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On Modified Bernstein–Kantorovich Type Operators on a Varying Symmetric Interval and Their Approximation Behaviour

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Abstract. In this study, we consider modified Bernstein–Kantorovich-type positive linear operators defined on a varying symmetric interval. First, we present the basic moment properties of these operators. Using these properties, we then investigate their approximation behaviour. By applying the Bohman–Korovkin theorem, we show that the operators converge uniformly to continuous functions. In addition, we obtain several approximation estimates via the modulus of continuity, the Lipschitz class, the second-order modulus of continuity, and Peetre's K-functional. Moreover, a Voronovskaja-type asymptotic result is established for these operators. The theoretical results are also illustrated by graphical examples.

Key Words and Phrases: Approximation theory, linear positive operators, modulus of continuity, Peetre's K-functional

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1. Introduction

A proof of the well-known Weierstrass approximation theorem [19] was given by Bernstein [2]. The simple and useful sequence of operators used by Bernstein in this proof is given as follows:

$$B_k(v,t) = \sum_{m=0}^{k} \binom{k}{m} t^m (1-t)^{k-m} v\left(\frac{m}{k}\right). \tag{1.1}$$

Bernstein polynomials, which are obtained from the binomial distribution, are used in many scientific fields because of their simple structure. After Bohman [3] and Korovkin [14] independently established important theorems with the same

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implication, research on Bernstein polynomials increased significantly. Many generalizations and modifications of these polynomials have been introduced and remain active research topics. One such modification was introduced by Kantorovich [13] for integrable (measurable) functions:

$$\begin{aligned} B_k^*(v,t) &= (k+1)\sum_{m=0}^{k}\binom{k}{m}t^m(1-t)^{k-m}\int_{\frac{m}{k+1}}^{\frac{m+1}{k+1}} v(u)\,du \\ &= (k+1)\sum_{m=0}^{k}\binom{k}{m}t^m(1-t)^{k-m}\int_0^1 v\left(\frac{m+u}{k+1}\right)\,du. \end{aligned} \tag{1.2}$$

Here, $t \in [0,1]$ and $v \in L_p[0,1]$.

Several studies related to (1.1) and (1.2) can be found in [6–13]. In 2012, Çilo [14] studied approximation properties of Bernstein polynomials on the symmetric interval $[-1,1]$. Related approximation and rate-of-approximation results were also presented by Çilo, Ural, and İzgi [15]. Later, in 2014, Kahvecibaşı [16] studied Bernstein–Kantorovich-type modifications on the symmetric interval.

$$C_k(v,t) = \sum_{m=0}^{k} r_k^m(t)v(2\frac{m}{k}-1) \tag{1.3}$$

where $r_{k,m}(t) = \frac{1}{2^k}\binom{k}{m}(1+t)^m(1\text{-}t)^{k-m}$, $t \in [-1,1]$.

$$C_k^*(v,t) = \frac{k+1}{2}\sum_{m=0}^{k} r_k^m(t)\int_{2\frac{m}{k+1}-1}^{2\frac{m+1}{k+1}-1} v(u)\,du. \tag{1.4}$$

Ural [18] studied the following polynomials in her master's thesis in 2012:

$$U_k(v,t) = \frac{k}{k-2}\sum_{m=0}^{k} p_k^m(t)v(\frac{(k-2)m+k}{k^2}\ ) \tag{1.5}$$

where $p_{k,m}(t) = \frac{1}{2^k}\binom{k}{m}(t-\frac{1}{k})^m(\frac{k-1}{k}-t)^{k-m}$, $t \in [\frac{1}{k},\frac{k-1}{k}]$ and $k \geq 3$.

Motivated by the above-mentioned studies, we define and investigate the following Bernstein–Kantorovich-type positive linear operator.

$$D_k(v,t) = \frac{k^2}{2(k+1)}\sum_{m=0}^{k} s_{k,m}(t)\int_{\left(2\frac{m}{k}-1\right)\frac{k+1}{k}}^{\left(2\frac{m+1}{k}-1\right)\frac{k+1}{k}} v(u)\,du. \tag{1.6}$$

where

$$s_{k,m}(t)=\binom{k}{m}\left(\frac{k}{2(k+1)}\right)^k\left(t+\frac{k+1}{k}\right)^m\left(\frac{k+1}{k}-t\right)^{k-m}\quad,t\in\left[-\frac{k+1}{k},\frac{k+1}{k}\right].$$

For $k\geq 1$, let

$$I_k=\left[-\frac{k+1}{k},\frac{k+1}{k}\right].$$

Then

$$\bigcup_{k\geq 1} I_k=[-2,2],\qquad \bigcap_{k\geq 1} I_k=[-1,1].$$

Throughout the paper, the functions on which D_k acts are assumed to be defined on $[-2,2]$, whereas the approximation properties are investigated for $t\in[-1,1]$.

