Multi-sublinear rough maximal operator on product total Morrey-Guliyev spaces

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Abstract. In this paper, we study the boundedness of the multi-sublinear maximal operator $M_{\Omega,m}$ with rough kernels $\Omega \in L^s(\mathbb{S}^{n-1})$, $1 < s \leq \infty$ on product total Morrey-Guliyev spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$. We obtain the boundedness of the operators $M_{\Omega,m}$ from product total Morrey-Guliyev spaces $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to total Morrey-Guliyev spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$.

Key Words and Phrases: total Morrey-Guliyev spaces, multi-sublinear maximal function, rough kenel

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1. Introduction

The classical Morrey spaces, introduced by Morrey [25] in 1938, have been studied intensively by various authors and together with weighted Lebesgue spaces play an important role in the theory of partial differential equations. They appeared to be quite useful in the study of local behavior of the solutions of elliptic differential equations and describe local regularity more precisely than Lebesgue spaces. See, for example, [7, 8, 9] for details. The boundedness of fractional integral operators on the classical Morrey spaces was studied by Adams [3], Chiarenza and Frasca [6], see also [9, 29]. In [6], by establishing a pointwise estimate of fractional integrals in terms of the Hardy-Littlewood maximal function, they showed the boundedness of fractional integral operators on the Morrey spaces, see also [29].

The total Morrey-Guliyev spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$, introduced by Guliyev [11], extend the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ by including the second parameter μ, which can be seen as the intermediate spaces between Lebesgue spaces and Morrey spaces. The norm in these spaces is defined by a combination of the norms of $L^{p,\lambda}(\mathbb{R}^n)$

and $L^{p,\mu}(\mathbb{R}^n)$, which allows a wider range of behavior. Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. The total Morrey-Guliyev spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$ are the set of all locally integrable functions f with the finite (quasi-)norm

$$\|f\|_{L^{p,\lambda,\mu}} = \sup_{x \in \mathbb{R}^n, \, t > 0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L^p(B(x,t))},$$

where $B(x, t)$ denotes the ball centered at x with radius $t > 0$. Here the norm in the case $\mu \leq \lambda$ is equal to the maximum of the norms of $L^{p,\lambda}(\mathbb{R}^n)$ and $L^{p,\mu}(\mathbb{R}^n)$. Total Morrey-Guliyev spaces can be viewed as generalizations of both classical and modified Morrey spaces. In particular, the case where $\lambda = \mu$ corresponds to classical Morrey space, and the case where $\mu = 0$ corresponds to modified Morrey space $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$, see [5, 10, 12, 13, 14, 15, 16, 17, 21, 22, 31, 32, 33].

Let $1 < s \leq \infty$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$. The multi-sublinear maximal operator $M_{\Omega,m}$ with rough kernels Ω is defined by

$$M_{\Omega,m}(\overrightarrow{f})(x) = \sup_{r > 0} \frac{1}{r^{nm}} \int_{B(\overrightarrow{y}, r)} |\Omega(\overrightarrow{y})| \prod_{j=1}^{m} |f_i(x - y_i)| d\overrightarrow{y}.$$

If $m = 1$, then $M_\Omega \equiv M_{\Omega,1}$ is the maximal operator with rough kernel Ω. When $m = 1$ and $\Omega \equiv 1$, then $M \equiv M_{1,1}$ is the classical Hardy-Littlewood maximal operator.

In this work, we prove the boundedness of the multi-sublinear maximal operator with rough kernels $M_{\Omega,m}$ from product total Morrey-Guliyev space $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to $L^{p,\lambda,\mu}(\mathbb{R}^n)$, if $p > s'$, $1 < p_1, \ldots, p_m < \infty$, $1/p = 1/p_1 + \ldots + 1/p_m$ and from the space $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to the weak space $L^{p,\lambda,\mu}(\mathbb{R}^n)$, if $p = s'$, $1 \leq p_1, \ldots, p_m < \infty$ and $1/p = 1/p_1 + \ldots + 1/p_m$ and at least one exponent p_i, $1 \leq i \leq m$ equals one, see [1, 2, 4, 19, 23, 24, 27, 28, 30].

