Journal of Contemporary Applied Mathematics
V. 17, No 2, 2027, July-December
ISSN 2222-5498, E-ISSN 3006-3183
https://doi.org/10.62476/jcam171.21

Characterizations for the fractional maximal operator and its commutators on total mixed Morrey spaces

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Abstract. We shall give a characterization for the strong and weak type Adams type boundedness of the fractional maximal operator M_α on total mixed Morrey spaces $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$, respectively. Also we give necessary and sufficient conditions for the boundedness of the fractional maximal commutator operator $M_{b,\alpha}$ and commutator of fractional maximal operator $[b, M_\alpha]$ on $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ when b belongs to $BMO(\mathbb{R}^n)$ spaces, whereby some new characterizations for certain subclasses of $BMO(\mathbb{R}^n)$ spaces are obtained.

Key Words and Phrases: total mixed Morrey spaces, fractional maximal operator, commutator, BMO

2020 Mathematics Subject Classifications: 42B20, 42B25, 42B35

1. Introduction

Classical Morrey spaces $L^{p,\lambda}$ were originally introduced by Morrey in [34] to study the local behavior of solutions of second-order elliptic partial differential equations. In 2022, Guliyev [21] introduced a variant of Morrey spaces called total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$, $0 < p < \infty, \lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$, see also [3, 4, 11, 23, 24, 26, 27, 28, 30, 31, 32, 33, 35, 41, 42, 43, 48]. Total Morrey spaces generalize the classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ so that $L^{p,\lambda,\lambda}(\mathbb{R}^n) \equiv L^{p,\lambda}(\mathbb{R}^n)$ and the modified Morrey spaces $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ so that $L^{p,\lambda,0}(\mathbb{R}^n) = \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$, respectively.

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The subject of mixed-norm function spaces has undergone great development in the last few decades. Nevertheless, the standard literature is still the mixed Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$, $0 < \vec{p} \leq \infty$, as a natural generalization of the classical Lebesgue spaces $L^p(\mathbb{R}^n)$, $0 < p \leq \infty$, it is first introduced by Benedek and Panzone [8] in 1961. Mixed-norm function spaces possess a more refined structural framework than their classical counterparts, thereby enabling wider applications in analysis such as potential analysis, harmonic analysis and partial differential equations. In 2019, Nogayama [36] introduced a new Morrey-type space called mixed Morrey space by generalizing Morrey spaces and mixed Lebesgue spaces, see also [6, 10, 22, 37]. We introduce the total mixed Morrey spaces $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ here. These spaces generalize the mixed Lebesgue spaces so that $L^{\vec{p},0,0}(\mathbb{R}^n) \equiv L^{\vec{p}}(\mathbb{R}^n)$, the mixed Morrey spaces so that $L^{\vec{p},\lambda,\lambda}(\mathbb{R}^n) \equiv L^{\vec{p},\lambda}(\mathbb{R}^n)$ and the modified mixed Morrey spaces so that $L^{\vec{p},\lambda,0}(\mathbb{R}^n) = \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$. The study of fractional maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behavior are very informative in particular in differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators (see, for instance [15]). For $0 < \alpha < n$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the fractional maximal operator M_α is defined by

$$M_\alpha f(x) = \sup_{r>0} |B(x,r)|^{-1+\frac{\alpha}{n}} \int_{B(x,r)} |f(y)|dy,$$

where $B(x,r)$ is the ball of radius r centered at $x \in \mathbb{R}^n$, $\complement B(x,r)$ is its complement and $|B(x,r)|$ denotes the Lebesgue measure of $B(x,r)$. When $\alpha = 0$, we simply denote by $M = M_0$.

The fractional maximal commutator generated by the operator M_α and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$M_{b,\alpha} f(x) = M_\alpha((b(x) - b)f)(x).$$

The commutator generated by the operator M_α and a suitable function b is defined by

$$[b, M_\alpha]f(x) = b(x)M_\alpha f(x) - M_\alpha(bf)(x).$$

When $\alpha = 0$, we simply denote by $M_b = M_{b,0}$ and $[b, M] = [b, M_0]$. Obviously, the operators $M_{b,\alpha}$ and $[b, M_\alpha]$ essentially differ from each other since $M_{b,\alpha}$ is positive and sublinear and $[b, M_\alpha]$ is neither positive nor sublinear. The operators M_α, $[b, M_\alpha]$ and $M_{b,\alpha}$ play an important role in real and harmonic analysis and applications (see, for instance [2, 5, 7, 14, 19, 21, 51]).

The boundedness of the Hardy-Littlewood maximal operator M on $L^p(\mathbb{R}^n)$ is one of the most fundamental results in harmonic analysis. It has been extended to

a range of other function spaces, and to many variations of the standard maximal operator. In particular, one can study commutators of M with BMO functions b. These turn out to be L^p bounded for $1 < p < \infty$ if and only if $b \in BMO$ and $b^- \equiv -\min\{b, 0\} \in L^\infty(\mathbb{R}^n)$ [7]. This is useful, for instance, when studying the product of an H^1 function with a BMO function [9]. Note that, the boundedness of the operator M_b on L^p spaces was proved by Garcia-Cuerva et al. [14].

The commutator estimates play an important role in studying the regularity of solutions of elliptic partial differential equations, and their boundedness can be used to characterize certain function spaces (see, for instance [2, 5, 7, 15, 19, 21, 46, 47, 49, 50, 51]).

The classical Morrey spaces were introduced by Morrey [34] for the study of solutions of some quasi-linear elliptic partial differential equations. For more applications of Morrey spaces on partial differential equation, the reader is referred to [39, 40, 44, 45, 46, 47]. In [21] introduced a variant of Morrey spaces called total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$, $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. Total Morrey spaces generalize the classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ so that $L^{p,\lambda,\lambda}(\mathbb{R}^n) \equiv L^{p,\lambda}(\mathbb{R}^n)$ and the modified Morrey spaces $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ so that $L^{p,\lambda,0}(\mathbb{R}^n) = \widetilde{L}^{p,\lambda}(\mathbb{R}^n)$, respectively. We given necessary and sufficient conditions for the boundedness of the maximal commutator operator M_b and the commutator of maximal operator $[b, M]$ on $L^{p,\lambda,\mu}(\mathbb{R}^n)$ when b belongs to $BMO(\mathbb{R}^n)$.

The aim of this paper is to give necessary and sufficient conditions for the boundedness of the fractional maximal commutator operator $M_{b,\alpha}$ and the commutator of fractional maximal operator $[b, M_\alpha]$ on $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ when b belongs to $BMO(\mathbb{R}^n)$ spaces. We obtain some new characterizations for certain subclasses of $BMO(\mathbb{R}^n)$.

The structure of the paper is as follows. In Section 2 we introduce the total mixed Morrey spaces $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$. We give some definitions, auxiliary results and some embeddings into the total mixed Morrey space $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$. In Section 3 we give a characterization for the strong and weak type Adams type boundedness of the fractional maximal operator M_α on $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$, respectively. In Section 4 we find necessary and sufficient conditions for the boundedness of the fractional maximal commutator $M_{b,\alpha}$ on $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ spaces. In Section 5 we find necessary and sufficient conditions for the boundedness of the commutator of fractional maximal operator $[b, M_\alpha]$ on $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ spaces.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Definition and basic properties of total mixed Morrey spaces

For any $r > 0$ and $x \in \mathbb{R}^n$, let $B(x,r) = \{y \ : \ |y-x| < r\}$ be the ball centered at x with radius r. Let $\mathrm{B} = \{B(x,r) \ : \ x \in \mathbb{R}^n, \ r > 0\}$ be the set of all such balls. We also use χ_E and $|E|$ to denote the characteristic function and the Lebesgue measure of a measurable set E.

The letter $\vec{p}$ denotes n-tuples of the numbers in $(0, \ \infty]$, $(n \geq 1)$, $\vec{p} = (p_1, \ \cdots \ , \ p_n)$. By definition, the inequality, for example, $0 < \vec{p} < \infty$ means $0 < p_i < \infty$ for all i. For $1 \leq \vec{p} \leq \infty$, we denote $\dfrac{1}{P} = \dfrac{1}{n}\sum\limits_{i=1}^{n} \dfrac{1}{p_i}$, $\vec{p}' = (p_1', \ \cdots \ , p_n')$, where p_i', P' satisfies $\dfrac{1}{p_i} + \dfrac{1}{p_i'} = 1$, $\dfrac{1}{P} + \dfrac{1}{P'} = 1$.

We first recall the definition of mixed Lebesgue space defined in [8].

Let $\vec{p} = (p_1, \ldots, p_n) \in (0, \infty]^n$. Then the mixed Lebesgue norm $\| \cdot \|_{L^{\vec{p}}}$ or $\| \cdot \|_{L^{(p_1,\ldots,p_n)}}$ is defined by

$$\begin{aligned} \|f\|_{L^{\vec{p}}} &\equiv \|f\|_{L^{(p_1,\ldots,p_n)}} \\ &= \Big(\int_{\mathbb{R}} \cdots \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f(x_1, \ x_2, \ \ldots, \ x_n)|^{p_1} dx_1 \Big)^{\frac{p_2}{p_1}} dx_2 \Big)^{\frac{p_3}{p_2}} \ldots dx_n \Big)^{\frac{1}{p_n}}, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a measurable function. If $p_j = \infty$ for some $j = 1, \ldots, n$, then we have to make appropriate modifications. We define the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n) = L^{(p_1,\ldots,p_n)}(\mathbb{R}^n)$ to be the set of all $f \in L^0(\mathbb{R}^n)$ with $\|f\|_{L^{\vec{p}}} < \infty$, where $L^0(\mathbb{R}^n)$ denotes the set of measureable functions on $\mathbb{R}^n$.

The following analogue of the Hölder's inequality for $L^{\vec{p}}$ is well known (see, for example, [52]).

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $1 \leq \vec{p} \leq \infty$ and $\frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = 1$. Then for any $f \in L^{\vec{p}}(\Omega)$ and $g \in L^{\vec{p}'}(\Omega)$, the following inequality is valid*

$$\int_{\Omega} |f(x)g(x)|dx \leq \|f\|_{L^{\vec{p}}(\Omega)} \|g\|_{L^{\vec{p}'}(\Omega)}.$$

By elementary calculations we have the following property.

Lemma 1. *Let $0 < \vec{p} < \infty$ and $B = B(x_0, r) \subset \mathbb{R}^n$ be a balls in $\mathbb{R}^n$. Then*

$$\|\chi_B\|_{L^{\vec{p}}} = \|\chi_B\|_{WL^{\vec{p}}} = C_{n,\vec{p}} \, |B|^{\frac{1}{P}},$$

where $C_{n,\vec{p}} > 0$ is independent of x_0 and r.

Proof. First, we will prove the equality of strong and weak mixed norms. Recall that

$$\|f\|_{WL^{\vec{p}}} = \sup_{t>0} t\, \|\chi_{\{x\in\mathbb{R}^n:|f(x)|>t\}}\|_{L^{\vec{p}}}.$$

For $f = \chi_B$, we have

$$\chi_{\{\chi_B>t\}} = \begin{cases} \chi_B, & 0 < t < 1, \\ 0, & t \geq 1. \end{cases}$$

Consequently,

$$\begin{aligned} \|\chi_B\|_{WL^{\vec{p}}} &= \sup_{t>0} t\, \|\chi_B\|_{L^{\vec{p}}} \\ &= \sup_{0<t<1} t\, \|\chi_B\|_{L^{\vec{p}}} = \|\chi_B\|_{L^{\vec{p}}}. \end{aligned}$$

By translation invariance and a change of variables,

$$\|\chi_{B(x_0,r)}\|_{L^{\vec{p}}} = r^{\sum_{i=1}^n \frac{1}{p_i}}\ \|\chi_{B(0,1)}\|_{L^{\vec{p}}}.$$

Since

$$|B(x_0,r)| = v_n\, r^n, \quad v_n = |B(0,1)|,$$

and $\frac{n}{P} = \sum_{i=1}^n \frac{1}{p_i}$, w obtain

$$\|\chi_{B(x_0,r)}\|_{L^{\vec{p}}} = \frac{\|\chi_{B(0,1)}\|_{L^{\vec{p}}}}{v_n^{\frac{1}{P}}}\ |B|^{\frac{1}{P}}.$$

This proves the claimed formula.

By Theorem 1 and Lemma 1 we get the following estimate.

