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Results on Grundy Chromatic Number of Comb Product of Some Graphs

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Abstract. This work examines the Grundy chromatic number (First-Fit chromatic number) of comb product graphs constructed from standard graph classes such as paths, cycles, and complete graphs. Exact values are derived for combinations including $P_q \circ P_{q'}$, $P_q \circ K_t$, $K_t \circ K_t'$, $C_r \circ C_{r'}$, and $C_r \circ P_q$. The results highlight how base graph interactions impact greedy coloring strategies.

Key Words and Phrases: Grundy chromatic number, Comb graph, Path, cycle, complete graph.

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1. Introduction

Graph coloring is a core area in graph theory due to its theoretical richness and practical applications in diverse domains such as job scheduling, register allocation in compilers, channel assignment, and resource management in distributed networks [11, 9, 8]. In particular, *vertex coloring*, which involves assigning colors to vertices such that no two adjacent vertices receive the same color, remains a classical and well-studied problem.

Among the many variants of vertex coloring, the *Grundy chromatic number* (or First-Fit chromatic number), denoted $\Gamma(G)$ for a graph G, plays a central role in understanding the worst-case behavior of greedy coloring algorithms. The First-Fit algorithm colors vertices in a specified order, assigning each vertex the smallest positive integer not used by its colored neighbors. The Grundy chromatic number is the maximum number of colors that the First-Fit algorithm may use over all possible orderings of the vertices [13, 12, 3].

Unlike the chromatic number $\chi(G)$, which minimizes the number of colors needed, the Grundy number captures the worst-case scenario in online coloring settings. The study of Grundy numbers has been extended to several graph families and operations, such as Cartesian products [4], lexicographic products [10], and total graphs [7]. However, relatively little attention has been paid to the behavior of Grundy numbers in the context of *comb product graphs*, also known as *rooted product graphs*.

The *comb product*, originally defined by Godsil and McKay [6], constructs a new graph $G \circ H$ by taking one copy of a rooted graph H for each vertex v of a base graph G, and identifying the root of each copy with the corresponding vertex of G. This operation has proven valuable in exploring the structural properties of complex graph constructions [5]. Prior studies have focused primarily on chromatic and equitable coloring properties of such graphs [2], particularly when G and H are classical graphs like paths, cycles, or complete graphs.

In this paper, we aim to extend the investigation of Grundy chromatic numbers to comb product graphs of the form $P_q \circ P_{q'}$, $P_q \circ K_t$, $K_t \circ K_t'$, $C_r \circ C_{r'}$, and $C_r \circ P_q$. We seek to determine exact values or bounds for $\Gamma(G \circ H)$ in these settings and to identify how the structure of G and H influences the greedy coloring outcome.

Our motivation stems from recent work by Barani and Venkatachalam [2], who studied equitable colorings of comb product graphs but left open the question of its behavior under greedy strategies. By analyzing the interplay between local density (as induced by H) and global connectivity (as provided by G), we uncover new insights into the extremal behavior of Grundy numbers in hierarchical graph constructions.

2. Preliminaries

Let $G = (V, E)$ be a finite, simple, undirected graph. A proper coloring of G is an assignment of colors to vertices such that no two adjacent vertices share the same color. The *chromatic number* $\chi(G)$ is the minimum number of colors needed for a proper coloring.

The *Grundy chromatic number* $\Gamma(G)$ is defined as the largest number of colors that can appear when the First-Fit coloring algorithm is applied to G over all permutations of its vertex set [13]. Formally, for a vertex ordering $\pi = (v_1, v_2, \ldots, v_n)$, the First-Fit algorithm assigns to v_i the least positive integer not already used by its neighbors that occur earlier in the ordering. The maximum number of colors that can be used over all such orderings is $\Gamma(G)$.

Some known results [12, 12, 11]for classical graphs include:

- $\Gamma(P_n) = 3$ for $n \geq 4$.
- $\Gamma(C_n) = 3$ for $n \geq 4$.

- $\Gamma(K_n) = n$.

We now define the comb product formally. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, and let H be a graph with a distinguished root vertex r. The *comb product* $G \circ H$ is constructed by taking n disjoint copies of H, say $H_1, H_2, \ldots, H_n$, and identifying the root of H_i with the vertex v_i of G. The resulting graph inherits the structure of G at the root vertices, with local neighborhoods expanded by the structure of H.

Comb product graphs introduce an intricate balance between global graph structure and localized complexity, making them ideal candidates for studying coloring parameters like the Grundy number. Prior work has determined chromatic [5], equitable [2], and total colorings [1] in similar settings. However, the behavior of the First-Fit algorithm in such hierarchical constructions is largely uncharted.

This paper addresses this gap by providing exact values for $\Gamma(G \circ H)$ where $G, H \in \{P_n, C_n, K_n\}$ and presenting detailed combinatorial proofs for each case.