Throughout this paper, we assume the letter C always remains to denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $(\mathbb{R}^n)^m = \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ be the m-fold product space ($m \in \mathbb{N}$). For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x of radius r, and by $^\complement B(x, r)$ denote its complement. Let $|B(x, r)|$ be the Lebesgue measure of the ball $B(x, r)$. Also for

$\overrightarrow{x} = (x_1, \ldots, x_m) \in \mathbb{R}^{mn}$ and $r > 0$, we denote by $B(\overrightarrow{x}, r)$ the open ball centered at $\overrightarrow{x}$ of radius r, and $B(\overrightarrow{x}, r)$ We denote by $\overrightarrow{f}$ the m-tuple $(f_1, f_2, \ldots, f_m)$, $\overrightarrow{y} = (y_1, \ldots, y_m)$ and $d\overrightarrow{y} = dy_1 \cdots dy_n$.

Definition 1. *Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L^{p,\lambda}(\mathbb{R}^n)$ the classical Morrey space, by $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ the modified Morrey space [10], and by $L^{p,\lambda,\mu}(\mathbb{R}^n)$ the total Morrey-Guliyev space [11] the set of all classes of locally integrable functions f with the finite norms*

$$\|f\|_{L^{p,\lambda}} = \sup_{x \in \mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,t))},$$

$$\|f\|_{\widetilde{L}^{p,\lambda}} = \sup_{x \in \mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,t))},$$

$$\|f\|_{L^{p,\lambda,\mu}} = \sup_{x \in \mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L^p(B(x,t))},$$

respectively.

Definition 2. *Let $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak Morrey space $WL^{p,\lambda}(\mathbb{R}^n)$, the weak modified Morrey space $W\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ [10] and the weak total Morrey-Guliyev space $WL^{p,\lambda,\mu}(\mathbb{R}^n)$ [11] as the set of all locally integrable functions f with finite norms*

$$\|f\|_{WL^{p,\lambda}} = \sup_{x \in \mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{WL^p(B(x,t))},$$

$$\|f\|_{W\widetilde{L}^{p,\lambda}} = \sup_{x \in \mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{WL^p(B(x,t))},$$

$$\|f\|_{WL^{p,\lambda,\mu}} = \sup_{x \in \mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{WL^p(B(x,t))},$$

respectively.

Note that

$$L^{p,0,0}(\mathbb{R}^n) = \widetilde{L}^{p,0}(\mathbb{R}^n) = L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n),$$

$$WL^{p,0,0}(\mathbb{R}^n) = W\widetilde{L}^{p,0}(\mathbb{R}^n) = WL^{p,0}(\mathbb{R}^n) = WL^p(\mathbb{R}^n),$$

$$L^{p,\lambda,\lambda}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n), \quad L^{p,\lambda,0}(\mathbb{R}^n) = \widetilde{L}^{p,\lambda}(\mathbb{R}^n),$$

$$\|f\|_{WL^{p,\lambda,\mu}} \le \|f\|_{L^{p,\lambda,\mu}} \text{ and therefore } L^{p,\lambda,\mu} \subset WL^{p,\lambda,\mu}(\mathbb{R}^n)$$

and

$$L^{p,\lambda,\mu}(\mathbb{R}^n) \subset L^{p,\lambda}(\mathbb{R}^n) \text{ and } \|f\|_{L^{p,\lambda}} \le \|f\|_{L^{p,\lambda,\mu}},$$

$$L^{p,\lambda,\mu}(\mathbb{R}^n) \subset L^{p,\mu}(\mathbb{R}^n) \text{ and } \|f\|_{L^{p,\mu}} \le \|f\|_{L^{p,\lambda,\mu}}$$

$$\widetilde{L}^{p,\lambda}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \text{ and } \|f\|_{L^p} \le \|f\|_{\widetilde{L}^{p,\lambda}}$$

and if $\lambda < 0$ or $\lambda > n$, then $L^{p,\lambda}(\mathbb{R}^n) = \widetilde{L}^{p,\lambda}(\mathbb{R}^n) = WL^{p,\lambda}(\mathbb{R}^n) = W\widetilde{L}^{p,\lambda}(\mathbb{R}^n) = \Theta$, where $\Theta \equiv \Theta(\mathbb{R}^n)$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Lemma 1. *[11] If $0 < p < \infty$, $0 \le \lambda \le n$ and $0 \le \mu \le n$, then for $1 \le p < \infty$, $L^{p,\lambda,\mu}(\mathbb{R}^n)$ is a Banach space and for $0 < p < 1$, $L^{p,\lambda,\mu}(\mathbb{R}^n)$ is a quasi-Banach space.*

Lemma 2. *[11, 13] If $0 < p < \infty$, $0 \le \lambda \le n$ and $0 \le \mu \le n$, then*

$$L^{p,\lambda,\mu}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n) \cap L^{p,\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^{p,\mu}}\right\}.$$