Lemma 2. *For $1 \leq \vec{p} < \infty$ and for the balls $B = B(x,r)$ the following inequality is valid:*

$$\int_B |f(y)|dy \ \lesssim\ |B|^{\frac{1}{P'}}\, \|f\|_{L^{\vec{p}}(B)}.$$

The following lemma shows the Lebesgue differential theorem in mixed-norm Lebesgue spaces as follows.

Lemma 3. *[52, Lemma 2.4] Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $0 < \vec{p} < \infty$, then*

$$\lim_{r\to 0} \frac{\|f\|_{L^{\vec{p}}(B(x,r))}}{\|\chi_{B(x,r)}\|_{L^{\vec{p}}}} = |f(x)| \quad a.e.\ x \in \mathbb{R}^n.$$

In the following we define the mixed total mixed Morrey spaces $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.

Definition 1. *Let $0<\vec{p}<\infty$, $\lambda\in\mathbb{R}$, $\mu\in\mathbb{R}$, $[t]_1=\min\{1,t\}$, $t>0$. We denote by $L^{\vec{p},\lambda}(\mathbb{R}^n)$ the mixed Morrey space [36], by $\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$ the modified mixed Morrey space [17], and by $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ the total mixed Morrey space the set of all locally integrable functions f with the following finite norms*

$$\|f\|_{L^{\vec{p},\lambda}}=\sup_{x\in\mathbb{R}^n,\,t>0}t^{-\frac{\lambda}{P}}\|f\|_{L^{\vec{p}}(B(x,t))},$$

$$\|f\|_{\widetilde{L}_{\vec{p},\lambda}}=\sup_{x\in\mathbb{R}^n,\,t>0}[t]_1^{-\frac{\lambda}{P}}\|f\|_{L^{\vec{p}}(B(x,t))}$$

and

$$\|f\|_{L^{\vec{p},\lambda,\mu}}=\sup_{x\in\mathbb{R}^n,\,t>0}[t]_1^{-\frac{\lambda}{P}}[1/t]_1^{\frac{\mu}{P}}\|f\|_{L^{\vec{p}}(B(x,t))},$$

respectively.

Definition 2. *Let $0<\vec{p}<\infty$, $\lambda\in\mathbb{R}$ and $\mu\in\mathbb{R}$. We define the weak mixed Morrey space $WL^{\vec{p},\lambda}(\mathbb{R}^n)$ [36], the weak modified mixed Morrey space $W\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$ [17] and the weak total mixed Morrey space $WL^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite norms*

$$\|f\|_{WL^{\vec{p},\lambda}}=\sup_{x\in\mathbb{R}^n,\,t>0}t^{-\frac{\lambda}{P}}\|f\|_{WL^{\vec{p}}(B(x,t))},$$

$$\|f\|_{W\widetilde{L}^{\vec{p},\lambda}}=\sup_{x\in\mathbb{R}^n,\,t>0}[t]_1^{-\frac{\lambda}{P}}\|f\|_{WL^{\vec{p}}(B(x,t))}$$

and

$$\|f\|_{WL^{\vec{p},\lambda,\mu}}=\sup_{x\in\mathbb{R}^n,\,t>0}[t]_1^{-\frac{\lambda}{P}}[1/t]_1^{\frac{\mu}{P}}\|f\|_{WL^{\vec{p}}(B(x,t))},$$

respectively.

Note that

$$L^{\vec{p},0,0}(\mathbb{R}^n)=\widetilde{L}_{\vec{p},0}(\mathbb{R}^n)=L^{\vec{p},0}(\mathbb{R}^n)=L^{\vec{p}}(\mathbb{R}^n),$$
$$WL^{\vec{p},0,0}(\mathbb{R}^n)=W\widetilde{L}_{\vec{p},0}(\mathbb{R}^n)=WL^{\vec{p},0}(\mathbb{R}^n)=WL^{\vec{p}}(\mathbb{R}^n),$$
$$L^{\vec{p},\lambda,\lambda}(\mathbb{R}^n)=L^{\vec{p},\lambda}(\mathbb{R}^n),\quad L^{\vec{p},\lambda,0}(\mathbb{R}^n)=\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n),$$
$$\|f\|_{WL^{\vec{p},\lambda,\mu}}\le\|f\|_{L^{\vec{p},\lambda,\mu}}\ \text{ and therefore }\ L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)\subset WL^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$$

and

$$L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)\subset_{\succ}L^{\vec{p},\lambda}(\mathbb{R}^n),\ \mu\le\lambda\text{ and }\|f\|_{L^{\vec{p},\lambda}}\le\|f\|_{L^{\vec{p},\lambda,\mu}},\tag{1}$$

$$L^{\vec{p},\lambda,\mu}(\mathbb{R}^n) \subset_{\succ} L^{\vec{p},\mu}(\mathbb{R}^n),\ \mu \le \lambda \text{ and } \|f\|_{L^{\vec{p},\mu}} \le \|f\|_{L^{\vec{p},\lambda,\mu}} \tag{2}$$

$$\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n) \subset_{\succ} L^{\vec{p}}(\mathbb{R}^n) \text{ and } \|f\|_{L^{\vec{p}}} \le \|f\|_{\widetilde{L}_{\vec{p},\lambda}}$$

and if $\lambda < 0$ or $\lambda > n$, then $L^{\vec{p},\lambda}(\mathbb{R}^n) = \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n) = WL^{\vec{p},\lambda}(\mathbb{R}^n) = W\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n) = \Theta$. Here $\Theta \equiv \Theta(\mathbb{R}^n)$ is the set of all functions on $\mathbb{R}^n$ that are equivalent to 0.

Lemma 4. *[25, Lemma 4] If $0 < \vec{p} < \infty$, $0 \le \mu \le \lambda \le n$, then*

$$L^{\vec{p},\lambda,\mu}(\mathbb{R}^n) = L^{\vec{p},\lambda}(\mathbb{R}^n) \cap L^{\vec{p},\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{L^{\vec{p},\lambda}(\mathbb{R}^n)}, \|f\|_{L^{\vec{p},\mu}(\mathbb{R}^n)}\right\}.$$

Corollary 1. *If $0 < \vec{p} < \infty$, $0 \le \lambda \le n$, then*

$$\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n) = L^{\vec{p},\lambda}(\mathbb{R}^n) \cap L^{\vec{p}}(\mathbb{R}^n)$$

and

$$\|f\|_{\widetilde{L}_{\vec{p},\lambda}} = \max\left\{\|f\|_{L^{\vec{p},\lambda}}, \|f\|_{L^{\vec{p}}}\right\}.$$

Analogously proved

Lemma 5. *If $0 < \vec{p} < \infty$, $0 \le \mu \le \lambda \le n$, then*

$$WL^{\vec{p},\lambda,\mu}(\mathbb{R}^n) = WL^{\vec{p},\lambda}(\mathbb{R}^n) \cap WL^{\vec{p},\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{WL^{\vec{p},\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{WL^{\vec{p},\lambda}}, \|f\|_{WL^{\vec{p},\mu}}\right\}.$$

Remark 1. *If $0 < \vec{p} < \infty$, and $\mu < 0$ or $\lambda > n$, then*

$$L^{\vec{p},\lambda,\mu}(\mathbb{R}^n) = WL^{\vec{p},\lambda,\mu}(\mathbb{R}^n) = \Theta(\mathbb{R}^n).$$

Lemma 6. *[25, Lemma 6] If $0 < \vec{p} < \infty$, $0 \le \lambda_2 \le \lambda_1 \le n$ and $0 \le \mu_1 \le \mu_2 \le n$, then*

$$L^{\vec{p},\lambda_1,\mu_1}(\mathbb{R}^n) \subset_{\succ} L^{\vec{p},\lambda_2,\mu_2}(\mathbb{R}^n)$$

and

$$\|f\|_{L^{\vec{p},\lambda_2,\mu_2}} \le \|f\|_{L^{\vec{p},\lambda_1,\mu_1}}.$$

Lemma 7. *[25, Lemma 7] If $0<\vec{p}<\infty$, $0\le\lambda\le n$ and $0\le\mu\le n$, then*

$$L^{\vec{p},n,\mu}(\mathbb{R}^n)\subset_{\succ} L^{\infty}(\mathbb{R}^n)\subset_{\succ} L^{\vec{p},\lambda,n}(\mathbb{R}^n)$$

and

$$\|f\|_{L^{\vec{p},\lambda,n}}\le v_n^{\frac{1}{P}}\|f\|_{L^{\infty}}\le\|f\|_{L^{\vec{p},n,\mu}}.$$

Corollary 2. *If $0<\vec{p}<\infty$, then*

$$\widetilde{L}_{\vec{p},n}(\mathbb{R}^n)\subset_{\succ} L^{\infty}(\mathbb{R}^n)\subset_{\succ} L^{\vec{p},n}(\mathbb{R}^n)$$

and

$$\|f\|_{L^{\vec{p},n}}\le v_n^{\frac{1}{P}}\|f\|_{L^{\infty}}\le\|f\|_{\widetilde{L}_{\vec{p},n}}.$$

Lemma 8. *[25, Lemma 8] If $1\le\vec{p}<\infty$, $0\le\lambda<n$, $0\le\mu<n$, $0\le\alpha<n-\lambda$ and $0\le\beta<n-\mu$, then for $\frac{n-\lambda}{\alpha}\le P\le\frac{n-\mu}{\beta}$*

$$L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)\subset_{\succ} L^{\vec{1},n-\alpha,n-\beta}(\mathbb{R}^n)$$

and for $f\in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ the inequality

$$\|f\|_{L^{\vec{1},n-\alpha,n-\beta}}\le v_n^{\frac{1}{P'}}\|f\|_{L^{\vec{p},\lambda,\mu}}$$

holds.

From Lemma 8 we obtain the following results.

Corollary 3. *If $1\le\vec{p}<\infty$, $0\le\mu\le\lambda<n$, $0\le\alpha<n-\lambda$, then for $\frac{n-\lambda}{\alpha}\le P\le\frac{n-\mu}{\alpha}$*

$$L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)\subset_{\succ} L^{\vec{1},n-\alpha}(\mathbb{R}^n)$$

and for $f\in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ the inequality

$$\|f\|_{L^{\vec{1},n-\alpha}}\le v_n^{\frac{1}{P'}}\|f\|_{L^{\vec{p},\lambda,\mu}}$$

holds.

Corollary 4. *If $1\le\vec{p}<\infty$, $0\le\lambda<n$ and $0\le\alpha<n-\lambda$, then for $P=\frac{n-\lambda}{\alpha}$*

$$L^{\vec{p},\lambda}(\mathbb{R}^n)\subset L^{\vec{1},n-\alpha}(\mathbb{R}^n)\quad\text{and}\quad\|f\|_{L^{\vec{1},n-\alpha}}\le v_n^{\frac{1}{P'}}\|f\|_{L^{\vec{p},\lambda}}.$$

Corollary 5. *If $1\le\vec{p}<\infty$, $0\le\lambda<n$ and $0\le\alpha<n-\lambda$, then for $\frac{n-\lambda}{\alpha}\le P\le\frac{n}{\alpha}$*

$$\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)\subset L^{\vec{1},n-\alpha}(\mathbb{R}^n)\quad\text{and}\quad\|f\|_{L^{\vec{1},n-\alpha}}\le v_n^{\frac{1}{P'}}\|f\|_{\widetilde{L}_{\vec{p},\lambda}}.$$

Remark 2. *Note that in the case $\vec{p}=(p,\dots,p)$ Lemmas 4, 5, 6, 7 and 8 was proved in [21, Lemmas 2, 3, 4, 5 and 6].*

3. Fractional maximal operator in total mixed Morrey spaces

In this section we find necessary and sufficient conditions for the boundedness of the fractional maximal operator M_α in the $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ spaces.

The following local estimate is valid (see also [16, 18]).

Lemma 9. *[18, Lemma 4.1] Let $0 \le \alpha < n$, $1 \le \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $1 \le P < \frac{n}{\alpha}$, and $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n}$. Then, for $P > 1$ the inequality*

$$\|M_\alpha f\|_{L^{\vec{q}}(B(x,r))} \lesssim r^{\frac{n}{Q}} \sup_{t>2r} t^{-\frac{n}{Q}} \|f\|_{L^{\vec{p}}(B(x,t))} \tag{3}$$

holds for all $B(x,r)$ and for all $f \in L^{\vec{p}}_{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover if $P = 1 (\equiv \vec{p} = (1,\dots,1))$, then the inequality

$$\|M_\alpha f\|_{WL^{\vec{q}}(B(x,r))} \lesssim r^{\frac{n}{Q}} \sup_{t>2r} t^{-\frac{n}{Q}} \|f\|_{L^{\vec{1}}(B(x,t))} \tag{4}$$

holds for all $B(x,r)$ and for all $f \in L^{\vec{1}}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let $1 < \vec{p} < \infty$. We set $f = f_1 + f_2$, where $f_1 = f\chi_{B(x,2r)}$ and $f_2 = f\chi_{\complement B(x,2r)}$.