The *Grundy chromatic number* $\Gamma(G)$ of a graph G is the largest number of colors that can be used by the First-Fit coloring algorithm over all possible vertex orderings. It represents the worst-case performance of greedy coloring and is always greater than or equal to the chromatic number $\chi(G)$. This section explores the Grundy chromatic number of comb product graphs formed using paths, cycles, and complete graphs.

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, and let H be a rooted graph with a fixed root $r \in V(H)$. The *comb product* $G \circ H$ is constructed by taking one copy of H for each vertex $v_i \in G$, and identifying the root r of the i-th copy of H with v_i.

3. Main Results

Theorem 1. *For all integers $q, q' \geq 4$, the Grundy chromatic number of the comb product $P_q \circ P_{q'}$ is given by*

$$\Gamma(P_q \circ P_{q'}) = 3. \tag{1}$$

Proof. Let P_q be the base path with vertex set

$$V(P_q) = \{v_1, v_2, \ldots, v_q\} \quad \text{and} \quad E(P_q) = \{v_i v_{i+1} \mid 1 \leq i \leq q-1\}.$$

Attach to each vertex $v_i \in V(P_q)$ a copy of $P_{q'}$ such that v_i acts as the root (first vertex) of the attached path. Denote the additional vertices in the i-th attached path as

$$\{u_{i,j} \mid 2 \leq j \leq q'\}, \quad \text{with } u_{i,1} := v_i.$$

Then the vertex set of the comb product is

$$V(P_q \circ P_{q'}) = \bigcup_{i=1}^{q} \big(\{v_i\} \cup \{u_{i,j} \mid 2 \leq j \leq q'\}\big). \tag{2}$$

The edge set consists of:

- the base path edges: $v_i v_{i+1}$ for $1 \leq i \leq q-1$,
- the attachment edges: $v_i u_{i,2}$ for $1 \leq i \leq q$,
- the internal path edges of each attached $P_{q'}$: $u_{i,j}u_{i,j+1}$ for $2 \leq j < q'$.

Now, we define a greedy First-Fit coloring function $f : V \to \mathbb{N}$.

Color vertices in each copy of $P_{q'}$ (excluding root) in the order $u_{i,q'}, u_{i,q'-1}, \ldots, u_{i,2}$ using the pattern:

$$f(u_{i,j}) = \begin{cases} 1 & \text{if } j \equiv 0 \pmod 3, \\ 2 & \text{if } j \equiv 1 \pmod 3, \\ 3 & \text{if } j \equiv 2 \pmod 3. \end{cases}$$

Then, color the root v_i after all its neighbors using the lowest available color. Since each v_i is connected to at most 3 differently colored neighbors (in path and in its copy), it will get a color from $\{1, 2, 3\}$.

We now show that the maximum number of colors used under this ordering is exactly 3. The vertices $u_{i,j}$ require at most 3 colors in the greedy sequence since each has at most 2 neighbors (being part of a path). Similarly, each v_i connects to:

- Two vertices from the base path (v_{i-1}, v_{i+1}),
- One vertex $u_{i,2}$ from the attached path.

Thus, in the worst-case ordering (where all neighbors are colored first), v_i sees at most 3 distinct colors, forcing $\Gamma(P_q \circ P_{q'}) \leq 3$.

To show $\Gamma \geq 3$, we consider a worst-case ordering where $u_{i,j}$ are colored before v_i, and $q, q' \geq 4$ ensures such a configuration exists with 3 colors appearing on the neighbors of v_i.

Therefore,

$$\Gamma(P_q \circ P_{q'}) = 3.$$

Theorem 2. *For $q \geq 2$, $t \geq 2$, the Grundy chromatic number of the comb product $P_q \circ K_t$ is:*

$$\Gamma(P_q \circ K_t) = t + 1. \tag{3}$$

Proof.

Let P_q be the base path graph with vertex set and edge set defined as

$$V(P_q) = \{v_1, v_2, \ldots, v_q\}, \quad E(P_q) = \{v_i v_{i+1} \mid 1 \leq i < q\}.$$

Let K_t denote a complete graph, where each vertex $v_i \in V(P_q)$ serves as the root of a copy of K_t by identifying $v_i \equiv u_{i,1}$. The remaining vertices in the i-th copy are denoted by $u_{i,2}, \ldots, u_{i,t}$.

Thus, the vertex set of the comb product $P_q \circ K_t$ is given by:

$$V(P_q \circ K_t) = \bigcup_{i=1}^{q} \big(\{v_i\} \cup \{u_{i,j} \mid 2 \leq j \leq t\}\big). \tag{4}$$

Each vertex v_i is adjacent to:

- $t - 1$ vertices from the i-th copy of K_t, namely $u_{i,2}, \ldots, u_{i,t}$;
- Up to two neighboring vertices in the base path P_q, specifically v_{i-1} and v_{i+1}, if they exist.

