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Commutators of the maximal function with BMO functions on total mixed Morrey spaces

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Abstract. In this paper, we introduce the total mixed Morrey spaces $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ and establish some basic properties and embeddings. We prove the boundedness of the maximal commutator operator M_b and the commutator of the maximal operator $[b, M]$ on total mixed Morrey spaces $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$. Using the boundedness results, we obtain some new characterizations for certain subclasses of the $BMO(\mathbb{R}^n)$ space.

Key Words and Phrases: total mixed Morrey spaces, maximal operator, commutators, BMO spaces.

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1. Introduction

Classical Morrey spaces $L_{p,\lambda}$ were originally introduced by Morrey in [19] to study the local behavior of solutions of second-order elliptic partial differential equations. In 2022, Guliyev [13] introduced a variant of Morrey spaces called total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, $0 < p < \infty, \lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$, see also [6, 15, 16, 18, 22, 23]. Total Morrey spaces generalize the classical Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ so that $L_{p,\lambda,\lambda}(\mathbb{R}^n) \equiv L_{p,\lambda}(\mathbb{R}^n)$ and the modified Morrey spaces $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ so that $L_{p,\lambda,0}(\mathbb{R}^n) = \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$, respectively. The subject of mixed-norm function spaces has undergone great development in the last few decades. Nevertheless, the standard literature is still the mixed Lebesgue spaces $L_{\vec{p}}(\mathbb{R}^n)$, $0 < \vec{p} \le \infty$, as a natural generalization of the classical Lebesgue spaces $L_p(\mathbb{R}^n)$, $0 < p \le \infty$, it is first introduced by Benedek and Panzone [3] in 1961. Mixed-norm function spaces possess a more refined structural framework than their classical counterparts, thereby enabling wider applications in analysis such as potential analysis, harmonic analysis and partial differential equations. In 2019, Nogayama [20] introduced a new Morrey-type space called mixed Morrey space by generalizing

Morrey spaces and mixed Lebesgue spaces, see also [1, 5, 14, 21]. We introduce the total mixed Morrey spaces $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ here. These spaces generalize the mixed Lebesgue spaces so that $L_{\vec{p},0,0}(\mathbb{R}^n) \equiv L_{\vec{p}}(\mathbb{R}^n)$, the mixed Morrey spaces so that $L_{\vec{p},\lambda,\lambda}(\mathbb{R}^n) \equiv L_{\vec{p},\lambda}(\mathbb{R}^n)$ and the modified mixed Morrey spaces so that $L_{\vec{p},\lambda,0}(\mathbb{R}^n) = \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$.

The classical Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|dy,$$

where $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$. The sharp maximal function of Fefferman and Stein $M^\sharp f$ is defined by

$$M^\sharp f(x) = \sup_{B \ni x} |B|^{-1} \int_B |f(y) - f_B|dy,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x. These operators M and $M^\sharp$ play an essential role in real and harmonic analysis. The maximal commutator of M with a locally integrable function b is defined by

$$M_b f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |b(x) - b(y)||f(y)|dy.$$

A (nonlinear) commutator of maximal operator M with a locally integrable function b is defined by

$$[b, M]f(x) = b(x)Mf(x) - M(bf)(x).$$

Obviously, the operators M_b and $[b, M]$ are significantly different from each other, since M_b is positive and sublinear, while $[b, M]$ is neither positive nor sublinear.

Commutator estimates play an important role in studying the regularity of solutions of second-order elliptic partial differential equations, and their boundedness can be used to characterize some function spaces (see, for instance [7, 9, 24, 25, 27]). The M_b operator is used to examine the commutators of singular integral operators with the symbol BMO (see [8, 26]). Note that the boundedness of the operator M_b on L_p spaces was proved by Garcia-Cuerva et al. in [8]. The nonlinear commutator $[b, M]$ of the maximal operator is used to study the product of a function in H_1 and a function in BMO (see [4]). In [2] Bastero et al. studied the necessary and sufficient conditions for the boundedness of $[b, M]$ on L_p spaces.

In this paper we introduce the total mixed Morrey spaces $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$. We give basic properties of the spaces $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ and study some embeddings into the Morrey space $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$. We obtain the boundedness of maximal commutator operator M_b and commutator of maximal operator $[b, M]$ in total mixed Morrey spaces $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$. We give some characterizations for some subclasses of the BMO space by using boundedness results.

The paper is organized as follows. In Section 2 we give basic properties of the spaces $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ and study some embeddings into the total mixed Morrey space $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$. In Section 3 we find necessary and sufficient conditions for the boundedness of the maximal commutator M_b on $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ spaces. In Section 4 we find necessary and sufficient conditions for the boundedness of the commutator of maximal operator $[b, M]$ on $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Definition and basic properties of total mixed Morrey spaces

For any $r > 0$ and $x \in \mathbb{R}^n$, let $B(x, r) = \{y \ : \ |y - x| < r\}$ be the ball centered at x with radius r. Let $\mathrm{B} = \{B(x, r) \ : \ x \in \mathbb{R}^n, \ r > 0\}$ be the set of all such balls. We also use χ_E and $|E|$ to denote the characteristic function and the Lebesgue measure of a measurable set E.

The letter $\vec{p}$ denotes n-tuples of the numbers in $(0, \ \infty]$, $(n \ge 1)$, $\vec{p} = (p_1, \ \cdots \ , \ p_n)$. By definition, the inequality, for example, $0 < \vec{p} < \infty$ means $0 < p_i < \infty$ for all i. For $1 \le \vec{p} \le \infty$, we denote $\dfrac{1}{P} = \dfrac{1}{n}\sum\limits_{i=1}^{n}\dfrac{1}{p_i}$, $\vec{p}\,' = (p_1', \ \cdots \ , p_n')$, where p_i', P' satisfies $\dfrac{1}{p_i} + \dfrac{1}{p_i'} = 1$, $\dfrac{1}{P} + \dfrac{1}{P'} = 1$.

We first recall the definition of mixed Lebesgue space defined in [3].