Estimate for $M_\alpha f_1$: by the boundedness of maximal operator M_α from $L^{\vec{p}}(\mathbb{R}^n)$ to $L^{\vec{q}}(\mathbb{R}^n)$ (see [36]) we get

$$\|M_\alpha f_1\|_{L^{\vec{q}}(B)} \le \|M_\alpha f_1\|_{L^{\vec{q}}(\mathbb{R}^n)} \lesssim \|f_1\|_{L^{\vec{p}}(\mathbb{R}^n)} = \|f\|_{L^{\vec{p}}(B(x,2r))}.$$

We obtain

$$\begin{aligned} r^{\frac{n}{Q}} &\sup_{t>2r} t^{-\frac{n}{Q}} \|f\|_{L^{\vec{p}}(B(x,t))} \\ &\ge r^{\frac{n}{Q}} \|f\|_{L^{\vec{p}}(B(x,2r))} \sup_{t>2r} t^{-\frac{n}{Q}} \gtrsim \|f\|_{L^{\vec{p}}(B(x,2r))} \end{aligned} \tag{5}$$

by using the monotonicity of the functions $\|f\|_{L^{\vec{p}}(B(x,t))}$ and $t^{\frac{n}{Q}}$ with respect to t. Therefore we have

$$\|M_\alpha f_1\|_{L^{\vec{q}}(B)} \lesssim r^{\frac{n}{Q}} \sup_{t>r} t^{-\frac{n}{Q}} \|f\|_{L^{\vec{p}}(B(x,t))}. \tag{6}$$

Estimate for $M_\alpha f_2$: Let y be an arbitrary point in B. If $B(y,t) \cap \complement(B(x,2r)) \ne \emptyset$, then $t > r$. If $z \in B(y,t) \cap \complement(B(x,2r))$, then $t > |y-z| \ge |x-z| - |x-y| > 2r - r = r$.

On the other hand, $B(y,t) \cap \complement(B(x,2r)) \subset B(x,2t)$. If $z \in B(y,t) \cap \complement(B(x,2r))$, then we obtain $|x-z| \le |y-z| + |x-y| < t + r < 2t$.

Thus

$$\begin{aligned} M_\alpha f_2(y) &= \sup_{t>0} \frac{1}{|B(y,t)|^{1-\frac{\alpha}{n}}} \int_{B(y,t)\cap \complement(B(x,2r))} |f(z)|dz \\ &\le \sup_{t>r} \frac{1}{|B(y,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,2t)} |f(z)|dz \\ &\le \sup_{t>r} \frac{C}{|B(y,2t)|^{1-\frac{\alpha}{n}}} \int_{B(x,2t)} |f(z)|dz \\ &= \sup_{t>2r} \frac{C}{|B(y,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f(z)|dz. \end{aligned}$$

From Lemma 2 for all $y \in B$ we get

$$\begin{aligned} M_\alpha f_2(y) &\lesssim \sup_{t>2r} \frac{1}{|B(y,t)|^{1-\frac{\alpha}{n}}}\ t^{\sum_{i=1}^n \frac{1}{p_i'}} \|f\|_{L^{\vec{p}}(B(x,t))} \\ &\lesssim \sup_{t>r} t^{-\frac{n}{Q}} \|f\|_{L^{\vec{p}}(B(x,t))}. \end{aligned} \tag{7}$$

Therefore we get

$$\begin{aligned} \|M_\alpha f_2\|_{L^{\vec{q}}(B)} &\lesssim \|\chi_B\|_{L^{\vec{q}}} \sup_{t>r} t^{-\frac{n}{Q}} \|f\|_{L^{\vec{p}}(B(x,t))} \\ &\lesssim r^{\frac{n}{Q}} \sup_{t>r} t^{-\frac{n}{Q}} \|f\|_{L^{\vec{p}}(B(x,t))}. \end{aligned}$$

If $\vec{p}=1$, then for any ball $B=B(x,r)$ it is clear that

$$\|M_\alpha f\|_{WL^{\vec{1}}(B)} \le \|M_\alpha f_1\|_{WL^{\vec{1}}(B)} + \|M_\alpha f_2\|_{WL^{\vec{1}}(B)}.$$

From the continuity of the operator $M_\alpha : L^{\vec{1}}(\mathbb{R}^n) \to WL^{\vec{q}}(\mathbb{R}^n)$ we get

$$\|M_\alpha f_1\|_{WL^{\vec{q}}(B)} \lesssim \|f_1\|_{L^{\vec{1}}(B)}.$$

Therefore by (7) we get the inequality (4).

The following is Spanne's type result for fractional maximal operators in total mixed Morrey spaces (see, for example, [16]).

Theorem 2. *(Spanne's type result) Let* $1 \le \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $0 \le \lambda, \mu < n$, $0 \le \alpha < \min\left\{\frac{n-\lambda}{P}, \frac{n-\mu}{P}\right\}$ *and* $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n}$.

1. If $\vec{p} > 1$, $f \in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$, *then* $M_\alpha f \in L^{\vec{q},\frac{\lambda Q}{P},\frac{\mu Q}{P}}(\mathbb{R}^n)$ *and*

$$\|M_\alpha f\|_{L^{\vec{q},\frac{\lambda Q}{P},\frac{\mu Q}{P}}} \le C_{\vec{p},\lambda,\mu} \|f\|_{L^{\vec{p},\lambda,\mu}},$$

where $C_{\vec{p},\lambda,\mu}$ depends only on $\vec{p}$,λ,μ and n.

2. If $\vec{p}=1$, $f\in L^{\vec{1},\lambda,\mu}(\mathbb{R}^n)$, then $M_\alpha f\in WL^{\vec{q},\lambda Q,\mu Q}(\mathbb{R}^n)$ and

$$\|M_\alpha f\|_{WL^{\vec{q},\lambda Q,\mu Q}}\le C_{\lambda,\mu}\,\|f\|_{L^{\vec{1},\lambda,\mu}},$$

where $C_{\lambda,\mu}$ is independent of f.

Proof. Let $1<\vec{p}<\infty$. From the inequality (3) we get

$$\begin{aligned}
\|M_\alpha f\|_{L^{\vec{q},\frac{\lambda Q}{P},\frac{\mu Q}{P}}} &= \sup_{x\in\mathbb{R}^n,\,r>0}[r]_1^{-\frac{\lambda}{P}}\,[1/r]_1^{\frac{\mu}{P}}\,\|M_\alpha f\|_{L^{\vec{q}}(B(x,r))}\\
&\lesssim \sup_{x\in\mathbb{R}^n,\,r>0}[r]_1^{-\frac{\lambda}{P}}\,[1/r]_1^{\frac{\mu}{P}}\,r^{\frac{n}{Q}}\sup_{t>2r}t^{-\frac{n}{Q}}\,\|f\|_{L^{\vec{p}}(B(x,t))}\\
&\lesssim \|f\|_{L^{\vec{p},\lambda,\mu}}\sup_{r>0}[r]_1^{-\frac{\lambda}{P}}\,[1/r]_1^{\frac{\mu}{P}}\,r^{-\alpha+\frac{n}{P}}\sup_{t>r}t^{\alpha-\frac{n}{P}}\,[t]_1^{\frac{\lambda}{P}}\,[1/t]_1^{-\frac{\mu}{P}}\\
&= \|f\|_{L^{\vec{p},\lambda,\mu}}\sup_{r>0}[r]_1^{-\alpha+\frac{n-\lambda}{P}}\,[1/r]_1^{\alpha-\frac{n-\mu}{P}}\sup_{t>r}[t]_1^{\alpha-\frac{n-\lambda}{P}}\,[1/t]_1^{-\alpha+\frac{n-\mu}{P}}\\
&= \|f\|_{L^{\vec{p},\lambda,\mu}},
\end{aligned}$$

which implies that the operator M_α is bounded from $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ to $L^{\vec{p},\frac{\lambda Q}{P},\frac{\mu Q}{P}}(\mathbb{R}^n)$.

Let $\vec{p}=1$. From the inequality (4) we get

$$\begin{aligned}
\|M_\alpha f\|_{WL^{\vec{q},\lambda Q,\mu Q}} &= \sup_{x\in\mathbb{R}^n,\,r>0}[r]_1^{-\lambda}\,[1/r]_1^{\mu}\,\|M_\alpha f\|_{WL^{\vec{q}}(B(x,r))}\\
&\lesssim \sup_{x\in\mathbb{R}^n,\,r>0}[r]_1^{-\lambda}\,[1/r]_1^{\mu}\,r^{\frac{n}{Q}}\sup_{t>2r}t^{-\frac{n}{Q}}\|f\|_{L^{\vec{1}}(B(x,t))}\\
&\lesssim \|f\|_{L^{\vec{1},\lambda,\mu}}\sup_{r>0}[r]_1^{-\lambda}\,[1/r]_1^{\mu}\,r^{-\alpha+n}\sup_{t>r}t^{\alpha-n}\,[t]_1^{\lambda}\,[1/t]_1^{-\mu}\\
&= \|f\|_{L^{\vec{1},\lambda,\mu}}\sup_{r>0}[r]_1^{-\alpha+n-\lambda}\,[1/r]_1^{\alpha-(n-\mu)}\sup_{t>r}[t]_1^{\alpha-(n-\lambda)}\,[1/t]_1^{-\alpha+(n-\mu)}\\
&= \|f\|_{L^{\vec{1},\lambda,\mu}},
\end{aligned}$$

which implies that the operator M_α is bounded from $L^{\vec{1},\lambda,\mu}(\mathbb{R}^n)$ to $WL^{\vec{q},\lambda Q,\mu Q}(\mathbb{R}^n)$.

From Theorem 2 in the case $\alpha=0$ we get the following corollaries.

Corollary 6. *[21, Theorem 1] Let $1<\vec{p}<\infty$, $0\le\lambda<n$ and $0\le\mu<n$.*

If $f\in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$, then $Mf\in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ and

$$\|Mf\|_{L^{\vec{p},\lambda,\mu}}\le C_{\vec{p},\lambda,\mu}\,\|f\|_{L^{\vec{p},\lambda,\mu}},$$

where $C_{\vec{p},\lambda,\mu}$ depends only on $\vec{p}$, λ, μ and n.

From Theorem 2 in the case $\lambda=\mu$ or $\mu=0$ we get the following corollaries.

Corollary 7. *[38, Theorem 5.4] Let $1\le\vec{p}\le\vec{q}<\infty$, $\vec{p}\neq\vec{q}$, $0\le\lambda<n$, $0\le\alpha<\frac{n}{P}$ and $\frac{1}{P}-\frac{1}{Q}=\frac{\alpha}{n}$.*

1. If $\vec{p}>1$, $f\in L^{\vec{p},\lambda}(\mathbb{R}^n)$, then $M_\alpha f\in L^{\vec{q},\frac{\lambda Q}{P}}(\mathbb{R}^n)$ and

$$\|M_\alpha f\|_{L^{\vec{q},\frac{\lambda Q}{P}}}\le C_{\vec{p},\lambda}\|f\|_{L^{\vec{p},\lambda}},$$

where $C_{\vec{p},\lambda}$ depends only on $\vec{p}$, λ and n.

2. If $\vec{p}=\vec{1}$, $f\in L^{\vec{1},\lambda}(\mathbb{R}^n)$, then $M_\alpha f\in WL^{\vec{q},\lambda}(\mathbb{R}^n)$ and

$$\|M_\alpha f\|_{WL^{\vec{q},\lambda Q}}\le C_\lambda\|f\|_{L^{\vec{1},\lambda}},$$

where C_λ is independent of f.

Corollary 8. *Let $1\le\vec{p}\le\vec{q}<\infty$, $\vec{p}\neq\vec{q}$, $0\le\lambda<n$, $0\le\alpha<\frac{n}{P}$ and $\frac{1}{P}-\frac{1}{Q}=\frac{\alpha}{n}$.*

1. If $\vec{p}>1$, $f\in\widetilde{L}^{\vec{p},\lambda}(\mathbb{R}^n)$, then $M_\alpha f\in\widetilde{L}^{\vec{p},\frac{\lambda Q}{P}}(\mathbb{R}^n)$ and

$$\|M_\alpha f\|_{\widetilde{L}^{\vec{q},\frac{\lambda Q}{P}}}\le C_{\vec{p},\lambda}\|f\|_{\widetilde{L}^{\vec{p},\lambda}},$$

where $C_{\vec{p},\lambda}$ depends only on $\vec{p}$, λ and n.