We define a First-Fit (greedy) coloring algorithm that assigns colors based on a carefully chosen vertex ordering.

- For each $i = 1$ to q, first color all vertices of K_t except the root v_i (i.e., $u_{i,2}, \ldots, u_{i,t}$).
- Then, color the root v_i.

Since K_t is complete, the $t-1$ non-root vertices $u_{i,2}, \ldots, u_{i,t}$ are mutually adjacent and require $t - 1$ distinct colors. The root v_i is adjacent to all these vertices and may additionally be adjacent to up to two previously colored path neighbors (e.g., v_{i-1} and/or v_{i+1}).

Thus, v_i may be adjacent to up to t different colors when it's its turn to be colored.

$$f(u_{i,j}) = j - 1, \quad \text{for } j = 2, 3, \ldots, t,$$
$$f(v_i) = t + 1.$$

In the worst-case First-Fit ordering, the root vertex v_i may be adjacent to:

- $t-1$ distinct colors from its own K_t copy,
- up to 2 additional colors from adjacent base vertices.

Therefore, the maximum color needed is $t+1$. Since this coloring is achievable under a specific ordering, and since no vertex has degree more than $t+1$, the Grundy number is:

$$\Gamma(P_q \circ K_t) = t + 1.$$

Theorem 3. *For $t, t' \geq 2$, the Grundy chromatic number of the comb product $K_t \circ K_{t'}$ is:*

$$\Gamma(K_t \circ K_{t'}) = t \cdot t'. \tag{5}$$

Proof. Let K_t be the base complete graph with vertex set

$$V(K_t) = \{v_1, v_2, \ldots, v_t\}.$$

For each vertex $v_i \in V(K_t)$, attach a copy of the complete graph $K_{t'}$ such that v_i serves as the root. Denote the vertex set of the i-th copy as

$$V_i = \{u_{i,j} \mid 1 \leq j \leq t'\}, \quad \text{with } u_{i,1} := v_i.$$

Then the total vertex set of the comb product $K_t \circ K_{t'}$ is

$$V(K_t \circ K_{t'}) = \bigcup_{i=1}^{t} V_i = \bigcup_{i=1}^{t} \{u_{i,1}, u_{i,2}, \ldots, u_{i,t'}\}. \tag{6}$$

Each set V_i induces a complete subgraph isomorphic to $K_{t'}$. Additionally, the root vertices $\{u_{1,1}, u_{2,1}, \ldots, u_{t,1}\}$—which are the original vertices of K_t—form a complete graph among themselves.

We define a First-Fit coloring based on a specific vertex ordering:

- For each $i = 1$ to t, color all the non-root vertices of the i-th copy of $K_{t'}$, i.e., $\{u_{i,2}, \ldots, u_{i,t'}\}$.
- Then, color the root vertices $\{u_{1,1}, u_{2,1}, \ldots, u_{t,1}\}$ last, in any order.

Since each $K_{t'}$ is complete, its t' vertices need t' distinct colors under any ordering. So, in each copy, the non-root vertices will receive colors 1 through $t'-1$, and the root $u_{i,1}$ can receive color t'.

However, since the root vertices form a complete graph K_t, and each root $u_{i,1}$ is already adjacent to $t'-1$ colored vertices in its own copy, placing the root vertices at the end forces the final root to be adjacent to:

- $(t-1)$ already colored root vertices (from K_t),
- $(t'-1)$ vertices in its own $K_{t'}$ copy.

Thus, the last root vertex is adjacent to $(t-1) \cdot t' + (t'-1)$ colored vertices in the worst-case. This totals to $t \cdot t' - 1$, requiring $t \cdot t'$ colors.

Since the First-Fit coloring algorithm assigns the smallest available color not used by neighbors, and the maximum degree any vertex can have is $t \cdot t' - 1$, the worst-case coloring requires exactly $t \cdot t'$ colors.

Hence, we conclude:

$$\Gamma(K_t \circ K_{t'}) = t \cdot t'.$$

Theorem 4. *For cycles C_r and $C_{r'}$ with $r, r' \geq 4$, the Grundy chromatic number of the comb product $C_r \circ C_{r'}$ is:*

$$\Gamma(C_r \circ C_{r'}) = 4. \tag{7}$$

Proof. Let C_r be the base cycle with vertex set

$$V(C_r) = \{v_1, v_2, \dots, v_r\} \quad \text{and edge set} \quad E(C_r) = \{v_i v_{i+1} \mid 1 \leq i < r\} \cup \{v_r v_1\}.$$

To each vertex $v_i \in V(C_r)$, attach a copy of the cycle $C_{r'}$, with vertex set

$$V_i = \{u_{i,j} \mid 1 \leq j \leq r'\}, \quad \text{where } u_{i,1} := v_i \text{ (the root)}.$$

The total vertex set of the comb product $C_r \circ C_{r'}$ is then

$$V(C_r \circ C_{r'}) = \bigcup_{i=1}^{r} V_i = \bigcup_{i=1}^{r} \{u_{i,1}, u_{i,2}, \dots, u_{i,r'}\}. \tag{8}$$

Each subgraph V_i induces a cycle isomorphic to $C_{r'}$, and the root vertices $\{u_{1,1}, u_{2,1}, \dots, u_{r,1}\}$, which correspond to $V(C_r)$, form the base cycle C_r.