Let $e_m(y)=y^m$ and $\theta_m(t)=D_k((y-t)^m,t)$, $m=0,1,2,\ldots$.

Lemma 1.

$$D_k\left(e_0(y),t\right)=1$$

$$D_k\left(e_1(y),t\right)=t+\frac{k+1}{k^2}$$

$$D_k\left(e_2(y),t\right)=t^2-\frac{t^2}{k}+\frac{2(k+1)t}{k^2}+\frac{(3k+4)(k+1)^2}{3k^4}$$

$$D_k\left(e_3(y),t\right)=$$

$$t^3-\frac{(3k-2)t^3}{k^2}+\frac{3\left(k^2-1\right)t^2}{k^3}+\frac{(3k+2)(k+1)^2t}{k^4}+\frac{(3k+2)(k+1)^3}{k^6}$$

$$D_k\left(e_4(y),t\right)=$$

$$t^4-\frac{\left(6k^2-11k+6\right)t^4}{k^3}+\frac{4(k-2)\left(k^2-1\right)t^3}{k^4}+\frac{6(k-1)(k+1)^2t^2}{k^4}+\frac{12(k+1)^3}{k^5}$$

$$+\frac{(15k^2+30k+16)(k+1)^4}{5k^8}$$

$$\theta_0(t)=1$$

$$\theta_1(t)=\frac{k+1}{k^2}$$

$$\theta_2(t) = -\frac{t^2}{k} + \frac{(3k+4)(k+1)^2}{3k^4}$$

$$\theta_4(t) = \frac{3(k-2)t^4}{k^3} + \frac{8(k+1)t^3}{k^4} - \frac{6(k+1)^2t^2}{k^4} - \frac{8(k+1)^3t}{k^6} + \frac{(15k^2+30k+16)(k+1)^4}{5k^8}$$

Lemma 2.

$$\lim_{k\to\infty} k\theta_1(t) = 1$$

$$\lim_{k\to\infty} k\theta_2(t) = -t^2 + 1$$

$$\lim_{k\to\infty} k^2\theta_4(t) = 3t^4 - 6t^2 + 3.$$

We denote by $C[-2,2]$ the space of all real-valued continuous functions on $[-2,2]$, equipped with the supremum norm

$$\|v\|_\infty = \sup_{y\in[-2,2]} |v(y)|.$$

Moreover,

$$C^2[-2,2] = \left\{v \in C[-2,2] : v', v'' \in C[-2,2]\right\}.$$

2. Main Results

In this section, we present the local approximation properties of the operator defined in (1.6).

Lemma 3. *Let $v \in C[-2,2]$ and $t \in [-1,1]$. Then*

$$|D_k(v,t)| \le \|v\|_\infty.$$

Proof. By the definition of (1.6) and Lemma 1.1,

$$\begin{aligned}
|D_k(v,t)| &= \left| \frac{k^2}{2(k+1)} \cdot \sum_{m=0}^{k} s_{k,m}(t) \int_{\left(2\frac{m}{k}-1\right)\frac{k+1}{k}}^{\left(2\frac{m+1}{k}-1\right)\frac{k+1}{k}} v(u)du \right| \\
&\le \frac{k^2}{2(k+1)} \cdot \sum_{m=0}^{k} s_{k,m}(t) \int_{\left(2\frac{m}{k}-1\right)\frac{k+1}{k}}^{\left(2\frac{m+1}{k}-1\right)\frac{k+1}{k}} |v(u)|\, du \\
&\le \frac{k^2}{2(k+1)} \cdot \sum_{m=0}^{k} s_{k,m}(t) \int_{\left(2\frac{m}{k}-1\right)\frac{k+1}{k}}^{\left(2\frac{m+1}{k}-1\right)\frac{k+1}{k}} \|v\|_\infty\, du = \|v\|_\infty.
\end{aligned}$$

Theorem 1. *Let $v \in C[-2,2]$. Then*

$$\lim_{k\to\infty} \|D_k(v,\cdot) - v\|_{\infty,[-1,1]} = 0.$$

That is, $D_k(v,t)$ converges uniformly to $v(t)$ on $[-1,1]$.