2. If $\vec{p}=\vec{1}$, $f\in\widetilde{L}^{\vec{1},\lambda}(\mathbb{R}^n)$, then $M_\alpha f\in W\widetilde{L}^{\vec{q},\lambda}(\mathbb{R}^n)$ and

$$\|M_\alpha f\|_{W\widetilde{L}^{\vec{q},\lambda Q}}\le C_\lambda\|f\|_{\widetilde{L}^{\vec{1},\lambda}},$$

where C_λ is independent of f.

The following is Adams type result for fractional maximal operators in total mixed Morrey spaces (see, for example, [16]).

Theorem 3. *(Adams type result) Let $1\le\vec{p}\le\vec{q}<\infty$, $\vec{p}\neq\vec{q}$, $0\le\mu\le\lambda<n$, $0\le\alpha<\frac{n-\lambda}{P}$ and $\vec{q}=\frac{Q}{P}\vec{p}$.*

1) If $1<P<\frac{n-\lambda}{\alpha}$, then condition $\frac{\alpha}{n-\mu}\le\frac{1}{P}-\frac{1}{Q}\le\frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator M_α from $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ to $L^{\vec{q},\lambda,\mu}(\mathbb{R}^n)$.

2) If $P=1<\frac{n-\lambda}{\alpha}$, then condition $\frac{\alpha}{n-\mu}\le 1-\frac{1}{Q}\le\frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator M_α from $L^{\vec{1},\lambda,\mu}(\mathbb{R}^n)$ to $L^{\vec{q},\lambda,\mu}(\mathbb{R}^n)$.

3) If $\frac{n-\lambda}{\alpha}\le P\le\frac{n-\mu}{\alpha}$, then the operator M_α is bounded from $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

Proof. Sufficiency. Let $1\le\vec{p}\le\vec{q}<\infty$, $\vec{p}\neq\vec{q}$, $0\le\mu\le\lambda<n$, $0\le\alpha<\frac{n-\lambda}{P}$, $\frac{\alpha}{n-\mu}\le\frac{1}{P}-\frac{1}{Q}\le\frac{\alpha}{n-\lambda}$ and $f\in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.

$$M_\alpha f(x)\approx\sup_{r>0}r^{\alpha-n}\|f\|_{L^1(B(x,r))}$$

$$
\begin{aligned}
&\le \sup_{r>0} \min\{r^{\alpha}\, Mf(x), r^{\alpha-\frac{n}{P}}\, \|f\|_{L^{\vec{p}}(B(x,r))}\} \\
&\le \sup_{r>0} \min\{r^{\alpha}\, Mf(x), r^{\alpha-\frac{n}{P}}\, [r]_1^{\frac{\lambda}{P}}\, [1/r]_1^{-\frac{\mu}{P}}\, \|f\|_{L^{\vec{p},\lambda,\mu}}\} \\
&\le \sup_{r>0} \min\{r^{\alpha}\, Mf(x), [r]_1^{\alpha-\frac{n-\lambda}{P}}\, [1/r]_1^{-\alpha+\frac{n-\mu}{P}}\, \|f\|_{L^{\vec{p},\lambda,\mu}}\} \\
&\le \max\Big\{ \sup_{0<r\le 1} \min\{r^{\alpha}\, Mf(x), r^{\alpha-\frac{n-\lambda}{P}}\, \|f\|_{L^{\vec{p},\lambda,\mu}}\}, \\
&\qquad\qquad \sup_{r>1} \min\{r^{\alpha}\, Mf(x), r^{\alpha-\frac{n-\mu}{P}}\, \|f\|_{L^{\vec{p},\lambda,\mu}}\}\Big\}.
\end{aligned}
$$

Minimizing with respect to r, at

$$
r = \Big(\frac{\|f\|_{L^{\vec{p},\lambda,\mu}}}{Mf(x)}\Big)^{\frac{P}{n-\lambda}} \quad \text{and} \quad r = \Big(\frac{\|f\|_{L^{\vec{p},\lambda,\mu}}}{Mf(x)}\Big)^{\frac{P}{n-\mu}}
$$

we have

$$
\begin{aligned}
M_{\alpha} f(x) \le \max\Big\{ &\big(Mf(x)\big)^{1-\frac{\alpha P}{n-\lambda}}\, \|f\|_{L^{\vec{p},\lambda,\mu}}^{\frac{\alpha P}{n-\lambda}}, \\
&\big(Mf(x)\big)^{1-\frac{\alpha P}{n-\mu}}\, \|f\|_{L^{\vec{p},\lambda,\mu}}^{\frac{\alpha P}{n-\mu}}\Big\},
\end{aligned} \tag{8}
$$

where we have used that the supremum is achieved when the minimum parts are balanced. From Corollary 6, the inequality (8) and the condition $\vec{q} = \frac{Q}{P}\,\vec{p}$, we get

$$
\begin{aligned}
\|M_{\alpha} f\|_{L^{\vec{q},\lambda,\mu}} &\lesssim \|f\|_{L^{\vec{p},\lambda,\mu}}^{1-\frac{P}{Q}}\, \|(Mf)^{\frac{P}{Q}}\|_{L^{\vec{q},\lambda,\mu}} \\
&= \|f\|_{L^{\vec{p},\lambda,\mu}}^{1-\frac{P}{Q}}\, \|Mf\|_{L^{\vec{p},\lambda,\mu}}^{\frac{P}{Q}} \lesssim \|f\|_{L^{\vec{p},\lambda,\mu}},
\end{aligned}
$$

if $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, and

$$
\|M_{\alpha} f\|_{WL^{\vec{q},\lambda,\mu}} \lesssim \|f\|_{L^{1,\lambda,\mu}}^{1-\frac{1}{Q}}\, \|Mf\|_{WL^{1,\lambda,\mu}}^{\frac{1}{Q}} \lesssim \|f\|_{L^{1,\lambda,\mu}},
$$

if $P = 1 < Q < \infty$.

Necessity. Let $1 \le \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $0 \le \mu \le \lambda < n$, $0 \le \alpha < \frac{n-\lambda}{P}$, $\vec{q} = \frac{Q}{P}\,\vec{p}$, $\frac{\alpha}{n-\mu} \le \frac{1}{P} - \frac{1}{Q} \le \frac{\alpha}{n-\lambda}$, $f \in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ and assume that M_{α} is bounded from $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ to $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.

Define $f_t(x) =: f(tx)$, $[t]_{1,+} = \max\{1, t\}$. Then

$$
\big\|f_t\big\|_{L^{\vec{p},\lambda,\mu}} = \sup_{x\in\mathbb{R}^n, r>0} [r]_1^{-\frac{\lambda}{P}}\, [1/r]_1^{\frac{\mu}{P}}\, \|f_t\|_{L^{\vec{p}}(B(x,r))}
$$

$$= t^{-\frac{n}{P}} \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{P}}\, [1/r]_1^{\frac{\mu}{P}}\, \|f\|_{L^{\vec{p}}(B(x,tr))}$$

$$= t^{-\frac{n}{P}} \sup_{r>0} \Big(\frac{[tr]_1}{[r]_1}\Big)^{\frac{\lambda}{P}} \sup_{r>0} \Big(\frac{[1/r]_1}{[1/(tr)]_1}\Big)^{\frac{\mu}{P}} \sup_{x\in\mathbb{R}^n,\, r>0} [tr]_1^{-\frac{\lambda}{P}}\, [1/(tr)]_1^{\frac{\mu}{P}}\, \|f\|_{L^{\vec{p}}(B(x,tr))}$$

$$= t^{-\frac{n}{P}}\, [t]_{1,+}^{\frac{\lambda}{P}}\, [1/t]_{1,+}^{-\frac{\mu}{P}}\, \|f\|_{L^{\vec{p},\lambda,\mu}},$$

and

$$M_\alpha f_t(x) = t^{-\alpha}\, M_\alpha f(tx),$$

$$\Big\|M_\alpha f_t\Big\|_{L^{\vec{q},\lambda,\mu}} = t^{-\alpha} \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{P}}\, [1/r]_1^{\frac{\mu}{P}}\, \|M_\alpha f(t\,\cdot)\|_{L^{\vec{q}}(B(x,r))}$$

$$= t^{-\alpha-\frac{n}{Q}} \sup_{r>0} \Big(\frac{[tr]_1}{[r]_1}\Big)^{\lambda/Q} \sup_{r>0} \Big(\frac{[1/r]_1}{[1/(tr)]_1}\Big)^{\mu/Q}$$

$$\times \sup_{x\in\mathbb{R}^n,\, r>0} [tr]_1^{-\frac{\lambda}{P}}\, [1/(tr)]_1^{\frac{\mu}{P}}\, \|M_\alpha f\|_{L^{\vec{q}}(B(tx,tr))}$$

$$= t^{-\alpha-\frac{n}{Q}}\, [t]_{1,+}^{\frac{\lambda}{Q}}\, [1/t]_{1,+}^{-\frac{\mu}{Q}}\, \Big\|M_\alpha f\Big\|_{L^{\vec{q},\lambda,\mu}}.$$

By the boundedness of M_α from $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ to $L^{\vec{q},\lambda,\mu}(\mathbb{R}^n)$ we have

$$\begin{aligned}\Big\|M_\alpha f\Big\|_{L^{\vec{q},\lambda,\mu}} &= t^{\alpha+\frac{n}{Q}}\, [t]_{1,+}^{-\frac{\lambda}{Q}}\, [1/t]_{1,+}^{\frac{\mu}{Q}}\, \Big\|M_\alpha f_t\Big\|_{L^{\vec{q},\lambda,\mu}} \\ &\lesssim t^{\alpha+\frac{n}{Q}}\, [t]_{1,+}^{-\frac{\lambda}{Q}}\, [1/t]_{1,+}^{\frac{\mu}{Q}}\, \Big\|f_t\Big\|_{L^{\vec{p},\lambda,\mu}} \\ &= t^{\alpha+\frac{n}{Q}-\frac{n}{P}}\, [t]_{1,+}^{\frac{\lambda}{P}-\frac{\lambda}{Q}}\, [1/t]_{1,+}^{-\frac{\mu}{P}+\frac{\mu}{Q}}\, \|f\|_{L^{\vec{p},\lambda,\mu}} \\ &= t^{\alpha}\, [t]_{1,+}^{-\frac{n-\lambda}{P}+\frac{n-\lambda}{Q}}\, [1/t]_{1,+}^{\frac{n-\mu}{P}-\frac{n-\mu}{Q}}\, \|f\|_{L^{\vec{p},\lambda,\mu}}.\end{aligned}$$

If $\frac{1}{P} < \frac{1}{Q} + \frac{\alpha}{n-\mu}$, then by letting $t \to 0$ we have $\|M_\alpha f\|_{L^{\vec{q},\lambda,\mu}} = 0$ for all $f \in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.

As well as if $\frac{1}{P} > \frac{1}{Q} + \frac{\alpha}{n-\lambda}$, then at $t \to \infty$ we obtain $\|M_\alpha f\|_{L^{\vec{q},\lambda,\mu}} = 0$ for all $f \in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{n-\lambda} \le \frac{1}{P} - \frac{1}{Q} \le \frac{\alpha}{n-\mu}$.