Corollary 1. *If $0 < p < \infty$, $0 \le \lambda \le n$, then*

$$\widetilde{L}^{p,\lambda}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$$

and

$$\|f\|_{\widetilde{L}^{p,\lambda}} = \max\left\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^p}\right\}.$$

Lemma 3. *[11, 13] If $0 < p < \infty$, $0 \le \lambda \le n$ and $0 \le \mu \le n$, then*

$$WL^{p,\lambda,\mu}(\mathbb{R}^n) = WL^{p,\lambda}(\mathbb{R}^n) \cap WL^{p,\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{WL^{p,\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{WL^{p,\lambda}}, \|f\|_{WL^{p,\mu}}\right\}.$$

Remark 1. *If $0 < p < \infty$, and $\lambda < 0$ or $\lambda > n$ or $\mu < 0$ or $\mu > n$, then*

$$L^{p,\lambda,\mu}(\mathbb{R}^n) = WL^{p,\lambda,\mu}(\mathbb{R}^n) = \Theta(\mathbb{R}^n).$$

3. Boundedness of multi-sublinear maximal operator $M_{\Omega,m}$ on product total Morrey-Guliyev spaces

In this part, we investigate the boundedness of multi-sublinear maximal operator $M_{\Omega,m}$ on product total Morrey-Guliyev spaces.

The boundedness of Hardy-Littlewood maximal operator on the total Morrey-Guliyev spaces was studied by author [11]. Their results can be summarized as follows.

Theorem 1. *[11] 1. If $f \in L^{1,\lambda,\mu}(\mathbb{R}^n)$, $0 \le \lambda < n$ and $0 \le \mu < n$, then $Mf \in WL^{1,\lambda,\mu}(\mathbb{R}^n)$ and*

$$\|Mf\|_{WL^{1,\lambda,\mu}} \le C_{1,\lambda,\mu}\|f\|_{L^{1,\lambda,\mu}}, \tag{1}$$

where $C_{1,\lambda,\mu}$ is independent of f.

*   2. If $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$, $1 < p < \infty$, $0 \le \lambda < n$ and $0 \le \mu < n$, then $Mf \in L^{p,\lambda,\mu}(\mathbb{R}^n)$ and*

$$\|Mf\|_{L^{p,\lambda,\mu}} \le C_{p,\lambda,\mu}\|f\|_{L^{p,\lambda,\mu}}, \tag{2}$$

where $C_{p,\lambda,\mu}$ depends only on p,λ,μ and n.

Lemma 4. *[18] Let $1 < s \le \infty$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$, p be the harmonic mean of $p_1,\ldots,p_m > 1$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \times \ldots \times L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$*

$$M_{\Omega,m}\boldsymbol{f}(x) \le C_0 \prod_{j=1}^m \left[M\big(f_j^{\frac{p}{p}}\big)(x) \right]^{\frac{p}{s'p_j}}, \tag{3}$$

where $C_0 = (mn)^{-\frac{1}{s}}\|\Omega\|_{L_s(\mathbb{S}^{mn-1})}$.

When $m \ge 2$ and $\Omega \in L^s(\mathbb{S}^{mn-1})$, we find out $M_{\Omega,m}$ also have the same properties by providing the following multi-version of the Theorem 1.

Theorem 2. *Let $1 < s \le \infty$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$, p be the harmonic mean of $p_1,\ldots,p_m > 1$ and*

$$\frac{\lambda}{p} = \sum_{j=1}^m \frac{\lambda_j}{p_j} \quad \text{for } 0 \le \lambda_j < n$$

$$\frac{\mu}{p} = \sum_{j=1}^m \frac{\mu_j}{p_j} \quad \text{for } 0 \le \mu_j < n. \tag{4}$$

(i) If $p > s'$, then the operator $M_{\Omega,m}$ is bounded from product total Morrey-Guliyev space $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to total Morrey-Guliyev space $L^{p,\lambda,\mu}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\boldsymbol{f} \in L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$

$$\|M_{\Omega,m}\boldsymbol{f}\|_{L^{p,\lambda,\mu}} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j,\mu_j}}.$$

(ii) *If $p = s'$, then the operator $M_{\Omega,m}$ is bounded from product total Morrey-Guliyev space $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to weak total Morrey-Guliyev space $WL^{p,\lambda,\mu}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\boldsymbol{f} \in L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$*

$$\|M_{\Omega,m}\boldsymbol{f}\|_{WL^{p,\lambda,\mu}} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}}.$$

Proof.