Proof. By the Bohman–Korovkin theorem, it is sufficient to verify the convergence for the test functions

$$e_0(y) = 1, \qquad e_1(y) = y, \qquad e_2(y) = y^2.$$

Using Lemma 1.1, we obtain

$$\lim_{k\to\infty} \|D_k(e_0,\cdot) - e_0\|_{\infty,[-1,1]} = 0,$$

$$\lim_{k\to\infty} \|D_k(e_1,\cdot) - e_1\|_{\infty,[-1,1]} = 0,$$

and

$$\lim_{k\to\infty} \|D_k(e_2,\cdot) - e_2\|_{\infty,[-1,1]} = 0.$$

Hence, by the Bohman–Korovkin theorem,

$$\lim_{k\to\infty} \|D_k(v,\cdot) - v\|_{\infty,[-1,1]} = 0.$$

In approximation theory, the rate at which an operator approaches a function is important. A standard tool for estimating this rate is the modulus of continuity.

The first-order modulus of continuity is defined by

$$\omega_1(v,\mu) = \sup_{\substack{y,z\in[a,b]\\ |y-z|\le\mu}} |v(y) - v(z)| \tag{2.1}$$

The definition and useful properties of $\omega_1(v,\mu)$ can be found in many sources; see, for example, [1].

Theorem 2. *Let $v \in C[-2,2]$ and $t \in [-1,1]$. Then, for the operators defined by* (1.6)*, we have*

$$|D_k(v,t) - v(t)| \le (1+\sqrt{10})\,\omega_1\left(v, \frac{1}{\sqrt{k}}\right), \qquad k \ge 1.$$

Proof. For $v \in C[-2,2]$, the first-order modulus of continuity satisfies

$$|v(y)-v(t)| \leq \left(1+\frac{|y-t|}{\mu}\right)\omega_1(v,\mu), \qquad \mu>0.$$

Since D_k is a positive linear operator and $D_k(1,t)=1$, we obtain

$$\begin{aligned}|D_k(v,t)-v(t)| &\leq D_k(|v(y)-v(t)|,t)\\ &\leq \left(1+\frac{1}{\mu}D_k(|y-t|,t)\right)\omega_1(v,\mu).\end{aligned}$$

By the Cauchy–Bunyakovsky–Schwarz inequality,

$$D_k(|y-t|,t) \leq \sqrt{D_k((y-t)^2,t)}\sqrt{D_k(1,t)}.$$

Since $D_k(1,t)=1$, it follows that

$$D_k(|y-t|,t) \leq \sqrt{D_k((y-t)^2,t)}.$$

From Lemma 1.1,

$$D_k((y-t)^2,t) = -\frac{t^2}{k}+\frac{(3k+4)(k+1)^2}{3k^4}.$$

Since $t \in [-1,1]$, we have

$$D_k((y-t)^2,t) \leq \frac{(3k+4)(k+1)^2}{3k^4}.$$

Moreover, for $k \geq 1$,

$$\frac{(3k+4)(k+1)^2}{3k^4} \leq \frac{10}{k}.$$

Therefore,

$$D_k(|y-t|,t) \leq \sqrt{\frac{10}{k}}.$$

Consequently,

$$|D_k(v,t)-v(t)| \leq \left(1+\frac{1}{\mu}\sqrt{\frac{10}{k}}\right)\omega_1(v,\mu).$$

Choosing

$$\mu = \frac{1}{\sqrt{k}},$$

we obtain

$$|D_k(v,t)-v(t)| \leq (1+\sqrt{10})\omega_1\left(v,\frac{1}{\sqrt{k}}\right).$$

This completes the proof.

If a function v satisfies the condition

$$|v(y)-v(z)| \leq M|y-z|^{\mu}, \qquad (0<\mu\leq 1) \tag{2.2}$$

then v is said to belong to the Lipschitz class, denoted by $Lip_M(\mu)$.