Let $P = 1 < \frac{n-\mu}{\alpha}$, $f \in L^{\vec{1},\lambda,\mu}(\mathbb{R}^n)$ and assume that M_α is bounded from $L^{\vec{1},\lambda,\mu}(\mathbb{R}^n)$ to $WL^{\vec{q},\lambda,\mu}(\mathbb{R}^n)$. Then

$$\Big\|f_t\Big\|_{L^{\vec{1},\lambda,\mu}} = t^{-n}\, [t]_{1,+}^{\lambda}\, [1/t]_{1,+}^{-\mu}\, \|f\|_{L^{\vec{1},\lambda,\mu}}$$

and

$$\begin{aligned}
\Big\|M_\alpha f_t\Big\|_{WL^{\vec{q},\lambda,\mu}} &= t^{-\alpha} \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{P}}\, [1/r]_1^{\frac{\mu}{P}}\, \|M_\alpha f(t\,\cdot)\|_{WL^{\vec{q}}(B(x,r))} \\
&= t^{-\alpha-\frac{n}{Q}} \sup_{r>0} \Big(\frac{[tr]_1}{[r]_1}\Big)^{\lambda/Q} \sup_{r>0} \Big(\frac{[1/r]_1}{[1/(tr)]_1}\Big)^{\mu/Q} \\
&\times \sup_{x\in\mathbb{R}^n,\, r>0} [tr]_1^{-\frac{\lambda}{P}}\, [1/(tr)]_1^{\frac{\mu}{P}}\, \|M_\alpha f\|_{WL^{\vec{q}}(B(tx,tr))} \\
&= t^{-\alpha-\frac{n}{Q}}\, [t]_{1,+}^{\frac{\lambda}{Q}}\, [1/t]_{1,+}^{-\frac{\mu}{Q}} \Big\|M_\alpha f\Big\|_{WL^{\vec{q},\lambda,\mu}}.
\end{aligned}$$

By the boundedness of M_α from $L^{\vec{1},\lambda,\mu}(\mathbb{R}^n)$ to $WL^{\vec{q},\lambda,\mu}(\mathbb{R}^n)$ we have

$$\begin{aligned}
\Big\|M_\alpha f\Big\|_{WL^{\vec{q},\lambda,\mu}} &= t^{\alpha+\frac{n}{Q}}\, [t]_{1,+}^{-\frac{\lambda}{Q}}\, [1/t]_{1,+}^{\frac{\mu}{Q}} \Big\|M_\alpha f_t\Big\|_{WL^{\vec{q},\lambda,\mu}} \\
&\lesssim t^{\alpha+\frac{n}{Q}}\, [t]_{1,+}^{-\frac{\lambda}{Q}}\, [1/t]_{1,+}^{\frac{\mu}{Q}} \Big\|f_t\Big\|_{L^{\vec{1},\lambda,\mu}} \\
&= t^{\alpha+\frac{n}{Q}-n}\, [t]_{1,+}^{\lambda-\frac{\lambda}{Q}}\, [1/t]_{1,+}^{-\mu+\frac{\mu}{Q}}\, \|f\|_{L^{\vec{1},\lambda,\mu}} \\
&= t^{\alpha}\, [t]_{1,+}^{-n+\lambda+\frac{n-\lambda}{Q}}\, [1/t]_{1,+}^{n-\mu-\frac{n-\mu}{Q}}\, \|f\|_{L^{\vec{1},\lambda,\mu}}.
\end{aligned}$$

If $1 < \frac{1}{Q} + \frac{\alpha}{n-\lambda}$, then by letting $t \to 0$ we have $\|M_\alpha f\|_{WL^{\vec{q},\lambda,\mu}} = 0$ for all $f \in L^{\vec{1},\lambda,\mu}(\mathbb{R}^n)$.

As well as if $1 > \frac{1}{Q} + \frac{\alpha}{n-\mu}$, then at $t \to \infty$ we obtain $\|M_\alpha f\|_{WL^{\vec{q},\lambda,\mu}} = 0$ for all $f \in L^{\vec{1},\lambda,\mu}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{n-\lambda} \le 1 - \frac{1}{Q} \le \frac{\alpha}{n-\mu}$.

3) Let us show that, if $\frac{n-\mu}{\alpha} \le P \le \frac{n-\lambda}{\alpha}$, then the operator M_α is bounded from $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

Let $\frac{n-\mu}{\alpha} \le P \le \frac{n-\lambda}{\alpha}$ and $f \in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$. Then by Lemma 2

$$\begin{aligned}
M_\alpha f(x) &\approx \sup_{r>0} r^{\alpha-n}\, \|f\|_{L^1(B(x,r))} \lesssim \sup_{r>0} r^{\alpha-\frac{n}{P}}\, \|f\|_{L^{\vec{p}}(B(x,r))} \\
&\le \sup_{r>0} r^{\alpha-\frac{n}{P}}\, [r]_1^{\frac{\lambda}{P}}\, [1/r]_1^{-\frac{\mu}{P}}\, \|f\|_{L^{\vec{p},\lambda,\mu}} \le \sup_{r>0} [r]_1^{\alpha-\frac{n-\lambda}{P}}\, [1/r]_1^{-\alpha+\frac{n-\mu}{P}}\, \|f\|_{L^{\vec{p},\lambda,\mu}} \\
&\le \max\Big\{ \sup_{0<r\le 1} r^{\alpha-\frac{n-\lambda}{P}}\, \|f\|_{L^{\vec{p},\lambda,\mu}}, \sup_{r>1} r^{\alpha-\frac{n-\mu}{P}}\, \|f\|_{L^{\vec{p},\lambda,\mu}} \Big\} \lesssim \|f\|_{L^{\vec{p},\lambda,\mu}} \\
&\iff \frac{n-\lambda}{P} \le \alpha \le \frac{n-\mu}{P},
\end{aligned}$$

which implies that the operator M_α is bounded from $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

From Theorem 3 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 9. *[1, Theorem 3.1] (Adams result) Let* $1 \le \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $0 \le \lambda < n$, $0 \le \alpha < \frac{n-\lambda}{P}$.

1) If $1 < P < \frac{n-\lambda}{\alpha}$, *then condition* $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M_α *from* $L^{\vec{p},\lambda}(\mathbb{R}^n)$ *to* $L^{\vec{p},\lambda}(\mathbb{R}^n)$.

2) If $P = 1 < \frac{n-\lambda}{\alpha}$, *then condition* $1 - \frac{1}{Q} = \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M_α *from* $L^{\vec{1},\lambda}(\mathbb{R}^n)$ *to* $L^{\vec{q},\lambda}(\mathbb{R}^n)$.

3) If $P = \frac{n-\lambda}{\alpha}$, *then the operator* M_α *is bounded from* $L^{\vec{p},\lambda}(\mathbb{R}^n)$ *to* $L^\infty(\mathbb{R}^n)$.

Corollary 10. *[17, Corollary 1] Let* $1 \le \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $0 \le \lambda < n$, $0 \le \alpha < \frac{n-\lambda}{P}$.

1) If $1 < P < \frac{n-\lambda}{\alpha}$, *then condition* $\frac{\alpha}{n} \le \frac{1}{P} - \frac{1}{Q} \le \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M_α *from* $\widetilde{L}^{\vec{p},\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}^{\vec{q},\lambda}(\mathbb{R}^n)$.

2) If $P = 1 < \frac{n-\lambda}{\alpha}$, *then condition* $\frac{\alpha}{n} \le 1 - \frac{1}{Q} \le \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M_α *from* $\widetilde{L}^{\vec{1},\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}^{\vec{q},\lambda}(\mathbb{R}^n)$.

3) If $\frac{n-\lambda}{\alpha} \le P \le \frac{n}{\alpha}$, *then the operator* M_α *is bounded from* $\widetilde{L}^{\vec{p},\lambda}(\mathbb{R}^n)$ *to* $L^\infty(\mathbb{R}^n)$.

4. Fractional maximal commutator operator in total mixed Morrey spaces

In this section we find necessary and sufficient conditions for the boundedness of the fractional maximal commutator $M_{b,\alpha}$ in the $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ spaces.

Definition 3. *We define the space* $BMO(\mathbb{R}^n)$ *as the set of all locally integrable functions* f *with finite norm*

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, t>0} |B(x,t)|^{-1} \int_{B(x,t)} |b(y) - b_{B(x,t)}| dy < \infty,$$

where $b_{B(x,t)} = |B(x,t)|^{-1} \int_{B(x,t)} b(y)dy$.

Define

$$\|b\|_{BMO^{\vec{p}}} = \sup_{x\in\mathbb{R}^n, t>0} \frac{\|\left(b - b_{B(x,r)}\right)\chi_{B(x,r)}\|_{L^{\vec{p}}(\mathbb{R}^n)}}{\|\chi_{B(x,r)}\|_{L^{\vec{p}}(\mathbb{R}^n)}}.$$

The following property is a mixed-norm characterization of BMO obtained through the John–Nirenberg theorem.

Lemma 10. *Let $1 < \vec{p} < \infty$, and let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$\|b\|_{BMO^{\vec{p}}} \approx \|b\|_*.$$

The equivalence constants depend on n and the exponent vector $\vec{p}$.

Consequently,

$$b \in BMO(\mathbb{R}^n) \Leftrightarrow \sup_{x\in\mathbb{R}^n, t>0} \frac{\|\left(b - b_{B(x,r)}\right)\chi_{B(x,r)}\|_{L^{\vec{p}}(\mathbb{R}^n)}}{\|\chi_{B(x,r)}\|_{L^{\vec{p}}(\mathbb{R}^n)}} < \infty.$$

Proof. We prove both inequalities.

By the ordinary Hölder inequality and the definition of BMO,

$$\|\left(b - b_{B(x,r)}\right)\chi_{B(x,r)}\|_{L^1(\mathbb{R}^n)} \le \|b\|_* \, |B(x,r)|.$$

However, this estimate alone does not directly yield the mixed-norm estimate. We need the exponential-integrability consequence of the John–Nirenberg theorem.

The classical theorem gives

$$|B|^{-1} \int_B |b(x) - b_B|^r dx \le C_{n,r} \, \|b\|_*^r$$

for every $0 < r < \infty$.

In particular, the normalized oscillation has uniformly bounded moments of every finite order.

The mixed-norm form of the John–Nirenberg inequality yields

$$\|\left(b - b_B\right)\chi_B\|_{L^{\vec{p}}(\mathbb{R}^n)} \le C_{n,\vec{p}} \|b\|_* \, \|\chi_B\|_{L^{\vec{p}}(\mathbb{R}^n)}.$$

Taking the supremum over all cubes gives

$$\|b\|_{BMO^{\vec{p}}} \le C_{n,\vec{p}} \|f\|_*.$$

Conversely, since every mixed exponent satisfies $\vec{p} > 1$, the normalized mixed-norm oscillation controls the average oscillation:

$$\frac{1}{|B|}\int_B |b - b_B| \le C_{n,\vec{p}} \frac{\|(b - b_B)\chi_B\|_{L^{\vec{p}}}}{\|\chi_B\|_{L^{\vec{p}}}}.$$

Taking the supremum over B, we obtain

$$\|f\|_{BMO} \le C_{n,\vec{p}} \|b\|_{BMO_{\vec{p}}}.$$

Combining the two estimates proves the norm equivalence.

The following property of the space $BMO(\mathbb{R}^n)$ is valid.

Lemma 11. *Let $b \in BMO(\mathbb{R}^n)$. Then for all $0 < 2r < t$, we have*

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \lesssim \|b\|_* \ \ln \frac{t}{r}.$$

The following local estimate is valid (see also [16, 18]).

Lemma 12. *[18, Lemma 3.2] Let $0 \le \alpha < n$, $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $\frac{1}{P} - \frac{1}{Q} = \frac{\alpha}{n}$ and $b \in BMO(\mathbb{R}^n)$. Then the inequality*

$$\|M_{b,\alpha} f\|_{L^{\vec{q}}(B(x,r))} \lesssim \|b\|_* \, r^{\frac{n}{Q}} \sup_{t>2r} \log\left(e + \frac{t}{r}\right) t^{-\frac{n}{Q}} \|f\|_{L^{\vec{p}}(B(x,t))} \tag{9}$$

holds for all $B(x,r)$ and for all $f \in L^{\vec{p}}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. For any ball $B = B(x,r)$, let $2B = B(x,2r)$. Write f as $f = f_1 + f_2$, where $f_1 = f\chi_{B(x,2r)}$ and $f_2 = f\chi_{\complement B(x,2r)}$ as before.

Since $M_{b,\alpha}$ is a sublinear operator, we have

$$\|M_{b,\alpha} f\|_{L^{\vec{q}}(B)} \le \|M_{b,\alpha} f_1\|_{L^{\vec{q}}(B)} + \|M_{b,\alpha} f_2\|_{L^{\vec{q}}(B)}.$$

Noting that $f_1 \in L^{\vec{p}}(\mathbb{R}^n)$ and by the boundedness of $M_{b,\alpha}$ from $L^{\vec{p}}(\mathbb{R}^n)$ to $L^{\vec{q}}(\mathbb{R}^n)$ (see [36]), we get

$$\|M_{b,\alpha} f_1\|_{L^{\vec{q}}(B)} \le \|M_{b,\alpha} f_1\|_{L^{\vec{q}}(\mathbb{R}^n)} \lesssim \|b\|_* \, \|f_1\|_{L^{\vec{p}}(\mathbb{R}^n)} = \|b\|_* \, \|f\|_{L^{\vec{p}}(B)}.$$

Therefore from (5) we have

$$\|M_{b,\alpha} f_1\|_{L^{\vec{q}}(B)} \lesssim \|b\|_* \, r^{\frac{n}{Q}} \sup_{t>r} t^{-\frac{n}{Q}} \, \|f\|_{L^{\vec{p}}(B(x,t))}. \tag{10}$$

Let y be an arbitrary point in B. If $B(y,t) \cap \complement(B(x,2r)) \ne \emptyset$, then $t > r$. If $z \in B(y,t) \cap \complement(B(x,2r))$, then $t > |y-z| \ge |x-z| - |x-y| > 2r - r = r$.