Let $\vec{p} = (p_1, \ldots, p_n) \in (0, \infty]^n$. Then the mixed Lebesgue norm $\| \cdot \|_{L_{\vec{p}}}$ or $\| \cdot \|_{L_{(p_1,\ldots,p_n)}}$ is defined by

$$\|f\|_{L_{\vec{p}}} \equiv \|f\|_{L_{(p_1,\ldots,p_n)}} = \Big(\int_{\mathbb{R}} \cdots \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f(x_1, \ x_2, \ \ldots, \ x_n)|^{p_1} dx_1 \Big)^{\frac{p_2}{p_1}} dx_2 \Big)^{\frac{p_3}{p_2}} \ldots dx_n \Big)^{\frac{1}{p_n}},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a measurable function. If $p_j = \infty$ for some $j = 1, \ldots, n$, then we have to make appropriate modifications. We define the mixed Lebesgue space $L_{\vec{p}}(\mathbb{R}^n) = L_{(p_1,\ldots,p_n)}(\mathbb{R}^n)$ to be the set of all $f \in L_0(\mathbb{R}^n)$ with $\|f\|_{L_{\vec{p}}} < \infty$, where $L_0(\mathbb{R}^n)$ denotes the set of measurable functions on $\mathbb{R}^n$.

The following analogue of the Hölder's inequality for $L_{\vec{p}}$ is well known (see, for example, [29]).

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $1 \le \vec{p} \le \infty$ and $\frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = 1$. Then for any $f \in L_{\vec{p}}(\Omega)$ and $g \in L_{\vec{p}'}(\Omega)$, the following inequality is valid*

$$\int_\Omega |f(x)g(x)|dx \le \|f\|_{L_{\vec{p}}(\Omega)} \|g\|_{L_{\vec{p}'}(\Omega)}.$$

By elementary calculations we have the following property.

Lemma 1. *Let $0 < \vec{p} < \infty$ and B be a ball in $\mathbb{R}^n$. Then*

$$\|\chi_B\|_{L_{\vec{p}}} = \|\chi_B\|_{WL_{\vec{p}}} = |B|^{\frac{1}{P}}.$$

By Theorem 1 and Lemma 1 we get the following estimate.

Lemma 2. *For $1 \le \vec{p} < \infty$ and for the balls $B = B(x,r)$ the following inequality is valid:*

$$\int_B |f(y)|dy \le |B|^{\frac{1}{P'}} \|f\|_{L_{\vec{p}}(B)}.$$

The following lemma is the Lebesgue differentiation theorem in mixed-norm Lebesgue spaces.

Lemma 3. [*29*, *Lemma* 2.4] *Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$ and $0 < \vec{p} < \infty$, then*

$$\lim_{r\to 0} \|\chi_{B(x,r)}\|_{L_{\vec{p}}}^{-1} \|f\|_{L_{\vec{p}}(B(x,r))} = |f(x)| \quad a.e.\ x \in \mathbb{R}^n.$$

In the following we define the mixed total Morrey spaces $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.

Definition 1. *Let $0 < \vec{p} < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1,t\}$, $t > 0$. We denote by $L_{\vec{p},\lambda}(\mathbb{R}^n)$ the mixed Morrey space [20], by $\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$ the modified mixed Morrey space [12], and by $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ the total mixed Morrey space the set of all locally integrable functions f with the following finite norms*

$$\|f\|_{L_{\vec{p},\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{P}} \|f\|_{L_{\vec{p}}(B(x,t))},$$

$$\|f\|_{\widetilde{L}_{\vec{p},\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{P}} \|f\|_{L_{\vec{p}}(B(x,t))}$$

and

$$\|f\|_{L_{\vec{p},\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{P}} [1/t]_1^{\frac{\mu}{P}} \|f\|_{L_{\vec{p}}(B(x,t))},$$

respectively.

Definition 2. *Let $0 < \vec{p} < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak mixed Morrey space $WL_{\vec{p},\lambda}(\mathbb{R}^n)$ [20], the weak modified mixed Morrey space $W\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$ [12] and the weak total mixed Morrey space $WL_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite norms*

$$\|f\|_{WL_{\vec{p},\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{P}} \|f\|_{WL_{\vec{p}}(B(x,t))},$$

$$\|f\|_{W\widetilde{L}^{\vec{p},\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{P}} \|f\|_{WL_{\vec{p}}(B(x,t))}$$

and

$$\|f\|_{WL_{\vec{p},\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{P}} [1/t]_1^{\frac{\mu}{P}} \|f\|_{WL_{\vec{p}}(B(x,t))},$$

respectively.

Note that

$$L_{\vec{p},0,0}(\mathbb{R}^n) = \widetilde{L}_{\vec{p},0}(\mathbb{R}^n) = L_{\vec{p},0}(\mathbb{R}^n) = L_{\vec{p}}(\mathbb{R}^n),$$
$$WL_{\vec{p},0,0}(\mathbb{R}^n) = W\widetilde{L}_{\vec{p},0}(\mathbb{R}^n) = WL_{\vec{p},0}(\mathbb{R}^n) = WL_{\vec{p}}(\mathbb{R}^n),$$
$$L_{\vec{p},\lambda,\lambda}(\mathbb{R}^n) = L_{\vec{p},\lambda}(\mathbb{R}^n), \quad L_{\vec{p},\lambda,0}(\mathbb{R}^n) = \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n),$$
$$\|f\|_{WL_{\vec{p},\lambda,\mu}} \le \|f\|_{L_{\vec{p},\lambda,\mu}} \text{ and therefore } L_{\vec{p},\lambda,\mu}(\mathbb{R}^n) \subset WL_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$$

and

$$L_{\vec{p},\lambda,\mu}(\mathbb{R}^n) \subset\!\succ L_{\vec{p},\lambda}(\mathbb{R}^n),\ \mu \le \lambda \text{ and } \|f\|_{L_{\vec{p},\lambda}} \le \|f\|_{L_{\vec{p},\lambda,\mu}}, \tag{1}$$

$$L_{\vec{p},\lambda,\mu}(\mathbb{R}^n) \subset\!\succ L_{\vec{p},\mu}(\mathbb{R}^n),\ \mu \le \lambda \text{ and } \|f\|_{L_{\vec{p},\mu}} \le \|f\|_{L_{\vec{p},\lambda,\mu}} \tag{2}$$

$$\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n) \subset\!\succ L_{\vec{p}}(\mathbb{R}^n) \text{ and } \|f\|_{L_{\vec{p}}} \le \|f\|_{\widetilde{L}_{\vec{p},\lambda}}$$

and if $\lambda < 0$ or $\lambda > n$, then $L_{\vec{p},\lambda}(\mathbb{R}^n) = \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n) = WL_{\vec{p},\lambda}(\mathbb{R}^n) = W\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n) = \Theta$. Here $\Theta \equiv \Theta(\mathbb{R}^n)$ is the set of all functions on $\mathbb{R}^n$ that are equivalent to 0.

