(x) \eta(x)^2\, dx.$$

This inequality provides a recursive control of the measure of super-level sets. More precisely, for suitable $k' < k$ one obtains

$$|A_k| \le C \frac{1}{(k-k')^2} |A_{k'}|^{1+\delta},$$

for some $\delta > 0$. To implement the iteration, define an increasing sequence of levels k_n converging to k_∞, and set $Y_n := |A_{k_n}|$. The recursive inequality then takes the form

$$Y_{n+1} \le C 2^{2(n+1)} Y_n^{1+\delta}.$$

This is the standard nonlinear recurrence used in the Moser/De Giorgi iteration. The key point is that the exponent $1 + \delta > 1$ forces rapid decay of the sequence Y_n, provided the initial measure Y_0 is sufficiently small. By choosing k_0 large enough, one ensures Y_0 is small, and the iteration lemma implies $Y_n \to 0$ as $n \to \infty$. Consequently, $u(x) \le k_\infty$ almost everywhere in $B_R(x_0)$, which shows that $u \in L^\infty_{\mathrm{loc}}(\Omega)$. Finally, combining local boundedness with oscillation decay arguments based on the nonlocal Caccioppoli inequality and fractional Sobolev embeddings, one proves that

$$\operatorname{osc}_{B_\rho(x_0)} u \le C \left(\frac{\rho}{R} \right)^\alpha \operatorname{osc}_{B_R(x_0)} u,$$

for some $\alpha \in (0,1)$. This establishes that $u \in C^\alpha_{\mathrm{loc}}(\Omega)$, i.e. weak solutions are locally Hölder continuous.

The following section addresses boundary regularity for solutions to the fractional elliptic problem.

4.3. Boundary Regularity of Solutions

We now address the regularity of weak solutions up to the boundary of the domain. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^{1,1}$ regularity. We consider the fractional semilinear problem (1) with $0 < s < 1$, $q > 1$, and $f \in L^\infty(\Omega)$.

Theorem 5 (Boundary Hölder Regularity)**.** *Let Ω be a bounded $C^{1,1}$ domain and $f \in L^\infty(\Omega)$. Then any weak solution $u \in W_0^{s,2}(\Omega)$ to problem* (1) *satisfies:*

$$u(x) \le C\, d_\Omega(x)^s \quad \textit{and} \quad \frac{u(x)}{d_\Omega(x)^s} \in C^\alpha(\overline{\Omega}),$$

for some $\alpha \in (0,1)$, where $d_\Omega(x) := dist(x, \partial\Omega)$.

Proof. We follow an approach based on barrier functions and the boundary Harnack principle. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^{1,1}$ boundary. Define the distance function

$$d_\Omega(x) := \mathrm{dist}(x, \partial\Omega),$$

which measures the Euclidean distance from a point $x \in \Omega$ to the boundary $\partial\Omega$. Since Ω is $C^{1,1}$, the function d_Ω is smooth in a neighborhood of the boundary, and in particular,

$$d_\Omega \in C^1(\overline{\Omega}) \quad \text{and} \quad |\nabla d_\Omega(x)| = 1 \quad \text{near } \partial\Omega.$$

We define the barrier function

$$\phi(x) := d_\Omega(x)^s,$$

for some fixed $s \in (0,1)$. Then $\phi \in W_0^{s,2}(\Omega)$, and it vanishes on $\partial\Omega$ in the trace sense.

According to regularity theory and explicit computations (see Ros-Oton and Serra [11]), for domains with $C^{1,1}$ boundary, the function $\phi(x) = d_\Omega(x)^s$ satisfies

$$(-\Delta)^s \phi(x) \ge c > 0 \quad \text{for } x \in \Omega \text{ sufficiently close to } \partial\Omega,$$

where c depends on the geometry of Ω and the exponent s. This inequality implies that ϕ acts as a supersolution near the boundary and can be used in comparison arguments, maximum principles, and boundary regularity proofs. In summary, the function $\phi(x) = d_\Omega(x)^s$ serves as a valid barrier near the boundary of Ω, satisfying:

- $\phi \in W_0^{s,2}(\Omega)$,
- $\phi(x) > 0$ in Ω, and $\phi(x) = 0$ on $\partial\Omega$,

- $(-\Delta)^s\phi(x) \geq c > 0$ near $\partial\Omega$.