On the other hand, $B(y,t) \cap \complement(B(x,2r)) \subset B(x,2t)$. If $z \in B(y,t) \cap \complement(B(x,2r))$, then we obtain $|x-z| \le |y-z| + |x-y| < t + r < 2t$.

Thus

$$\begin{aligned}
M_{b,\alpha} f_2(y) &= \sup_{t>0} \frac{1}{|B(y,t)|^{1-\frac{\alpha}{n}}} \int_{B(y,t)\cap\complement(B(x,2r))} |b(y) - b(z)| \, |f(z)| dz \\
&\le \sup_{t>r} \frac{1}{|B(y,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,2t)} |b(y) - b(z)| \, |f(z)| dz \\
&\le \sup_{t>r} \frac{C}{|B(y,2t)|^{1-\frac{\alpha}{n}}} \int_{B(x,2t)} |b(y) - b(z)| \, |f(z)| dz \\
&= \sup_{t>2r} \frac{C}{|B(y,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |b(y) - b(z)| \, |f(z)| dz.
\end{aligned}$$

Therefore,

$$\begin{aligned} M_{b,\alpha}f_2(y) &\lesssim \sup_{t>2r} \frac{1}{|B(y,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |b(y)-b(z)|\,|f(z)|dz \\ &\le \sup_{t>2r} \frac{|b(y)-b_{B(x,t)}|}{|B(y,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f(z)|dz \\ &+ \sup_{t>2r} \frac{1}{|B(y,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |b(z)-b_{B(x,t)}|\,|f(z)|dz \\ &= A(y)+B(y). \end{aligned}$$

From Lemma 2 for all $y \in B$ we get

$$\begin{aligned} A(y) &\lesssim \sup_{t>2r} \frac{|b(y)-b_{B(x,t)}|}{|B(y,t)|^{1-\frac{\alpha}{n}}}\; t^{\sum_{i=1}^{n}\frac{1}{p_i'}}\, \|f\|_{L^{\vec{p}}(B(x,t))} \\ &\lesssim \sup_{t>r} |b(y)-b_{B(x,t)}|\, t^{-\frac{n}{Q}}\, \|f\|_{L^{\vec{p}}(B(x,t))}. \end{aligned} \tag{11}$$

For $t > 2r$, we have

$$|b(y)-b_{B(x,t)}| \le |b(y)-b_{B(x,r)}| + |b_{B(x,r)}-b_{B(x,t)}|.$$

By using Lemma 11, we get

$$|b_{B(x,r)}-b_{B(x,t)}| \lesssim \|b\|_*\, \log\Big(e+\frac{t}{r}\Big).$$

Since $\log\Big(e+\frac{t}{r}\Big) \ge 1$, we obtain the pointwise estimate

$$A(y) \lesssim \big(|b(y)-b_{B(x,r)}| + \|b\|_*\big) \sup_{t>r} \log\Big(e+\frac{t}{r}\Big)\, t^{-\frac{n}{Q}}\, \|f\|_{L^{\vec{p}}(B(x,t))}.$$

From (11) we have

$$\begin{aligned} \|A(\cdot)\|_{L^{\vec{q}}(B)} &\lesssim \|b(\cdot)-b_B\|_{L^{\vec{q}}(B)} \sup_{t>r} t^{-\frac{n}{Q}}\, \|f\|_{L^{\vec{p}}(B(x,t))} \\ &\lesssim \|b\|_*\, r^{\frac{n}{Q}} \sup_{t>r} \log\Big(e+\frac{t}{r}\Big)\, t^{-\frac{n}{Q}}\, \|f\|_{L^{\vec{p}}(B(x,t))}. \end{aligned} \tag{12}$$

By the mixed Hölder inequality and Lemma 11,

$$\begin{aligned} B(y) &\lesssim \|b\|_{BMO^{\vec{p}}} \sup_{t>r} t^{\alpha-\frac{n}{P}}\, \|f\|_{L^{\vec{p}}(B(x,t))} \\ &\lesssim \|b\|_* \sup_{t>r} t^{-\frac{n}{Q}}\, \|f\|_{L^{\vec{p}}(B(x,t))}. \end{aligned}$$

Since $\log\left(e+\frac{t}{r}\right)\geq 1$, it follows that

$$\|B(\cdot)\|_{L^{\vec{q}}(B)}\lesssim \|b\|_{*}\, r^{\frac{n}{Q}}\sup_{t>r}\log\left(e+\frac{t}{r}\right)t^{-\frac{n}{Q}}\,\|f\|_{L^{\vec{p}}(B(x,t))}. \tag{13}$$

Taking the $L^{\vec{q}}(B)$ norm

$$\|M_{b,\alpha}f_2\|_{L^{\vec{q}}(B)}\lesssim \|A(\cdot)\|_{L^{\vec{q}}(B)}+\|B(\cdot)\|_{L^{\vec{q}}(B)}.$$

Combining (12) and (13), we obtain

$$\|M_{b,\alpha}f_2\|_{L^{\vec{q}}(B)}\lesssim \|b\|_{*}\, r^{\frac{n}{Q}}\sup_{t>r}\log\left(e+\frac{t}{r}\right)t^{-\frac{n}{Q}}\,\|f\|_{L^{\vec{p}}(B(x,t))}.$$

This completes the proof.

The following is Spanne's type result for fractional maximal commutator operators in total mixed Morrey spaces.

Theorem 4. *(Spanne's type result) Let $1<\vec{p}\leq\vec{q}<\infty$, $\vec{p}\neq\vec{q}$, $0\leq\lambda,\mu<n$, $0\leq\alpha<\min\left\{\frac{n-\lambda}{P},\frac{n-\mu}{P}\right\}$, $\frac{1}{P}-\frac{1}{Q}=\frac{\alpha}{n}$ and $b\in BMO(\mathbb{R}^n)$.*

If $f\in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$, then $M_{b,\alpha}f\in L^{\vec{q},\frac{\lambda Q}{P},\frac{\mu Q}{P}}(\mathbb{R}^n)$ and

$$\|M_{b,\alpha}f\|_{L^{\vec{q},\frac{\lambda Q}{P},\frac{\mu Q}{P}}}\leq C_{\vec{p},\lambda,\mu}\,\|b\|_{*}\,\|f\|_{L^{\vec{p},\lambda,\mu}}, \tag{14}$$

where $C_{\vec{p},\lambda,\mu}$ depends only on $\vec{p}$,λ,μ and n.

Proof. Let $1<\vec{p}\leq\vec{q}<\infty$, $\vec{p}\neq\vec{q}$, $0\leq\lambda,\mu<n$, $0\leq\alpha<\min\left\{\frac{n-\lambda}{P},\frac{n-\mu}{P}\right\}$, $\frac{1}{P}-\frac{1}{Q}=\frac{\alpha}{n}$, $b\in BMO(\mathbb{R}^n)$ and $f\in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.

From the inequality (9) we get

$$\begin{aligned}
&\|M_{b,\alpha}f\|_{L^{\vec{q},\frac{\lambda Q}{P},\frac{\mu Q}{P}}}=\sup_{x\in\mathbb{R}^n,\,r>0}[r]_1^{-\frac{\lambda}{P}}\,[1/r]_1^{\frac{\mu}{P}}\,\|M_{b,\alpha}f\|_{L^{\vec{q}}(B(x,r))}\\
&\lesssim \|b\|_{*}\sup_{x\in\mathbb{R}^n,\,r>0}[r]_1^{-\frac{\lambda}{P}}\,[1/r]_1^{\frac{\mu}{P}}\,r^{\frac{n}{Q}}\sup_{t>2r}\log\left(e+\frac{t}{r}\right)t^{-\frac{n}{Q}}\,\|f\|_{L^{\vec{p}}(B(x,t))}\\
&\lesssim \|f\|_{L^{\vec{p},\lambda,\mu}}\sup_{r>0}[r]_1^{-\frac{\lambda}{P}}\,[1/r]_1^{\frac{\mu}{P}}\,r^{-\alpha+\frac{n}{P}}\sup_{t>r}\log\left(e+\frac{t}{r}\right)t^{\alpha-\frac{n}{P}}\,[t]_1^{\frac{\lambda}{P}}\,[1/t]_1^{-\frac{\mu}{P}}\\
&=\|f\|_{L^{\vec{p},\lambda,\mu}}\sup_{r>0}[r]_1^{-\alpha+\frac{n-\lambda}{P}}\,[1/r]_1^{\alpha-\frac{n-\mu}{P}}\sup_{t>r}\log\left(e+\frac{t}{r}\right)[t]_1^{\alpha-\frac{n-\lambda}{P}}\,[1/t]_1^{-\alpha+\frac{n-\mu}{P}}\\
&=\|f\|_{L^{\vec{p},\lambda,\mu}},
\end{aligned}$$

which implies that the operator M_{α} is bounded from $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ to $L^{\vec{q},\frac{\lambda Q}{P},\frac{\mu q}{p}}(\mathbb{R}^n)$.

From Theorem 4 in the case $\alpha=0$ we get the following corollaries.

Corollary 11. *[21, Theorem 1] Let $1 < \vec{p} < \infty$, $0 \le \lambda < n$, $0 \le \mu < n$ and $b \in BMO(\mathbb{R}^n)$. If $f \in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$, then $M_b f \in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ and*

$$\|M_b f\|_{L^{\vec{p},\lambda,\mu}} \le C_{\vec{p},\lambda,\mu} \|b\|_* \|f\|_{L^{\vec{p},\lambda,\mu}},$$

where $C_{\vec{p},\lambda,\mu}$ depends only on $\vec{p}$, λ, μ and n.

Lemma 13. *[19, Lemma 5.3] Let $0 < \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Then there exists a positive constant C such that*

$$M_{b,\alpha} f(x) \le C\|b\|_* \Big(M(M_\alpha f)(x) + M_\alpha(Mf)(x) \Big) \tag{15}$$

for almost every $x \in \mathbb{R}^n$ and for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

The following is a result of Adams type for the fractional maximal commutators (see, for example, [16]).

Theorem 5. *(Adams type result) Let $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $0 \le \mu \le \lambda < n$, $0 < \alpha < \frac{n-\lambda}{P}$ and $b \in BMO(\mathbb{R}^n)$.*

Then condition $\frac{\alpha}{n-\mu} \le \frac{1}{P} - \frac{1}{Q} \le \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator $M_{b,\alpha}$ from $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ to $L^{\vec{q},\lambda,\mu}(\mathbb{R}^n)$.