We apply a First-Fit (greedy) coloring with a specific vertex ordering:

- For each $i = 1$ to r, color the vertices $u_{i,2}, \dots, u_{i,r'}$ first (i.e., all of $C_{r'}$ except the root).
- Then, color the root vertex $u_{i,1} = v_i$.

Each $C_{r'}$ with $r' \geq 4$ is a cycle, requiring at most 3 colors under greedy coloring. So, the vertices $u_{i,2}, \dots, u_{i,r'}$ will be colored using at most 3 colors.

When we reach the root vertex v_i, it is adjacent to:

- Two neighbors in the base cycle C_r: v_{i-1} and v_{i+1},
- Two neighbors in its own $C_{r'}$ copy: $u_{i,2}$ and $u_{i,r'}$.

In the worst-case vertex ordering, all these neighbors may have distinct colors. Therefore, v_i could be adjacent to 3 already-used colors.

Under the First-Fit strategy, v_i must receive a color not used by any of its neighbors. If its 3 neighbors (from attached cycle) and the two adjacent base-cycle vertices together cover 3 distinct colors, then v_i will need a 4th color.

Hence, the maximum number of colors required in this worst-case ordering is:

$$\Gamma(C_r \circ C_{r'}) = 4.$$

Theorem 5. *For $r \geq 3$ and $q \geq 2$, the Grundy chromatic number of the comb product $C_r \circ P_q$ is:*

$$\Gamma(C_r \circ P_q) = 4. \tag{9}$$

Proof. Let C_r be the base cycle with vertex set

$$V(C_r) = \{v_1, v_2, \ldots, v_r\} \quad \text{and edge set} \quad E(C_r) = \{v_i v_{i+1} \mid 1 \leq i < r\} \cup \{v_r v_1\}.$$

To each vertex $v_i \in V(C_r)$, attach a copy of the path P_q, denoted by

$$V_i = \{u_{i,j} \mid 1 \leq j \leq q\}, \quad \text{with } u_{i,1} := v_i.$$

Each attached path P_q has edge set

$$E_i = \{u_{i,j} u_{i,j+1} \mid 1 \leq j < q\}.$$

The total vertex set of the comb product $C_r \circ P_q$ is

$$V(C_r \circ P_q) = \bigcup_{i=1}^{r} V_i = \bigcup_{i=1}^{r} \{u_{i,1}, u_{i,2}, \ldots, u_{i,q}\}, \tag{10}$$

where each $u_{i,1}$ corresponds to a vertex of the base cycle and serves as the root of the attached path.

We apply the First-Fit coloring algorithm with the following vertex ordering:

- For each $i = 1$ to r, color the path vertices $u_{i,q}, u_{i,q-1}, \ldots, u_{i,2}$ first.
- Then, color the root vertex $u_{i,1} = v_i$.

Each P_q (with $q \geq 4$) has Grundy number at most 3, so the path vertices can be colored using colors from $\{1, 2, 3\}$.

When coloring the root $v_i = u_{i,1}$, it is adjacent to:

- Two base-cycle neighbors: v_{i-1}, v_{i+1},
- One path neighbor: $u_{i,2}$.

In a worst-case ordering, all three neighbors may be assigned different colors from $\{1, 2, 3\}$.

Since each root vertex v_i (appearing last in the ordering) may be adjacent to three differently colored vertices, the First-Fit algorithm will require a new color, namely color 4.

Thus, the maximum number of colors used in this greedy coloring is:

$$\Gamma(C_r \circ P_q) = 4.$$

4. Conclusion

In this paper, we analyzed the Grundy chromatic number of several comb product graphs, combining paths, cycles, and complete graphs. For each graph pair, we derived exact values of the Grundy number by considering worst-case vertex orderings and leveraging structural properties. The behavior of Grundy numbers in comb products involving more complex graphs such as wheel, helm, or double star graphs. Comparative analysis of Grundy numbers under other graph products (e.g., lexicographic, tensor). Algorithmic approaches for computing Grundy numbers efficiently in comb structures. Such studies could deepen our understanding of online and greedy coloring processes in graph theory and combinatorial optimization.

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