The barrier function plays a key role in establishing boundary regularity and constructing sub- and supersolutions in the theory of nonlocal partial differential equations. Let $u \in W_0^{s,2}(\Omega)$ be the weak solution to the fractional semilinear problem:

$$\begin{cases} (-\Delta)^s u + u^q = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $f \in L^\infty(\Omega)$, $q > 0$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{1,1}$ boundary. We introduce the barrier function

$$\phi(x) := d_\Omega(x)^s,$$

where $d_\Omega(x) := \operatorname{dist}(x, \partial\Omega)$. Since Ω has a $C^{1,1}$ boundary, it follows that $\phi \in W_0^{s,2}(\Omega)$, and from known regularity results (e.g., Ros-Oton and Serra), we have:

$$(-\Delta)^s\phi(x) \geq c > 0 \quad \text{near } \partial\Omega.$$

We now define the scaled barrier:

$$\psi(x) := C\, d_\Omega(x)^s = C\, \phi(x),$$

where $C > 0$ is chosen sufficiently large so that

$$(-\Delta)^s\psi(x) + \psi(x)^q \geq f(x) \quad \text{in } \Omega.$$

This is feasible because:

$$(-\Delta)^s\psi = C\,(-\Delta)^s\phi \geq Cc, \qquad \psi^q = C^q\,\phi^q \geq 0,$$

and since $f(x) \in L^\infty(\Omega)$, we have $f(x) \leq \|f\|_{L^\infty}$. Choosing C large enough ensures:

$$Cc + C^q\phi^q(x) \geq \|f\|_{L^\infty} \quad \text{in } \Omega.$$

Thus, ψ is a supersolution to the equation. Since both u and ψ vanish outside Ω, the fractional maximum principle implies:

$$u(x) \leq \psi(x) = C\, d_\Omega(x)^s \quad \text{in } \Omega.$$

This pointwise upper bound

$$u(x) \leq C\, d_\Omega(x)^s$$

provides quantitative control of u near the boundary. Such an estimate is crucial for boundary regularity and for analyzing solution behavior in singular perturbation problems. Motivated by this estimate, we define the normalized quotient function:

$$v(x) := \frac{u(x)}{d_\Omega(x)^s},$$

which captures the scaled behavior of the solution near the boundary and is instrumental in further regularity analysis. Since $u(x) \leq C\, d_\Omega(x)^s$, it follows that

$$v(x) := \frac{u(x)}{d_\Omega(x)^s} \leq C,$$

so $v \in L^\infty(\Omega)$. Near the boundary, $d_\Omega(x)^s \to 0$, but $u(x) \to 0$ at the same rate, ensuring that $v(x)$ remains bounded and well-defined up to $\partial\Omega$. The function v satisfies a nonlocal equation derived from the original PDE and the structure of the fractional Laplacian acting on products. While the explicit form of this equation is intricate, it is known from works such as Cabré-Sire [12, 13] and Xia-Yang [14] that the quotient $v = u/d_\Omega^s$ satisfies a nonlocal equation with bounded coefficients. The geometry of Ω and the regularity of d_Ω (due to the $C^{1,1}$ boundary) ensure that the nonlocal terms are well-behaved. Moreover, the nonlocal boundary Harnack inequality applies to such quotients and yields Hölder continuity up to the boundary. Therefore, we conclude that $v \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$ depending on s, N, q, and the regularity of Ω. This shows that u behaves like d_Ω^s near $\partial\Omega$, and its smoothness is determined by v. The result gives a sharp description of boundary behavior and completes the regularity analysis of the fractional semilinear problem. Combining interior regularity with boundary estimates from barrier arguments and comparison principles, we conclude that the weak solution $u \in W_0^{s,2}(\Omega)$ is both bounded and Hölder continuous up to the boundary.

$$\begin{cases} (-\Delta)^s u + u^q = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

with $f \in L^\infty(\Omega)$, satisfies the global regularity estimate:

$$u \in C^\alpha(\overline{\Omega}),$$

for some $\alpha \in (0,1)$ depending on s, N, q, and the geometry of Ω. Moreover, the boundary behavior of u is governed by the distance to the boundary. Specifically, there exists a constant $C > 0$ such that

$$0 \leq u(x) \leq C\, d_\Omega(x)^s \quad \text{for all } x \in \Omega,$$

and the quotient

$$v(x) := \frac{u(x)}{d_\Omega(x)^s} \in C^\alpha(\overline{\Omega}).$$

Therefore, we obtain the asymptotic behavior

$$u(x) \sim d_\Omega(x)^s \quad \text{as } x \to \partial\Omega,$$

in the sense that $u(x)/d_\Omega(x)^s$ remains bounded and Hölder continuous up to the boundary.