Lemma 4. *If $0 < \vec{p} < \infty$, $0 \le \mu \le \lambda \le n$, then*

$$L_{\vec{p},\lambda,\mu}(\mathbb{R}^n) = L_{\vec{p},\lambda}(\mathbb{R}^n) \cap L_{\vec{p},\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{L_{\vec{p},\lambda}(\mathbb{R}^n)}, \|f\|_{L_{\vec{p},\mu}(\mathbb{R}^n)}\right\}.$$

Proof. Let $f \in L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ and $0 \le \mu \le \lambda \le n$. Then from (1) and (2) we get $f \in L_{\vec{p},\lambda}(\mathbb{R}^n) \cap L_{\vec{p},\mu}(\mathbb{R}^n)$ and $\max\left\{\|f\|_{L_{\vec{p},\lambda}(\mathbb{R}^n)}, \|f\|_{L_{\vec{p},\mu}(\mathbb{R}^n)}\right\} \le \|f\|_{L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)}$.

Let $f \in L_{\vec{p},\lambda}(\mathbb{R}^n) \cap L_{\vec{p},\mu}(\mathbb{R}^n)$. Then

$$\|f\|_{L_{\vec{p},\lambda,\mu}} = \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{P}} [1/t]_1^{\frac{\mu}{P}} \|f\|_{L_{\vec{p}}(B(x,t))}$$
$$= \max\left\{ \sup_{x\in\mathbb{R}^n, 0<t\le 1} t^{-\frac{\lambda}{P}} \|f\|_{L_{\vec{p}}(B(x,t))}, \sup_{x\in\mathbb{R}^n, t>1} [1/t]_1^{\frac{\mu}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} \right\}$$
$$\le \max\left\{ \|f\|_{L_{\vec{p},\lambda}}, \|f\|_{L_{\vec{p},\mu}} \right\}.$$

Thus, $f \in L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ and $\|f\|_{L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)} \le \max\left\{ \|f\|_{L_{\vec{p},\lambda}}, \|f\|_{L_{\vec{p},\mu}} \right\}$.

Therefore $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n) = L_{\vec{p},\lambda}(\mathbb{R}^n) \cap L_{\vec{p},\mu}(\mathbb{R}^n)$ and $\max\left\{ \|f\|_{L_{\vec{p},\lambda,\mu}} = \|f\|_{L_{\vec{p},\lambda}}, \|f\|_{L_{\vec{p},\mu}} \right\}$.

Corollary 1. *If* $0 < \vec{p} < \infty$, $0 \le \lambda \le n$, *then*

$$\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n) = L_{\vec{p},\lambda}(\mathbb{R}^n) \cap L_{\vec{p}}(\mathbb{R}^n)$$

and

$$\|f\|_{\widetilde{L}_{\vec{p},\lambda}} = \max\left\{ \|f\|_{L_{\vec{p},\lambda}}, \|f\|_{L_{\vec{p}}} \right\}.$$

Analogously proved

Lemma 5. *If* $0 < \vec{p} < \infty$, $0 \le \mu \le \lambda \le n$, *then*

$$WL_{\vec{p},\lambda,\mu}(\mathbb{R}^n) = WL_{\vec{p},\lambda}(\mathbb{R}^n) \cap WL_{\vec{p},\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{WL_{\vec{p},\lambda,\mu}(\mathbb{R}^n)} = \max\left\{ \|f\|_{WL_{\vec{p},\lambda}}, \|f\|_{WL_{\vec{p},\mu}} \right\}.$$

Remark 1. *If* $0 < \vec{p} < \infty$, *and* $\mu < 0$ *or* $\lambda > n$, *then*

$$L_{\vec{p},\lambda,\mu}(\mathbb{R}^n) = WL_{\vec{p},\lambda,\mu}(\mathbb{R}^n) = \Theta(\mathbb{R}^n).$$

Lemma 6. *If* $0 < \vec{p} < \infty$, $0 \le \lambda_2 \le \lambda_1 \le n$ *and* $0 \le \mu_1 \le \mu_2 \le n$, *then*

$$L_{\vec{p},\lambda_1,\mu_1}(\mathbb{R}^n) \subset\!\succ L_{\vec{p},\lambda_2,\mu_2}(\mathbb{R}^n)$$

and

$$\|f\|_{L_{\vec{p},\lambda_2,\mu_2}} \le \|f\|_{L_{\vec{p},\lambda_1,\mu_1}}.$$

Proof. Let $f \in L_{\vec{p},\lambda_1,\mu_1}$, $0 < \vec{p} < \infty$, $0 \le \lambda_2 \le \lambda_1 \le n$, $0 \le \mu_1 \le \mu_2 \le n$. Then

$$\|f\|_{L_{\vec{p},\lambda_2,\mu_2}} = \max\Big\{ \sup_{x\in\mathbb{R}^n,\, 0<t\le 1} t^{-\frac{\lambda_1-\lambda_2}{P}} t^{-\frac{\lambda_1}{P}} \|f\|_{L_{\vec{p}}(B(x,t))},$$
$$\sup_{x\in\mathbb{R}^n,\, t\ge 1} t^{-\frac{\mu_1-\mu_2}{P}} t^{-\frac{\mu_1}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} \Big\} \le \|f\|_{L_{\vec{p},\lambda_1,\mu_1}}.$$

Lemma 7. *If $0<\vec{p}<\infty$, $0\le\lambda\le n$ and $0\le\mu\le n$, then*

$$L_{\vec{p},n,\mu}(\mathbb{R}^n)\subset\!\!\succ L_\infty(\mathbb{R}^n)\subset\!\!\succ L_{\vec{p},\lambda,n}(\mathbb{R}^n)$$

and

$$\|f\|_{L_{\vec{p},\lambda,n}}\le v_n^{\frac{1}{P}}\|f\|_{L_\infty}\le\|f\|_{L_{\vec{p},n,\mu}}.$$

Proof. Let $f\in L_\infty(\mathbb{R}^n)$. Then for all $x\in\mathbb{R}^n$ and $0<t\le 1$

$$t^{-\frac{\lambda}{P}}\,\|f\|_{L_{\vec{p}}(B(x,t))}\le v_n^{\frac{1}{P}}\,\|f\|_{L_\infty},\quad 0\le\lambda\le n$$

and for all $x\in\mathbb{R}^n$ and $t\ge 1$

$$t^{-\frac{n}{P}}\,\|f\|_{L_{\vec{p}}(B(x,t))}\le v_n^{\frac{1}{P}}\,\|f\|_{L_\infty}.$$