(i) If $p > s'$, by (3) and the Hölder's inequality, we get

$$\left([t]_1^{-\lambda} [1/t]_1^{\mu} \int_{B(x,t)} |M_{\Omega,m}\boldsymbol{f}(y)|^p dy\right)^{\frac{1}{p}}$$

$$\leq C_0\left([t]_1^{-\lambda} [1/t]_1^{\mu} \int_{B(x,t)} \Big| \prod_{j=1}^{m} \Big[M(f_j^{\frac{s'p_j}{p}})(y)\Big]^{\frac{p}{s'p_j}} \Big|^p dy\right)^{\frac{1}{p}}$$

$$\leq C_0 \prod_{j=1}^{m} \left(\frac{1}{[t]_1^{\lambda_j}} \int_{B(x,t)} \Big[M(f_j^{\frac{s'p_j}{p}})(y)\Big]^{\frac{p}{s'}} dy\right)^{\frac{1}{p_j}}.$$

Taking the p-th root of both sides and applying Theorem 1 with $p/s' > 1$ and $\big|f_j\big|^{\frac{s'p_j}{p}} \in L^{\frac{p}{s'},\lambda_j,\mu_j}(\mathbb{R}^n)$, we get

$$\|M_{\Omega,m}\boldsymbol{f}\|_{L^{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n, t>0} \left([t]_1^{-\lambda} [1/t]_1^{\mu} \int_{B(x,t)} |M_{\Omega,m}\boldsymbol{f}(y)|^p dy\right)^{\frac{1}{p}}$$

$$= C_0 \prod_{j=1}^{m} \Big\|M\big(f_j^{\frac{s'p_j}{p}}\big)\Big\|_{L^{\frac{p}{s'},\lambda_j,\mu_j}}^{\frac{1}{s'}\frac{p_j}{p_j}}$$

$$\leq C \prod_{j=1}^{m} \Big\|f_j^{\frac{s'p_j}{p}}\Big\|_{L^{\frac{p}{s'},\lambda_j,\mu_j}}^{\frac{1}{s'}\frac{p_j}{p_j}} = C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}},$$

which is the desired inequality.

(ii) If $p = s'$, for any $\tau > 0$, let $\varepsilon_0 = \tau$, $\varepsilon_m = 1$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1} > 0$ be arbitrary which will be chosen later. From the pointwise estimate (3), we get

$$\{y \in B(x,t) : |M_{\Omega,m}\boldsymbol{f}(y)| > \tau\}$$

$$\subset \bigcup_{j=1}^{m} \left\{ y \in B(x,t) : \left[M(f_j^{\frac{s'p_j}{p}})(y) \right]^{\frac{p}{s'p_j}} > \frac{\varepsilon_{j-1}}{\varepsilon_j \, [t]_1^{\frac{\lambda-\lambda_j}{p_j}} \, [1/t]_1^{-\frac{\mu-\mu_j}{p_j}}} \right\}.$$

Let us now take $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{m-1} > 0$ such that

$$\frac{\varepsilon_j}{\varepsilon_{j-1}} = \frac{\left[\prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}} \right]^{s'/p_j}}{\tau^{s'/p_j} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}}}, \qquad j = 1, 2, ..., m.$$

Then, applying Theorem 1 with $p/s' = 1$ and the fact $\left| f_j \right|^{p_j} \in L^{1,\lambda_j,\mu_j}(\mathbb{R}^n)$, we get

$$\left| \left\{ y \in B(x,t) : M_{\Omega,m}\mathbf{f}(y) \right| > \tau \right\} \right|$$

$$\leq C \sum_{j=1}^{m} \left| \left\{ y \in B(x,t) : M(f_j^{p_j})(y) > \left(\frac{\varepsilon_{j-1}}{\varepsilon_j \, [t]_1^{\frac{\lambda-\lambda_j}{p_j}} \, [1/t]_1^{-\frac{\mu-\mu_j}{p_j}}} \right)^{p_j} \right\} \right|$$

$$\leq C \sum_{j=1}^{m} [t]_1^{\lambda_j} \, [1/t]_1^{-\mu_j} \left(\frac{\varepsilon_j \, [t]_1^{\frac{\lambda-\lambda_j}{p_j}} \, [1/t]_1^{-\frac{\mu-\mu_j}{p_j}}}{\varepsilon_{j-1}} \right)^{p_j} \|f_j^{p_j}\|_{L^{1,\lambda_j,\mu_j}}$$

$$= C \sum_{j=1}^{m} [t]_1^{\lambda} \, [1/t]_1^{-\mu} \left(\frac{\varepsilon_j}{\varepsilon_{j-1}} \right)^{p_j} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}}^{p_j}$$