Remark:

This completes the regularity theory for the solution u, showing that it is globally Hölder continuous in $\overline{\Omega}$, with a precise boundary singularity of order s. Such results are fundamental in the study of nonlocal elliptic equations and provide a sharp understanding of the interplay between the fractional operator, the nonlinearity, and the geometry of the domain.

5. Conclusion

In this study, we examined existence and regularity in fractional elliptic shape optimization. Using Γ-convergence, we proved the existence of optimal shapes in the nonlocal setting. Our analysis established boundedness, Hölder continuity, and precise boundary behavior of solutions. These results reinforce the theory of fractional shape optimization and open paths for future work. Promising directions include numerical methods, sensitivity analysis, inverse problems, and optimal control. The link between nonlocal operators and geometric optimization remains a rich field for theory and applications.

References

[1] J. F. Bonder, A. Ritorto, and A. M. Salort. (**2018**). *A class of shape optimization problems for some non local operators*, Advances in Calculus of Variations, **11**(4), pp. 373-386. https://doi.org/10.1515/acv-2016-0065.

[2] L. Caffarelli and L. Silvestre. (**2007**). *An extension problem related to the fractional Laplacian*, Communications in Partial Differential Equations, **32**(8), pp. 1245-1260.

[3] A. L. Dalibard and D. Gérard-Varet. (**2013**). *On shape optimization problems involving the fractional Laplacian*, ESAIM: Control, Optimisation and Calculus of Variations, **19**(4), pp. 976-1013.

[4] M. Fall, I. Faye, A. Sy, and D. Seck. (**2021**). *On shape optimization theory with fractional Laplacian*, Applied and Computational Mathematics, **10**(3), pp. 56-68. doi:10.11648/j.acm.20211003.12.

[5] M. Fall, A. Sy, I. Faye, and D. Seck. (**2025**). *On shape optimization theory with fractional p-Laplacian operators*, Abstract and Applied Analysis, Volume 2025, Article ID 1932719, 12 pages. https://doi.org/10.1155/aaa/1932719.

[6] M. Fall, M. G. Ngom, G. I. Sadio, I. Faye, A. Sy, and D. Seck. (**2026**). *Shape and topological optimization for a fractional elliptic boundary problem*, Journal of Finsler Geometry and its Applications, **7**(1), pp. 45-65. https://doi.org/10.22098/jfga.2025.17676.1163.

[7] Fall, M., Kourouma, B., Faye, I., & Sy, A. (2026). Shape Derivative of the Fractional p-Laplacian Operators Via Minmax Method. Journal of Contemporary Applied Mathematics, 16(2).

[8] A. Henrot and M. Pierre. (**2005**). *Shape Variation and Optimization: A Geometric Analysis*, Springer Science & Business Media, **48**.

[9] X. Ros-Oton and J. Serra. (**2014**). *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, Journal of Pure and Applied Mathematics, **101**(3), pp. 275-302.

[10] X. Ros-Oton and J. Serra. (**2016**). *Boundary regularity for fully nonlinear integro-differential equations*, Duke Mathematical Journal, **165**(11), pp. 2079-2154.

[11] X. Ros-Oton and J. Serra. (**2016**) *Regularity theory for general stable operators*, Journal of Differential Equations, **260**(12), pp. 8675-8715.

[12] X. Cabré and Y. Sire. (**2014**). *Nonlinear equations for fractional Laplacians I: Regularity, maximum principles, and Hamiltonian estimates*, Annales de l'I.H.P. Analyse non linéaire, **31**(1), pp. 23-53. http://www.numdam.org/item/AIHPC-2014-31-1-23-0.

[13] X. Cabré and Y. Sire. (**2015**). *Nonlinear equations for fractional Laplacians II: Existence, uniqueness, and qualitative properties of solutions*, Transactions of the American Mathematical Society, **367**(2), pp. 911-941. https://hal.science/hal-01341584.

[14] J. Xia and J. Yang. (**2021**). *Regularity of solutions to nonlocal elliptic equations with measurable coefficients*, Journal of Differential Equations, **274**, pp. 1-35. doi:10.1016/j.jde.2020.09.012.

[15] M. Warma. (**2015**). *The fractional relative capacity and the fractional Laplacian with Neumann and Robin boundary conditions on open sets*, Potential Analysis, **42**(2), pp. 499-547.

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