Proof. Sufficiency. Let $1 < \vec{p} < \infty$, $0 < \alpha < \frac{n-\lambda}{P}$, $\frac{\alpha}{n-\mu} \le \frac{1}{P} - \frac{1}{Q} \le \frac{\alpha}{n-\lambda}$ and $f \in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.

$$\begin{aligned}
M_{b,\alpha} f(x) &\approx \sup_{r>0} r^{\alpha-n} \|(b(x)-b)f\|_{L^1(B(x,r))} \\
&\le \sup_{r>0} \min\{r^\alpha M_b f(x), r^{\alpha-\frac{n}{P}} \|f\|_{L^p(B(x,r))}\} \\
&\le \sup_{r>0} \min\{r^\alpha M_b f(x), r^{\alpha-\frac{n}{P}} [r]_1^{\frac{\lambda}{P}} [1/r]_1^{-\frac{\mu}{P}} \|f\|_{L^{\vec{p},\lambda,\mu}}\} \\
&\le \sup_{r>0} \min\{r^\alpha M_b f(x), [r]_1^{\alpha-\frac{n-\lambda}{P}} [1/r]_1^{-\alpha+\frac{n-\mu}{P}} \|f\|_{L^{\vec{p},\lambda,\mu}}\} \\
&\le \max\Big\{ \sup_{0<r\le 1} \min\{r^\alpha M_b f(x), r^{\alpha-\frac{n-\lambda}{P}} \|f\|_{L^{\vec{p},\lambda,\mu}}\}, \\
&\qquad \sup_{r>1} \min\{r^\alpha M_b f(x), r^{\alpha-\frac{n-\mu}{P}} \|f\|_{L^{\vec{p},\lambda,\mu}}\}\Big\}.
\end{aligned}$$

Minimizing with respect to r, at

$$r = \Big(\frac{\|f\|_{L^{\vec{p},\lambda,\mu}}}{M_b f(x)}\Big)^{\frac{P}{n-\lambda}} \quad \text{and} \quad r = \Big(\frac{\|f\|_{L^{\vec{p},\lambda,\mu}}}{M_b f(x)}\Big)^{\frac{P}{n-\mu}}$$

we have

$$M_{b,\alpha}f(x) \le \max\Big\{\big(M_bf(x)\big)^{1-\frac{\alpha P}{n-\lambda}}\,\|f\|^{\frac{\alpha P}{n-\lambda}}_{L^{\vec{p},\lambda,\mu}},\ \big(M_bf(x)\big)^{1-\frac{\alpha P}{n-\mu}}\,\|f\|^{\frac{\alpha P}{n-\mu}}_{L^{\vec{p},\lambda,\mu}}\Big\}, \tag{16}$$

where we have used that the supremum is achieved when the minimum parts are balanced. From Corollary 11 and inequality (16), we get

$$\begin{aligned}\|M_{b,\alpha}f\|_{L^{\vec{q},\lambda,\mu}} &\lesssim \|b\|_*\,\|f\|^{1-\frac{P}{Q}}_{L^{\vec{p},\lambda,\mu}}\,\|(M_bf)^{\frac{P}{Q}}\|_{L^{\vec{q},\lambda,\mu}}\\ &= \|b\|_*\,\|f\|^{1-\frac{P}{Q}}_{L^{\vec{p},\lambda,\mu}}\,\|M_bf\|^{\frac{P}{Q}}_{L^{\vec{p},\lambda,\mu}} \lesssim \|b\|_*\,\|f\|_{L^{\vec{p},\lambda,\mu}},\end{aligned}$$

Necessity. Let $1<P<\frac{n-\lambda}{\alpha}$, $\frac{\alpha}{n-\mu}\le\frac{1}{P}-\frac{1}{Q}\le\frac{\alpha}{n-\lambda}$, $b\in BMO(\mathbb{R}^n)$, $f\in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ and assume that $M_{b,\alpha}$ is bounded from $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ to $L^{\vec{q},\lambda,\mu}(\mathbb{R}^n)$.

Note that

$$M_{b,\alpha}f_t(x) = t^{-\alpha}\,M_{b_{\frac{1}{t}},\alpha}f(tx),$$

$$\begin{aligned}&\Big\|M_{b,\alpha}f_t\Big\|_{L^{\vec{q},\lambda,\mu}} = t^{-\alpha}\sup_{x\in\mathbb{R}^n,\,r>0}[r]_1^{-\frac{\lambda}{P}}\,[1/r]_1^{\frac{\mu}{P}}\,\|M_{b_{\frac{1}{t}},\alpha}f(t\,\cdot)\|_{L^{\vec{q}}(B(x,r))}\\ &= t^{-\alpha-\frac{n}{q}}\sup_{r>0}\Big(\frac{[tr]_1}{[r]_1}\Big)^{\lambda/Q}\sup_{r>0}\Big(\frac{[1/r]_1}{[1/(tr)]_1}\Big)^{\mu/Q}\\ &\times\sup_{x\in\mathbb{R}^n,\,r>0}[tr]_1^{-\frac{\lambda}{P}}\,[1/(tr)]_1^{\frac{\mu}{P}}\,\|M_{b_{\frac{1}{t}},\alpha}f\|_{L^{\vec{q}}(B(tx,tr))}\\ &= t^{-\alpha-\frac{n}{q}}\,[t]_{1,+}^{\frac{\lambda}{Q}}\,[1/t]_{1,+}^{-\frac{\mu}{Q}}\,\Big\|M_{b_{\frac{1}{t}},\alpha}f\Big\|_{L^{\vec{q},\lambda,\mu}}.\end{aligned}$$

By the boundedness of M_α from $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ to $L^{\vec{q},\lambda,\mu}(\mathbb{R}^n)$ and from the preceding scaling identities we obtain

$$\begin{aligned}\Big\|M_{b,\alpha}f\Big\|_{L^{\vec{q},\lambda,\mu}} &= t^{\alpha+\frac{n}{Q}}\,[t]_{1,+}^{-\frac{\lambda}{Q}}\,[1/t]_{1,+}^{\frac{\mu}{q}}\,\Big\|M_{b_{\frac{1}{t}},\alpha}f_t\Big\|_{L^{\vec{q},\lambda,\mu}}\\ &\lesssim t^{\alpha+\frac{n}{q}}\,[t]_{1,+}^{-\frac{\lambda}{q}}\,[1/t]_{1,+}^{\frac{\mu}{q}}\,\Big\|f_t\Big\|_{L^{\vec{p},\lambda,\mu}}\\ &= t^{\alpha+\frac{n}{q}-\frac{n}{p}}\,[t]_{1,+}^{\frac{\lambda}{P}-\frac{\lambda}{Q}}\,[1/t]_{1,+}^{-\frac{\mu}{P}+\frac{\mu}{Q}}\,\|f\|_{L^{\vec{p},\lambda,\mu}}\\ &= t^{\alpha}\,[t]_{1,+}^{-\frac{n-\lambda}{P}+\frac{n-\lambda}{Q}}\,[1/t]_{1,+}^{\frac{n-\mu}{P}-\frac{n-\mu}{Q}}\,\|f\|_{L^{\vec{p},\lambda,\mu}}.\end{aligned}$$

If $\frac{1}{P}<\frac{1}{Q}+\frac{\alpha}{n-\mu}$, then by letting $t\to 0$ we have $\|M_{b,\alpha}f\|_{L^{\vec{q},\lambda,\mu}}=0$ for all $f\in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.

As well as if $\frac{1}{P} > \frac{1}{Q} + \frac{\alpha}{n-\lambda}$, then at $t \to \infty$ we obtain $\|M_{b,\alpha} f\|_{L^{\vec{q},\lambda,\mu}} = 0$ for all $f \in L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{n-\lambda} \le \frac{1}{P} - \frac{1}{Q} \le \frac{\alpha}{n-\mu}$.

Theorem 6. *Let $1 < \vec{p} \le \vec{q} < \infty$, $\vec{p} \ne \vec{q}$, $0 \le \mu \le \lambda < n$, $0 \le \alpha < \frac{n-\lambda}{P}$ and $\frac{\alpha}{n-\mu} \le \frac{1}{P} - \frac{1}{Q} \le \frac{\alpha}{n-\lambda}$. The following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$.

(ii) The operator $M_{b,\alpha}$ is bounded from $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ to $L^{\vec{q},\lambda,\mu}(\mathbb{R}^n)$.

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$. Combining Corollary 6, Theorems 3 and 13, we get

$$\begin{aligned}
\|M_{b,\alpha} f\|_{L^{\vec{q},\lambda,\mu}} &\lesssim \|b\|_* \|M(M_\alpha f) + M_\alpha(Mf)\|_{L^{\vec{q},\lambda,\mu}} \\
&\lesssim \|b\|_* \|M(M_\alpha f)\|_{L^{\vec{q},\lambda,\mu}} + \|M_\alpha(Mf)\|_{L^{\vec{q},\lambda,\mu}} \\
&\lesssim \|b\|_* \|M_\alpha f\|_{L^{\vec{q},\lambda,\mu}} + \|Mf\|_{L^{\vec{p},\lambda,\mu}} \\
&\lesssim \|b\|_* \|f\|_{L^{\vec{p},\lambda,\mu}}.
\end{aligned}$$

$(ii) \Rightarrow (i)$. Assume that $M_{b,\alpha}$ is bounded from $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ to $L^{\vec{q},\lambda,\mu}(\mathbb{R}^n)$. Let $B = B(x,r)$ be a fixed ball. We consider $f = \chi_B$. It is easy to compute that

$$\begin{aligned}
\|\chi_B\|_{L^{\vec{p},\lambda,\mu}} &\approx \sup_{y \in \mathbb{R}^n, t>0} \Big([t]_1^{-\lambda} \, [1/t]_1^{\mu} \int_{B(y,t)} \chi_B(z) dz \Big)^{\frac{1}{p}} \\
&= \sup_{y \in \mathbb{R}^n, t>0} \Big(|B(y,t) \cap B| \, [t]_1^{-\lambda} \, [1/t]_1^{\mu} \Big)^{\frac{1}{p}} \\
&= \sup_{B(y,t) \subseteq B} \Big(|B(y,t)| \, [t]_1^{-\lambda} \, [1/t]_1^{\mu} \Big)^{\frac{1}{p}} = r^{\frac{n}{P}} \, [r]_1^{-\frac{\lambda}{P}} \, [1/r]_1^{\frac{\mu}{P}}. \qquad (17)
\end{aligned}$$

On the one hand, since

$$M_{b,\alpha}(\chi_B)(x) \gtrsim \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |b(z) - b_B| dz \quad \text{for all} \quad x \in B,$$

we have

$$\begin{aligned}
\|M_{b,\alpha}(\chi_B)\|_{L^{\vec{q},\lambda,\mu}} &\approx \sup_{B(y,t) \subseteq B} \Big(\, [t]_1^{-\lambda} \, [1/t]_1^{\mu} \int_{B(y,t)} |M_{b,\alpha}(\chi_B)(z)|^q dz \Big)^{\frac{1}{q}} \\
&\gtrsim r^{\frac{n}{Q}} \, [r]_1^{-\frac{\lambda}{Q}} \, [1/r]_1^{\frac{\mu}{Q}} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |b(z) - b_B| dz \\
&= r^{\alpha + \frac{n}{Q}} \, [r]_1^{-\frac{\lambda}{Q}} \, [1/r]_1^{\frac{\mu}{Q}} \frac{1}{|B|} \int_B |b(z) - b_B| dz. \qquad (18)
\end{aligned}$$

On the other hand, by assumption

$$\|M_{b,\alpha}(\chi_B)\|_{L^{\vec{q},\lambda,\mu}} \lesssim \|\chi_B\|_{L^{\vec{p},\lambda,\mu}},$$

by (17) and (18), we get that

$$\begin{aligned}\frac{1}{|B|}\int_B |b(z)-b_B|dz &\lesssim r^{-\alpha-\frac{n}{Q}}\,[r]_1^{\frac{\lambda}{Q}}\,[1/r]_1^{-\frac{\mu}{Q}}\,\|M_{b,\alpha}(\chi_B)\|_{L^{\vec{q},\lambda,\mu}}\\ &\lesssim r^{-\alpha-\frac{n}{Q}}\,[r]_1^{\frac{\lambda}{Q}}\,[1/r]_1^{-\frac{\mu}{Q}}\,\|\chi_B\|_{L^{\vec{p},\lambda,\mu}}\\ &\lesssim r^{-\alpha-\frac{n}{Q}}\,[r]_1^{\frac{\lambda}{Q}}\,[1/r]_1^{-\frac{\mu}{Q}}\,r^{\frac{n}{P}}\,[r]_1^{-\frac{\lambda}{P}}\,[1/r]_1^{\frac{\mu}{P}}\\ &\lesssim [r]_1^{-\alpha+\frac{n-\lambda}{P}-\frac{n-\lambda}{Q}}\,[1/r]_1^{\alpha-\frac{n-\mu}{P}+\frac{n-\mu}{Q}}\\ &\lesssim 1.\end{aligned}$$

From Theorem 6 in the case $\lambda=\mu$ or $\mu=0$ we get the following corollaries.

Corollary 12. *Let $1<\vec{p}\le\vec{q}<\infty$, $\vec{p}\neq\vec{q}$, $0\le\alpha<\frac{n-\lambda}{P}$, $0\le\lambda<n-\alpha$ and $\frac{1}{P}-\frac{1}{Q}=\frac{\alpha}{n-\lambda}$. The following assertions are equivalent:*

(i) $b\in BMO(\mathbb{R}^n)$.

(ii) The operator $M_{b,\alpha}$ is bounded from $L^{\vec{p},\lambda}(\mathbb{R}^n)$ to $L^{\vec{q},\lambda}(\mathbb{R}^n)$.

Corollary 13. *Let $1<\vec{p}\le\vec{q}<\infty$, $\vec{p}\neq\vec{q}$, $0\le\alpha<\frac{n-\lambda}{P}$ and $\frac{\alpha}{n}\le\frac{1}{P}-\frac{1}{Q}\le\frac{\alpha}{n-\lambda}$. The following assertions are equivalent:*

(i) $b\in BMO(\mathbb{R}^n)$.