Thus $f\in L_{\vec{p},\lambda,n}(\mathbb{R}^n)$ and

$$\|f\|_{L_{\vec{p},\lambda,n}}\le v_n^{\frac{1}{P}}\,\|f\|_{L_\infty}.$$

Let $f\in L_{\vec{p},n,\mu}(\mathbb{R}^n)$. By the Lebesgue's differentiation theorem we have (see Lemma 3)

$$\lim_{t\to 0}|B(x,t)|^{-\frac{1}{P}}\,\|f\|_{L_{\vec{p}}(B(x,t))}=|f(x)|\quad\text{for}\quad a.e.\ x\in\mathbb{R}^n.$$

Then for a.e. $x\in\mathbb{R}^n$

$$\begin{aligned}|f(x)|&=|B(x,t)|^{-\frac{1}{P}}\,\|f\|_{L_{\vec{p}}(B(x,t))}\\&\le v_n^{-\frac{1}{P}}\sup_{x\in\mathbb{R}^n,\,0<t\le1}t^{-\frac{n}{P}}\,\|f\|_{L_{\vec{p}}(B(x,t))}\\&\le v_n^{-\frac{1}{P}}\,\|f\|_{L_{\vec{p},n,\mu}}.\end{aligned}$$

Thus $f\in L_\infty(\mathbb{R}^n)$ and

$$\|f\|_{L_\infty}\le v_n^{-\frac{1}{P}}\,\|f\|_{L_{\vec{p},n,\mu}}.$$

Corollary 2. *If $0<\vec{p}<\infty$, then*

$$\widetilde{L}_{\vec{p},n}(\mathbb{R}^n)\subset\!\!\succ L_\infty(\mathbb{R}^n)\subset\!\!\succ L_{\vec{p},n}(\mathbb{R}^n)$$

and

$$\|f\|_{L_{\vec{p},n}}\le v_n^{\frac{1}{P}}\|f\|_{L_\infty}\le\|f\|_{\widetilde{L}_{\vec{p},n}}.$$

Lemma 8. *If* $0 \le \lambda < n$, $0 \le \mu < n$, $0 \le \alpha < n-\lambda$ *and* $0 \le \beta < n-\mu$, *then for* $\frac{n-\lambda}{\alpha} \le \vec{p} \le \frac{n-\mu}{\beta}$

$$L_{\vec{p},\lambda,\mu}(\mathbb{R}^n) \subset\!\!\succ L_{\vec{1},n-\alpha,n-\beta}(\mathbb{R}^n)$$

and for $f \in L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ *the inequality*

$$\|f\|_{L_{\vec{1},n-\alpha,n-\beta}} \le v_n^{\frac{1}{P'}} \|f\|_{L_{\vec{p},\lambda,\mu}}$$

holds.

Proof. Let $0 < \alpha < n$, $0 \le \lambda < n$, $f \in L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ and $\frac{n-\lambda}{\alpha} \le \vec{p} \le \frac{n-\mu}{\beta}$. By the Hölder's inequality (see Theorem 1) we have

$$\begin{aligned}
&\|f\|_{L_{\vec{1},n-\alpha,n-\beta}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{\alpha-n}\, [1/t]_1^{n-\beta}\, \|f\|_{L_1(B(x,t))} \\
&\sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{\alpha-n}\, [1/t]_1^{n-\beta}\, \|f\|_{L_{\vec{p}}(B(x,t))}\, \|1\|_{L_{\vec{p}'}(B(x,t))} \\
&\le v_n^{\frac{1}{P'}} \sup_{x\in\mathbb{R}^n,\, t>0} \left([t]_1\, t^{-1}\right)^{-\sum_{i=1}^n \frac{1}{p_i'}} [t]_1^{\alpha-\frac{n-\lambda}{P}}\, [1/t]_1^{n-\beta-\frac{\mu}{P}} \\
&\times [t]_1^{-\frac{\lambda}{P}}\, [1/t]_1^{\frac{\mu}{P}}\, \|f\|_{L_{\vec{p}}(B(x,t))} \\
&\le v_n^{\frac{1}{P'}} \|f\|_{L_{\vec{p},\lambda,\mu}} \sup_{t>0} \left([t]_1\, t^{-1}\right)^{\frac{n-\mu}{P}\big)-\beta}\, [t]_1^{\alpha-\frac{n-\lambda}{P}}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\sup_{t>0} \left([t]_1\, t^{-1}\right)^{\frac{n-\mu}{P}-\beta} [t]_1^{\alpha-\frac{n-\lambda}{P}} \\
&= \max\left\{ \sup_{0<t\le 1} t^{\alpha-\frac{n-\lambda}{P}\big)}, \sup_{t>1} t^{\beta-\frac{n-\mu}{P}\big)} \right\} < \infty \\
&\Longleftrightarrow \frac{n-\lambda}{\alpha} \le \vec{p} \le \frac{n-\mu}{\beta}.
\end{aligned}$$

Thus $f \in L_{\vec{1},n-\alpha,n-\beta}(\mathbb{R}^n)$ and

$$\|f\|_{L_{\vec{1},n-\alpha,n-\beta}} \le v_n^{\frac{1}{P'}} \|f\|_{L_{\vec{p},\lambda,\mu}}.$$

From Lemma 8 we obtain the following results.

Corollary 3. *If* $0 \le \mu \le \lambda < n$, $0 \le \alpha < n-\lambda$, *then for* $\frac{n-\lambda}{\alpha} \le \vec{p} \le \frac{n-\mu}{\alpha}$

$$L_{\vec{p},\lambda,\mu}(\mathbb{R}^n) \subset\!\!\succ L_{\vec{1},n-\alpha}(\mathbb{R}^n)$$

and for $f \in L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ the inequality

$$\|f\|_{L_{\vec{1},n-\alpha}} \leq v_n^{\frac{1}{P'}} \, \|f\|_{L_{\vec{p},\lambda,\mu}}$$

holds.