$$= C \sum_{j=1}^{m} [t]_1^{\lambda} \, [1/t]_1^{-\mu} \left[\left(\frac{\varepsilon_j}{\varepsilon_{j-1}} \right) \|f_j\|_{L^{p_j,\lambda_j,\mu_j}} \right]^{p_j}$$

$$= C \sum_{j=1}^{m} [t]_1^{\lambda} \, [1/t]_1^{-\mu} \left(\frac{1}{\tau} \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}} \right)^{s'}$$

$$= C[t]_1^{\lambda} \, [1/t]_1^{-\mu} \left(\frac{1}{\tau} \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}} \right)^{p}.$$

Hence, we obtain the following inequality

$$\|M_{\Omega,m}\mathbf{f}\|_{WL^{p,\lambda,\mu}}$$

$$= \sup_{\tau>0} \tau \sup_{x \in \mathbb{R}^n, t>0} \left([t]_1^{-\lambda} \, [1/t]_1^{\mu} \left| \left\{ y \in B(x,t) : |M_{\Omega,\alpha,m}\mathbf{f}(y)| > \tau \right\} \right| \right)^{\frac{1}{p}}$$

$$\leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}}.$$

This is the conclusion (ii) of Theorem 2.

In the case $\lambda = \mu$, $\lambda_j = \mu_j$, $j = 1, \ldots, n$ from Theorem 2 we get the following corollary

Corollary 2. *[18] Let $1 < s \leq \infty$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$, p be the harmonic mean of $p_1, \ldots, p_m > 1$ and*

$$\frac{\lambda}{p} = \sum_{j=1}^m \frac{\lambda_j}{p_j} \quad for \ \ 0 \leq \lambda_j < n. \tag{5}$$

(i) If $p > s'$, then the operator $M_{\Omega,m}$ is bounded from product total Morrey-Guliyev space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to total Morrey-Guliyev space $L^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|M_{\Omega,m}\boldsymbol{f}\|_{L^{p,\lambda}} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}.$$

(ii) If $p = s'$, then the operator $M_{\Omega,m}$ is bounded from product total Morrey-Guliyev space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to weak total Morrey-Guliyev space $WL^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|M_{\Omega,m}\boldsymbol{f}\|_{WL^{p,\lambda}} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}.$$

In the case $\mu = \mu_j = 0$, $j = 1, \ldots, n$ from Theorem 2 we get the following corollary

Corollary 3. *[18] Let $1 < s \leq \infty$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$, p be the harmonic mean of $p_1, \ldots, p_m > 1$ and satisfy (5).*

(i) If $p > s'$, then the operator $\mathcal{M}_{\Omega,m}$ is bounded from product modified Morrey space $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to modified Morrey space $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\boldsymbol{f} \in \widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}_{\Omega,m}\boldsymbol{f}\|_{\widetilde{L}^{p,\lambda}} \leq C \prod_{j=1}^m \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

(ii) If $p = s'$, then the operator $\mathcal{M}_{\Omega,m}$ is bounded from product modified Morrey space $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to weak modified Morrey space $W\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\boldsymbol{f} \in \widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}_{\Omega,m}\boldsymbol{f}\|_{W\widetilde{L}^{p,\lambda}} \leq C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

In the case $s = \infty$, $\Omega \in L^{\infty}(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$, from Theorem 2 we get the following corollary

Corollary 4. *[20, Theorem 2.2] Let p be the harmonic mean of $p_1, \ldots, p_m > 1$ and*

$$\frac{\lambda}{p} = \sum_{j=1}^{m} \frac{\lambda_j}{p_j} \ \ for \ \ 0 \leq \lambda_j < n, \frac{\mu}{p} = \sum_{j=1}^{m} \frac{\mu_j}{p_j} \ \ for \ \ 0 \leq \mu_j < n. \qquad (6)$$

(i) If $p > 1$, then the operator $\mathcal{M}$ is bounded from product total Morrey space $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to total Morrey space $L^{p,\lambda,\mu}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\vec{f} \in L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$

$$\|\mathcal{M}\vec{f}\|_{L^{p,\lambda,\mu}} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}}.$$

(ii) If $p = 1$, then the operator $\mathcal{M}$ is bounded from product total Morrey space $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to weak total Morrey space $WL^{p,\lambda,\mu}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\vec{f} \in L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$

$$\|\mathcal{M}\vec{f}\|_{WL^{p,\lambda,\mu}} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}}.$$

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