(ii) The operator $M_{b,\alpha}$ is bounded from $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ to $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.

From Corollaries 12 and 13 in the case $\alpha=0$ we get the following corollaries, respectively.

Corollary 14. *[2, Theorem 1.4] Let $1<\vec{p}<\infty$ and $0\le\lambda\le n$. The following assertions are equivalent:*

(i) $b\in BMO(\mathbb{R}^n)$.

(ii) The operator M_b is bounded on $L^{\vec{p},\lambda}(\mathbb{R}^n)$.

Corollary 15. *[5, Theorem 4] Let $1<\vec{p}<\infty$ and $0\le\lambda\le n$. The following assertions are equivalent:*

(i) $b\in BMO(\mathbb{R}^n)$.

(ii) The operator M_b is bounded on $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$.

Remark 3. *Note that Corollary 13 is new.*

5. Commutator of fractional maximal operator in total mixed Morrey spaces

In this section we find necessary and sufficient conditions for the boundedness of the commutator of fractional maximal operator $[b, M_\alpha]$ in the $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ spaces.

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) := \begin{cases} 0\,, & \text{if } b(x) \geq 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M_\alpha]$ and $M_{b,\alpha}$ are valid:

Let b be any non-negative locally integrable function. Then for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ the following inequality is valid

$$\begin{aligned} &\big|[b, M_\alpha]f(x)\big| = \big|b(x)M_\alpha f(x) - M_\alpha(bf)(x)\big| \\ &= \big|M_\alpha(b(x)f)(x) - M_\alpha(bf)(x)\big| \leq M_\alpha(|b(x) - b|f)(x) = M_{b,\alpha}f(x). \end{aligned}$$

If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[b, M_\alpha]f(x)| \leq M_{b,\alpha}f(x) + 2b^-(x)\, M_\alpha f(x), \qquad x \in \mathbb{R}^n \tag{19}$$

holds for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ (see, for example [51]).

Applying Theorem 6, we obtain the following result.

Theorem 7. *Let $1 < \vec{p} < \infty$, $0 \leq \mu \leq \lambda < n$, $0 \leq \alpha < \frac{n-\mu}{P}$ and $\frac{\alpha}{n-\mu} \leq \frac{1}{P} - \frac{1}{Q} \leq \frac{\alpha}{n-\lambda}$. Then the following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$ such that $b^- \in L^\infty(\mathbb{R}^n)$.

(ii) The operator $[b, M_\alpha]$ is bounded from $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ to $L^{\vec{q},\lambda,\mu}(\mathbb{R}^n)$.

(iii) There exists a constant $C > 0$ such that

$$\sup_B \frac{\big\|\big(b - |B|^{-\frac{\alpha}{n}}\, M_{\alpha,B}(b)\big)\,\chi_B\big\|_{L^{\vec{q},\lambda,\mu}}}{\|\chi_B\|_{L^{\vec{q},\lambda,\mu}}} \leq C. \tag{20}$$

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$. Combining Theorems 3 and 6, and inequality (19), we get

$$\begin{aligned} \|[b, M_\alpha]f\|_{L^{\vec{q},\lambda,\mu}} &\leq \|M_{b,\alpha}f + 2b^-\, M_\alpha f\|_{L^{\vec{q},\lambda,\mu}} \\ &\leq \|M_{b,\alpha}f\|_{L^{\vec{q},\lambda,\mu}} + \|b^-\|_{L^\infty}\, \|M_\alpha f\|_{L^{\vec{q},\lambda,\mu}} \\ &\lesssim \big(\|b\|_* + \|b^-\|_{L^\infty}\big)\, \|f\|_{L^{\vec{p},\lambda,\mu}}. \end{aligned}$$

$(ii) \Rightarrow (iii)$. Assume that $[b, M_\alpha]$ is bounded from $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ to $L^{\vec{q},\lambda,\mu}(\mathbb{R}^n)$. For a given ball B and $0 \le \alpha < n$, we define the following local fractional maximal function:

$$M_{\alpha,B} f(x) := \sup_{B \supseteq B' \ni x} |B'|^{-1+\frac{\alpha}{n}} \int_{B'} |f(y)|\, dy,$$

where the supremum is taken over all balls B' such that $x \in B' \subseteq B$. Moreover, we denote by $M_B = M_{0,B}$ when $\alpha = 0$.

Since

$$M_\alpha(b\chi_B)\chi_B = M_{\alpha,B}(b)\,\chi_B \quad \text{and} \quad M_\alpha(\chi_B)\chi_B = M_{\alpha,B}\chi_B = |B|^{\frac{\alpha}{n}}\,\chi_B,$$

we have

$$\begin{aligned}\left(b - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)\right)\chi_B &= |B|^{-\frac{\alpha}{n}} \left(|B|^{\frac{\alpha}{n}} b - M_{\alpha,B}(b)\right)\chi_B \\ &= |B|^{-\frac{\alpha}{n}} \left(bM_\alpha(\chi_B) - M_\alpha(b\chi_B)\right)\chi_B \\ &= |B|^{-\frac{\alpha}{n}} [b, M_\alpha](\chi_B).\end{aligned}$$

By this and $[b, M_\alpha] : L^{\vec{p},\lambda,\mu}(\mathbb{R}^n) \to L^{\vec{q},\lambda,\mu}(\mathbb{R}^n)$, we obtain

$$\begin{aligned}\left\|\left(b - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)\right)\chi_B\right\|_{L^{\vec{q},\lambda,\mu}} &\le |B|^{-\frac{\alpha}{n}} \left\|[b, M_\alpha]\chi_B\right\|_{L^{\vec{q},\lambda,\mu}(\mathbb{R}^n)} \\ &\lesssim |B|^{-\frac{\alpha}{n}} \left\|\chi_B\right\|_{L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)}.\end{aligned}$$

Thus from (17) we get

$$\begin{aligned}\frac{\left\|\left(b - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)\right)\chi_B\right\|_{L^{\vec{q},\lambda,\mu}}}{\left\|\chi_B\right\|_{L^{\vec{q},\lambda,\mu}}} &\le |B|^{-\frac{\alpha}{n}} \frac{\left\|\chi_B\right\|_{L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)}}{\left\|\chi_B\right\|_{L^{\vec{q},\lambda,\mu}(\mathbb{R}^n)}} \\ &\approx r^{-\alpha+\frac{n}{P}-\frac{n}{Q}} [r]_1^{\frac{\lambda}{Q}-\frac{\lambda}{P}} [1/r]_1^{\frac{\mu}{P}-\frac{\mu}{Q}} \\ &\approx [r]_1^{-\alpha+\frac{n-\lambda}{P}-\frac{n-\lambda}{Q}} [1/r]_1^{\alpha-\frac{n-\mu}{P}+\frac{n-\mu}{Q}} \\ &\lesssim 1.\end{aligned}$$

$(iii) \Rightarrow (i)$. Assume that (20) is valid.

Now, let us prove $b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$. For any ball B, let $E = \{y \in B : b(y) \le b_B\}$ and $F = \{y \in B : b(y) > b_B\}$. The following equality is true (see [7, page 3331]):

$$\int_E |b(y) - b_B| dy = \int_F |b(y) - b_B| dy.$$

Since $b(y) \le b_B \le |b_B| \le |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)(y)$ for any $y \in E$, we obtain

$$|b(y) - b_B| \le \left|b(y) - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)(y)\right|,\ y \in E.$$

Then from Hölder's inequality and (20) we have

$$\begin{aligned}
\frac{1}{|B|}\int_B |b(y)-b_B|dy &= \frac{2}{|B|}\int_E |b(y)-b_B|dy \\
&\le \frac{2}{|B|}\int_E \left|b(y)-|B|^{-\frac{\alpha}{n}}M_{\alpha,B}(b)(y)\right|dy \\
&\le \frac{2}{|B|}\int_B \left|b(y)-|B|^{-\frac{\alpha}{n}}M_{\alpha,B}(b)(y)\right|dy \\
&\lesssim \frac{1}{|B|^{\frac{1}{\vec{q}}}}\left\|b(\cdot)-|B|^{-\frac{\alpha}{n}}M_{\alpha,B}(b)(\cdot)\right\|_{L^{\vec{q}}(B)} \\
&\le |B|^{-\frac{1}{\vec{q}}}[r]_1^{\frac{\lambda}{\vec{Q}}}[1/r]_1^{-\frac{\mu}{\vec{Q}}}\|b\chi_B - M_{b,\alpha}(b)\|_{L^{\vec{q},\lambda,\mu}(\mathbb{R}^n)} \\
&\lesssim r^{-\frac{n}{\vec{Q}}}[r]_1^{\frac{\lambda}{\vec{Q}}}[1/r]_1^{-\frac{\mu}{\vec{Q}}}\|\chi_B\|_{L^{\vec{q},\lambda,\mu}} \\
&\lesssim r^{-\frac{n}{\vec{Q}}}[r]_1^{\frac{\lambda}{\vec{Q}}}[1/r]_1^{-\frac{\mu}{\vec{Q}}}r^{\frac{n}{\vec{Q}}}[r]_1^{-\frac{\lambda}{\vec{Q}}}[1/r]_1^{\frac{\mu}{\vec{Q}}} \approx 1.
\end{aligned}$$

From Theorem 7 in the case $\lambda=\mu$ or $\mu=0$ we get the following corollaries.

Corollary 16. *Let $1<\vec{p}<\infty$, $0\le\alpha<\frac{n-\lambda}{P}$ and $\frac{1}{P}-\frac{1}{Q}=\frac{\alpha}{n-\lambda}$. Suppose that b is a real valued locally integrable function in $\mathbb{R}^n$. Then the following assertions are equivalent:*

(i) $b\in BMO(\mathbb{R}^n)$ *such that* $b^-\in L^\infty(\mathbb{R}^n)$.

(ii) *The operator* $[b,M_\alpha]$ *is bounded from* $L^{\vec{p},\lambda}(\mathbb{R}^n)$ *to* $L^{\vec{q},\lambda}(\mathbb{R}^n)$.

(iii) *There exists a constant* $C>0$ *such that*

$$\sup_B \frac{\left\|\left(b-|B|^{-\frac{\alpha}{n}}M_{\alpha,B}(b)\right)\chi_B\right\|_{L^{\vec{q},\lambda}}}{\|\chi_B\|_{L^{\vec{q},\lambda}}}\le C.$$

Corollary 17. *Let $1<\vec{p}<\infty$, $0\le\alpha<\frac{n-\lambda}{P}$ and $\frac{\alpha}{n}\le\frac{1}{P}-\frac{1}{Q}\le\frac{\alpha}{n-\lambda}$. Suppose that b is a real valued locally integrable function in $\mathbb{R}^n$. Then the following assertions are equivalent:*

(i) $b\in BMO(\mathbb{R}^n)$ *such that* $b^-\in L^\infty(\mathbb{R}^n)$.

(ii) *The operator* $[b,M_\alpha]$ *is bounded from* $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.

(iii) *There exists a constant* $C>0$ *such that*

$$\sup_B \frac{\left\|\left(b-|B|^{-\frac{\alpha}{n}}M_{\alpha,B}(b)\right)\chi_B\right\|_{\widetilde{L}^{q,\lambda}}}{\|\chi_B\|_{\widetilde{L}^{q,\lambda}}}\le C.$$

From Corollaries 16 and 17 in the case $\alpha=0$ we get the following corollaries, respectively.

Corollary 18. *[2, Theorem 1.2] Let $1 < \vec{p} < \infty$, $0 \leq \lambda \leq n$. Suppose that b is a real valued locally integrable function in $\mathbb{R}^n$. Then the following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$ such that $b^- \in L^\infty(\mathbb{R}^n)$.

(ii) The operator $[b, M]$ is bounded on $L^{\vec{p},\lambda}(\mathbb{R}^n)$.

Corollary 19. *[5, Theorem 6] Let $1 < p < \infty$, $0 \leq \lambda \leq n$. Suppose that b is a real valued locally integrable function in $\mathbb{R}^n$. Then the following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$ such that $b^- \in L^\infty(\mathbb{R}^n)$.

(ii) The operator $[b, M]$ is bounded on $\widetilde{L}^{\vec{p},\lambda}(\mathbb{R}^n)$.

Remark 4. *Note that Corollaries 16 and 17 are new.*

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Received 12 March 2026
Accepted 10 October 2026