Corollary 4. *If $0 \leq \lambda < n$ and $0 \leq \alpha < n - \lambda$, then for $\vec{p} = \frac{n-\lambda}{\alpha}$*

$$L_{\vec{p},\lambda}(\mathbb{R}^n) \subset L_{\vec{1},n-\alpha}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{L_{\vec{1},n-\alpha}} \leq v_n^{\frac{1}{P'}} \, \|f\|_{L_{\vec{p},\lambda}}.$$

Corollary 5. *If $0 \leq \lambda < n$ and $0 \leq \alpha < n - \lambda$, then for $\frac{n-\lambda}{\alpha} \leq \vec{p} \leq \frac{n}{\alpha}$*

$$\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n) \subset L_{\vec{1},n-\alpha}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{L_{\vec{1},n-\alpha}} \leq v_n^{\frac{1}{P'}} \, \|f\|_{\widetilde{L}_{\vec{p},\lambda}}.$$

Remark 2. *Note that in the case $\vec{p} = (p, \ldots, p)$ Lemmas 4, 5, 6, 7 and 8 was proved in [13, Lemmas 2, 3, 4, 5 and 6].*

3. $L_{\vec{p},\lambda,\mu}$-boundedness of the maximal commutator operator M_b

In this section, we obtain necessary and sufficient conditions for the boundedness of the maximal commutator M_b on the total mixed Morrey spaces $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.

Firstly, in the following lemma we give two local estimates for the maximal operator M (see also [10, 11]).

Lemma 9. *Let $1 \leq \vec{p} < \infty$ and $B(x,r)$ be any ball in $\mathbb{R}^n$. If $\vec{p} > 1$, then the inequality*

$$\|Mf\|_{L_{\vec{p}}(B(x,r))} \lesssim r^{\frac{n}{P}} \sup_{t>2r} t^{-\frac{n}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} \tag{3}$$

holds for all $f \in L^{\text{loc}}_{\vec{p}}(\mathbb{R}^n)$.

Moreover if $\vec{p} = (1, 1, \ldots, 1)$, then the inequality

$$\|Mf\|_{WL_{\vec{1}}(B(x,r))} \lesssim r^n \sup_{t>2r} t^{-n} \|f\|_{L_{\vec{1}}(B(x,t))} \tag{4}$$

holds for all $f \in L^{\text{loc}}_{\vec{1}}(\mathbb{R}^n)$.

Proof. Let $1 < \vec{p} < \infty$. We set $f = f_1 + f_2$, where $f_1 = f\chi_{B(x,2r)}$ and $f_2 = f\chi_{B(x,2r)}$.

Estimate for Mf_1: by the boundedness of maximal operator M on $L_{\vec{p}}(\mathbb{R}^n)$ (see [20]) we get

$$\|Mf_1\|_{L_{\vec{p}}(B)} \leq \|Mf_1\|_{L_{\vec{p}}(\mathbb{R}^n)} \lesssim \|f_1\|_{L_{\vec{p}}(\mathbb{R}^n)} = \|f\|_{L_{\vec{p}}(B(x,2r))}.$$

We obtain

$$r^{\frac{n}{P}} \sup_{t>2r} t^{-\frac{n}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} \geq r^{\frac{n}{P}} \|f\|_{L_{\vec{p}}(B(x,2r))} \sup_{t>2r} t^{-\frac{n}{P}} \gtrsim \|f\|_{L_{\vec{p}}(B(x,2r))} \tag{5}$$

by using the monotonicity of the functions $\|f\|_{L_{\vec{p}}(B(x,t))}$ and $t^{\frac{n}{P}}$ with respect to t. Therefore we have

$$\|Mf_1\|_{L_{\vec{p}}(B)} \lesssim r^{\frac{n}{P}} \sup_{t>r} t^{-\frac{n}{P}} \|f\|_{L_{\vec{p}}(B(x,t))}. \tag{6}$$

Estimate for Mf_2: Let y be an arbitrary point in B. If $B(y,t)\cap(B(x,2r)) \neq \emptyset$, then $t > r$. If $z \in B(y,t)\cap(B(x,2r))$, then $t > |y-z| \geq |x-z|-|x-y| > 2r-r = r$.

On the other hand, $B(y,t)\cap(B(x,2r)) \subset B(x,2t)$. If $z \in B(y,t)\cap(B(x,2r))$, then we obtain $|x-z| \leq |y-z| + |x-y| < t+r < 2t$.

Thus

$$\begin{aligned} Mf_2(y) &= \sup_{t>0} \frac{1}{|B(y,t)|} \int_{B(y,t)\cap(B(x,2r))} |f(z)|dz \\ &\leq \sup_{t>r} \frac{1}{|B(y,t)|} \int_{B(x,2t)} |f(z)|dz \\ &\leq \sup_{t>r} \frac{C}{|B(y,2t)|} \int_{B(x,2t)} |f(z)|dz \\ &= \sup_{t>2r} \frac{C}{|B(y,t)|} \int_{B(x,t)} |f(z)|dz. \end{aligned}$$

From Lemma 2 for all $y \in B$ we get

$$\begin{aligned} Mf_2(y) &\lesssim \sup_{t>2r} \frac{1}{|B(y,t)|} \, t^{\sum_{i=1}^{n} \frac{1}{p_i'}} \|f\|_{L_{\vec{p}}(B(x,t))} \\ &\lesssim \sup_{t>r} t^{-\frac{n}{P}} \|f\|_{L_{\vec{p}}(B(x,t))}. \end{aligned} \tag{7}$$

Therefore we get

$$\begin{aligned} \|Mf_2\|_{L_{\vec{p}}(B)} &\lesssim \|\chi_B\|_{L_{\vec{p}}} \sup_{t>r} t^{-\frac{n}{P}} \|f\|_{L_{\vec{p}}(B(x,t))} \\ &\lesssim r^{\frac{n}{P}} \sup_{t>r} t^{-\frac{n}{P}} \|f\|_{L_{\vec{p}}(B(x,t))}. \end{aligned}$$

If $\vec{p} = 1$, then for any ball $B = B(x,r)$ it is clear that

$$\|Mf\|_{WL_{\vec{1}}(B)} \leq \|Mf_1\|_{WL_{\vec{1}}(B)} + \|Mf_2\|_{WL_{\vec{1}}(B)}.$$

From the continuity of the operator $M : L_{\vec{1}}(\mathbb{R}^n) \to WL_{\vec{1}}(\mathbb{R}^n)$ we get

$$\|Mf_1\|_{WL_{\vec{1}}(B)} \lesssim \|f_1\|_{L_{\vec{1}}(B)}.$$

Therefore by (7) we get the inequality (4).

Secondly, in the following theorem we prove the boundedness of the maximal operator M on the total mixed Morrey spaces.

Theorem 2. *1. If $f \in L_{\vec{1},\lambda,\mu}(\mathbb{R}^n)$, $0 \le \lambda < n$ and $0 \le \mu < n$, then $Mf \in WL_{\vec{1},\lambda,\mu}(\mathbb{R}^n)$ and*

$$\|Mf\|_{WL_{\vec{1},\lambda,\mu}} \le C_{1,\lambda,\mu}\, \|f\|_{L_{\vec{1},\lambda,\mu}}, \tag{8}$$

where $C_{1,\lambda,\mu}$ does not depend on f.

2. If $f \in L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$, $1 < \vec{p} < \infty$, $0 \le \lambda < n$ and $0 \le \mu < n$, then $Mf \in L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ and

$$\|Mf\|_{L_{\vec{p},\lambda,\mu}} \le C_{\vec{p},\lambda,\mu}\, \|f\|_{L_{\vec{p},\lambda,\mu}}, \tag{9}$$

where $C_{\vec{p},\lambda,\mu}$ depends only on $\vec{p}$,λ,μ and n.

Proof. Let $\vec{p} = (1, 1, \dots, 1)$. From the inequality (4) we have

$$\begin{aligned}
\|Mf\|_{WL_{\vec{1},\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\lambda}\, [1/t]_1^{\mu}\, \|Mf\|_{WL_{\vec{1}}(B(x,t))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\lambda}\, [1/t]_1^{\mu}\, t^n \sup_{\tau>2t} \tau^{-n}\, \|f\|_{L_{\vec{1}}(B(x,\tau))} \\
&\lesssim \|f\|_{L_{\vec{1},\lambda,\mu}} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\lambda}\, [1/t]_1^{\mu}\, t^n \sup_{\tau>t} \tau^{-n}\, [\tau]_1^{\lambda}\, [1/\tau]_1^{-\mu} \\
&= \|f\|_{L_{\vec{1},\lambda,\mu}} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{n-\lambda}\, [1/t]_1^{\mu-n} \sup_{\tau>t} [\tau]_1^{\lambda-n}\, [1/\tau]_1^{n-\mu} \\
&\approx \|f\|_{L_{\vec{1},\lambda,\mu}} \sup_{\tau>1} [\tau]_1^{\lambda-n}\, [1/\tau]_1^{n-\mu} = \|f\|_{L_{\vec{1},\lambda,\mu}} \sup_{\tau>1} \tau^{-n+\mu} \\
&= \|f\|_{L_{\vec{1},\lambda,\mu}}
\end{aligned}$$

which implies that the operator M is bounded from $L_{\vec{1},\lambda,\mu}(\mathbb{R}^n)$ to $WL_{\vec{1},\lambda,\mu}(\mathbb{R}^n)$.

If $1 < \vec{p} < \infty$, then from the inequality (3) we have

$$\begin{aligned}
\|Mf\|_{L_{\vec{p},\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{P}}\, [1/t]_1^{\frac{\mu}{P}}\, \|Mf\|_{L_{\vec{p}}(B(x,t))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{P}}\, [1/t]_1^{\frac{\mu}{P}}\, t^{\frac{n}{p}} \sup_{\tau>2t} \tau^{-\frac{n}{P}}\, \|f\|_{L_{\vec{p}}(B(x,\tau))}
\end{aligned}$$

$$\lesssim \|f\|_{L_{\vec{p},\lambda,\mu}} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{P}} \, [1/t]_1^{\frac{\mu}{P}} \, t^{\frac{n}{P}} \sup_{\tau>t} \tau^{-\frac{n}{P}} \, [\tau]_1^{\frac{\lambda}{P}} \, [1/\tau]_1^{-\frac{\mu}{P}}$$

$$= \|f\|_{L_{\vec{p},\lambda,\mu}} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{\frac{n-\lambda}{P}} \, [1/t]_1^{\frac{\mu-n}{P}} \sup_{\tau>t} [\tau]_1^{\frac{\lambda-n}{P}} \, [1/\tau]_1^{\frac{n-\mu}{P}}$$

$$\approx \|f\|_{L_{\vec{p},\lambda,\mu}} \sup_{\tau>1} [\tau]_1^{\frac{\lambda-n}{P}} \, [1/\tau]_1^{\frac{n-\mu}{P}} = \|f\|_{L_{\vec{p},\lambda,\mu}} \sup_{\tau>1} \tau^{-\frac{n-\mu}{P}}$$

$$= \|f\|_{L_{\vec{p},\lambda,\mu}}$$

which implies that the operator M is bounded on $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.

If we take $\lambda = \mu$ or $\mu = 0$ in Theorem 2, then we get the following results.

Corollary 6. *[20] 1. If $f \in L_{\vec{1},\lambda}(\mathbb{R}^n)$ and $0 \le \lambda < n$, then $Mf \in WL_{\vec{1},\lambda}(\mathbb{R}^n)$ and*

$$\|Mf\|_{WL_{\vec{1},\lambda}} \le C_{1,\lambda} \, \|f\|_{L_{\vec{1},\lambda}},$$

where $C_{1,\lambda}$ does not depend on f.

2. If $f \in L_{\vec{p},\lambda}(\mathbb{R}^n)$, $1 < \vec{p} < \infty$ and $0 \le \lambda < n$, then $Mf \in L_{\vec{p},\lambda}(\mathbb{R}^n)$ and

$$\|Mf\|_{L_{\vec{p},\lambda}} \le C_{\vec{p},\lambda} \, \|f\|_{L_{\vec{p},\lambda}},$$

where $C_{\vec{p},\lambda}$ depends only on p, λ and n.

Corollary 7. *1. If $f \in \widetilde{L}_{\vec{1},\lambda}(\mathbb{R}^n)$ and $0 \le \lambda < n$, then $Mf \in W\widetilde{L}_{\vec{1},\lambda}(\mathbb{R}^n)$ and*

$$\|Mf\|_{W\widetilde{L}_{\vec{1},\lambda}} \le C_{1,\lambda} \, \|f\|_{\widetilde{L}_{\vec{1},\lambda}},$$

where $C_{1,\lambda}$ does not depend on f.

2. If $f \in \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$, $1 < \vec{p} < \infty$ and $0 \le \lambda < n$, then $Mf \in \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$ and

$$\|Mf\|_{\widetilde{L}_{\vec{p},\lambda}} \le C_{\vec{p},\lambda} \, \|f\|_{\widetilde{L}_{\vec{p},\lambda}},$$

where $C_{\vec{p},\lambda}$ depends only on $\vec{p}$, λ and n.

Definition 3. *The space $BMO(\mathbb{R}^n)$ is defined as the set of all locally integrable functions f with finite norm*

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y) - f_{B(x,t)}| dy < \infty,$$

where $f_{B(x,t)} = |B(x,t)|^{-1} \int_{B(x,t)} f(y) dy$.

Theorem 3. [*17, Lemma* 1] *If $b \in BMO(\mathbb{R}^n)$, then for any $q \in (0,1)$, there exists a positive constant C such that*

$$M_q^\sharp\big(M_b f\big)(x) \le C\|b\|_* M^2 f(x) \tag{10}$$

for every $x \in \mathbb{R}^n$ and for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Finally, we give the following theorem, which is one of our main results.

Theorem 4. *Let $1 < \vec{p} < \infty$, $0 \le \lambda \le n$ and $0 \le \mu \le n$. The following assertions are equivalent:*
(*i*) $b \in BMO(\mathbb{R}^n)$.
(*ii*) *The operator M_b is bounded on $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.*

Proof. $(i) \Rightarrow (ii)$. Assume $b \in BMO(\mathbb{R}^n)$. Combining Theorems 2 and 3, we obtain

$$\begin{aligned}\|M_b f\|_{L_{\vec{p},\lambda,\mu}} &\lesssim \|M_q^\sharp\big(M_b f\big)\|_{L_{\vec{p},\lambda,\mu}} \\ &\lesssim \|b\|_* \|M^2 f\|_{L_{\vec{p},\lambda,\mu}} \lesssim \|b\|_* \|Mf\|_{L_{\vec{p},\lambda,\mu}} \lesssim \|b\|_* \|f\|_{L_{\vec{p},\lambda,\mu}}.\end{aligned}$$

$(ii) \Rightarrow (i)$. Suppose M_b is bounded on $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$. Let $B = B(x,r)$ be a fixed ball. We consider $f = \chi_B$. It is easy to compute that

$$\begin{aligned}\|\chi_B\|_{L_{\vec{p},\lambda,\mu}} &\approx \sup_{y\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{P}} [1/t]_1^{\frac{\mu}{P}} \|\chi_B\|_{L_{\vec{p}}(B(y,t))} \\ &= \sup_{y\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{P}} [1/t]_1^{\frac{\mu}{P}} |B(y,t)\cap B|^{\frac{1}{P}} \\ &= \sup_{B(y,t)\subseteq B} [t]_1^{-\frac{\lambda}{P}} [1/t]_1^{\frac{\mu}{P}} |B(y,t)|^{\frac{1}{P}} \\ &= r^{\frac{n}{P}} [r]_1^{-\frac{\lambda}{P}} [1/r]_1^{\frac{\mu}{P}}.\end{aligned} \tag{11}$$

On the other hand, since

$$M_b(\chi_B)(x) \gtrsim \frac{1}{|B|}\int_B |b(z) - b_B|dz \quad \text{for all} \quad x \in B,$$

we get

$$\begin{aligned}&\|M_b(\chi_B)\|_{L_{\vec{p},\lambda,\mu}} \approx \sup_{B(y,t)} [t]_1^{-\frac{\lambda}{P}} [1/t]_1^{\frac{\mu}{P}} \||M_b(\chi_B)\|_{L_{\vec{p}}(B(y,t))} \\ &\gtrsim r^{\frac{n}{P}} [r]_1^{-\frac{\lambda}{P}} [1/r]_1^{\frac{\mu}{P}} \frac{1}{|B|}\int_B |b(z) - b_B|dz.\end{aligned} \tag{12}$$

From the assumption

$$\|M_b(\chi_B)\|_{L_{\vec{p},\lambda,\mu}} \lesssim \|\chi_B\|_{L_{\vec{p},\lambda,\mu}},$$

by (11) and (12), we find that

$$\frac{1}{|B|}\int_B |b(z)-b_B|dz \lesssim 1.$$

If we take $\lambda = \mu$ or $\mu = 0$ in Theorem 4, then we get the following results.

Corollary 8. *Let $1 < \vec{p} < \infty$ and $0 \le \lambda \le n$. The following assertions are equivalent:*
(i) $b \in BMO(\mathbb{R}^n)$.
(ii) The operator M_b is bounded on $L_{\vec{p},\lambda}(\mathbb{R}^n)$.

Corollary 9. *Let $1 < \vec{p} < \infty$ and $0 \le \lambda \le n$. The following assertions are equivalent:*
(i) $b \in BMO(\mathbb{R}^n)$.
(ii) The operator M_b is bounded on $\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$.

Remark 3. *Note that in the case $\vec{p} = (p, \dots, p)$ Theorems 2 and 4 were proved in [13, Theorems 1, 3].*

4. $L_{\vec{p},\lambda,\mu}$-boundedness of the commutator of maximal operator $[b, M]$

In this section we find necessary and sufficient conditions for the boundedness of the commutator of maximal operator $[b, M]$ on the total mixed Morrey spaces $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.

Let b be a function b defined on $\mathbb{R}^n$. We denote

$$b^-(x) := \begin{cases} 0\ , & \text{if } b(x) \ge 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. It is clear that $b^+(x) - b^-(x) = b(x)$.

The following relations hold between $[b, M]$ and M_b:

Let b be any non-negative locally integrable function. Then for all $f \in L_1^{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ the inequality

$$\begin{aligned}\big|[b, M]f(x)\big| &= \big|b(x)Mf(x) - M(bf)(x)\big| \\ &= \big|M(b(x)f)(x) - M(bf)(x)\big| \le M(|b(x) - b|f)(x) = M_b f(x)\end{aligned}$$

holds.

If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[b,M]f(x)| \le M_b f(x) + 2b^-(x)\, Mf(x), \qquad x \in \mathbb{R}^n \tag{13}$$

holds for all $f \in L_1^{\rm loc}(\mathbb{R}^n)$ (see, for example [13, 28]).

Let $B = B(x,r)$ be a fixed ball. Denote by $M_B f$ the local maximal function of f:

$$M_B f(x) := \sup_{B' \ni x:\, B' \subset B} \frac{1}{|B'|} \int_{B'} |f(y)|\, dy, \ x \in \mathbb{R}^n.$$

Applying Theorem 4, we obtain the following result, which is another of our main results.

Theorem 5. *Let $1 < \vec{p} < \infty$, $0 \le \lambda \le n$ and $0 \le \mu \le n$. Assume that b is a real-valued locally integrable function on $\mathbb{R}^n$. Then the following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$ such that $b^- \in L_\infty(\mathbb{R}^n)$.

(ii) The operator $[b,M]$ is bounded on $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.

(iii) There exists a constant $C > 0$ such that

$$\sup_B \frac{\|\big(b - M_B(b)\big)\, \chi_B\|_{L_{\vec{p},\lambda,\mu}}}{\|\chi_B\|_{L_{\vec{p},\lambda,\mu}}} \le C. \tag{14}$$

Proof. $(i) \Rightarrow (ii)$. Assume that $b \in BMO(\mathbb{R}^n)$. Combining Theorems 2 and 4, and inequality (13), we obtain

$$\begin{aligned} \|[b,M]f\|_{L_{\vec{p},\lambda,\mu}} &\le \|M_b f + 2b^-\, Mf\|_{L_{\vec{p},\lambda,\mu}} \\ &\le \|M_b f\|_{L_{\vec{p},\lambda,\mu}} + \|b^-\|_{L_\infty} \|Mf\|_{L_{\vec{p},\lambda,\mu}} \\ &\lesssim \big(\|b\|_* + \|b^-\|_{L_\infty}\big) \|f\|_{L_{\vec{p},\lambda,\mu}}. \end{aligned}$$

$(ii) \Rightarrow (i)$. Assume that $[b,M]$ is bounded on $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.

Since

$$M(b\chi_B)\chi_B = M_B(b) \quad \text{and} \quad M(\chi_B)\chi_B = \chi_B,$$

we get

$$\big(b - M_B(b)\big)\, \chi_B = b\,\chi_B - M_B(b)\,\chi_B = bM(\chi_B) - M(b\chi_B) = [b,M]\chi_B.$$

Then

$$\|\big(b - M_B(b)\big)\, \chi_B\|_{L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)} = \|[b,M]\chi_B\|_{L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)}.$$

Therefore from Lemma 2 and equation (11) we find

$$\frac{1}{|B|}\int_B |b-M_B(b)| \le |B|^{-1+\frac{1}{P'}} \|b-M_B(b)\|_{L_{\vec{p}}(B)}$$
$$\le |B|^{-\frac{1}{P}} [r]_1^{\frac{\lambda}{P}} [1/r]_1^{-\frac{\mu}{P}} \|b\chi_B - M_B(b)\|_{L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)}$$
$$\lesssim r^{-\frac{n}{P}} [r]_1^{\frac{\lambda}{P}} [1/r]_1^{-\frac{\mu}{P}} \|[b,M]\chi_B\|_{L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)}$$
$$\lesssim r^{-\frac{n}{P}} [r]_1^{\frac{\lambda}{P}} [1/r]_1^{-\frac{\mu}{P}} \|\chi_B\|_{L_{\vec{p},\lambda,\mu}} \approx 1.$$

We set

$$E := \{x \in B : b(x) \le b_B\}, \quad F := \{x \in B : b(x) > b_B\}.$$

Since
$$\int_E |b(t)-b_B|\,dt = \int_F |b(t)-b_B|\,dt,$$
Considering the inequality $b(x) \le b_B \le M_B(b)$, $x \in E$, we find

$$\frac{1}{|B|}\int_B |b-b_B| = \frac{2}{|B|}\int_E |b-b_B|$$
$$\le \frac{2}{|B|}\int_E |b-M_B(b)| \le \frac{2}{|B|}\int_B |b-M_B(b)| \lesssim 1.$$

Consequently, $b \in BMO(\mathbb{R}^n)$.

To show that $b^- \in L_\infty(\mathbb{R}^n)$, note that $M_B(b) \ge |b|$. Hence

$$0 \le b^- = |b| - b^+ \le M_B(b) - b^+ + b^- = M_B(b) - b.$$

Thus
$$(b^-) \le c,$$
and by the Lebesgue differentiation theorem (Lemma 3) we get that

$$b^-(x) \le c \quad \text{for a.e. } x \in \mathbb{R}^n.$$

If we take $\lambda = \mu$ or $\mu = 0$ in Theorem 5, then we get the following results.

Corollary 10. *Let $1 < \vec{p} < \infty$ and $0 \le \lambda \le n$. Assume that b is a real-valued locally integrable function in $\mathbb{R}^n$. Then the following assertions are equivalent:*

(i) $b \in BMO(\mathbb{R}^n)$ such that $b^- \in L_\infty(\mathbb{R}^n)$.

(ii) The operator $[b,M]$ is bounded on $L_{\vec{p},\lambda}(\mathbb{R}^n)$.

Corollary 11. *Let* $1 < \vec{p} < \infty$ *and* $0 \le \lambda \le n$. *Assume that* b *is a real-valued locally integrable function in* $\mathbb{R}^n$. *Then the following assertions are equivalent:*

(*i*) $b \in BMO(\mathbb{R}^n)$ *such that* $b^- \in L_\infty(\mathbb{R}^n)$.

(*ii*) *The operator* $[b, M]$ *is bounded on* $\widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$.

Remark 4. *Note that in the case* $\vec{p} = (p, \ldots, p)$ *Theorem 5 was proven [13, Theorem 4].*

5. Conclusions

In this paper we introduce the total mixed Morrey spaces $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$. These spaces generalize the mixed Lebesgue spaces so that $L_{\vec{p},0,0}(\mathbb{R}^n) \equiv L_{\vec{p}}(\mathbb{R}^n)$, the mixed Morrey spaces so that $L_{\vec{p},\lambda,\lambda}(\mathbb{R}^n) \equiv L_{\vec{p},\lambda}(\mathbb{R}^n)$ and the modified mixed Morrey spaces so that $L_{\vec{p},\lambda,0}(\mathbb{R}^n) = \widetilde{L}_{\vec{p},\lambda}(\mathbb{R}^n)$. We give basic properties of the spaces $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ and study some embeddings into the Morrey space $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$. We obtain necessary and sufficient conditions for the boundedness of the maximal commutator operator M_b and commutator of maximal operator $[b, M]$ on $L_{\vec{p},\lambda,\mu}(\mathbb{R}^n)$. Using the boundedness results we obtain some new characterizations for certain subclasses of $BMO(\mathbb{R}^n)$.

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