Theorem 3. *Let $v \in Lip_M(\mu)$ and $t \in [-1,1]$. Then, for the operators (1.6), we have*

$$|D_k(v,t)-v(t)| \leq M\left(\frac{10}{k}\right)^{\frac{\mu}{2}}.$$

Proof. By using the linearity and positivity of the operator given in (1.6), together with (2.2), we obtain

$$|D_k(v,t)-v(t)| \leq D_k\left(|v(y)-v(t)|,t\right)$$

$$\leq D_k\left(M|y-t|^{\mu},t\right).$$

By applying the Cauchy–Bunyakovsky–Schwarz inequality and Lemma 1.1,

$$|D_k(v,t)-v(t)| \leq M[D_k\left([(y-t)^2,t\right)]^{\frac{\mu}{2}}$$

$$\leq M[-\frac{t^2}{k}+\frac{(3k+4)(k+1)^2}{3k^4}]^{\frac{\mu}{2}} \leq M\left(\frac{10}{k}\right)^{\frac{\mu}{2}}.$$

The second-order modulus of continuity is defined by

$$\omega_2(v,\mu)=\sup\left\{|v(y+2u)-2v(y+u)+v(y)| : |u|\leq\mu,\ y,y+u,y+2u\in[-2,2]\right\}. \tag{2.3}$$

Peetre's second-order K-functional is defined by

$$K_2(v,\delta)=\inf_{w\in C^2[-2,2]}\left\{\|v-w\|_{\infty}+\delta\|w''\|_{\infty}\right\}.$$

It is well known that there exists a constant $M>0$ such that

$$K_2(v,\delta)\leq M\omega_2(v,\sqrt{\delta}).$$

Theorem 4. *Let $v \in C[-2,2]$, $t \in [-1,1]$, and $k \geq 2$. Then there exists a positive constant M, independent of v and k, such that*

$$|D_k(v,t) - v(t)| \leq 8M\omega_2\left(v, \frac{1}{\sqrt{k}}\right) + 2\omega_1\left(v, \frac{1}{k}\right).$$

Proof. Let $w \in C^2[-2,2]$. By Taylor's formula,

$$w(y) = w(t) + (y-t)w'(t) + \int_t^y (y-u)w''(u)\,du. \tag{2.5}$$

We define the auxiliary linear operator

$$D_k^*(w,t) = D_k(w,t) + w(t) - w\left(D_k(e_1,t)\right). \tag{2.5$'$}$$

From Lemma 1.1,

$$D_k(e_1,t) = t + \frac{k+1}{k^2}.$$

Set

$$a_k = \frac{k+1}{k^2}.$$

Thus,

$$D_k(e_1,t) = t + a_k.$$

Moreover,

$$D_k^*(1,t) = 1$$

and

$$D_k^*(y-t,t) = D_k(y-t,t) - a_k = 0.$$

Applying D_k^* to Taylor's formula (2.5), we obtain

$$D_k^*(w,t) - w(t) = D_k^*\left(\int_t^y (y-u)w''(u)\,du, t\right).$$

Using the definition of D_k^*, this can be written as

$$D_k^*(w,t) - w(t) = D_k\left(\int_t^y (y-u)w''(u)\,du, t\right) - \int_t^{t+a_k} (t+a_k-u)w''(u)\,du.$$

Hence,

$$\begin{aligned}|D_k^*(w,t) - w(t)| \leq{}& D_k\left(\left|\int_t^y (y-u)w''(u)\,du\right|, t\right)\\ &+ \left|\int_t^{t+a_k} (t+a_k-u)w''(u)\,du\right|.\end{aligned}$$

Since

$$\left|\int_t^y (y-u)w''(u)\,du\right| \le \frac{1}{2}(y-t)^2\|w''\|_\infty,$$

and

$$\left|\int_t^{t+a_k} (t+a_k-u)w''(u)\,du\right| \le \frac{a_k^2}{2}\|w''\|_\infty,$$

we get

$$|D_k^*(w,t)-w(t)| \le \frac{\|w''\|_\infty}{2}\left\{D_k((y-t)^2,t)+a_k^2\right\}.$$

By Lemma 1.1,

$$D_k((y-t)^2,t) = -\frac{t^2}{k}+\frac{(3k+4)(k+1)^2}{3k^4}.$$

Therefore,

$$\begin{aligned}|D_k^*(w,t)-w(t)| &\le \frac{\|w''\|_\infty}{2}\left\{-\frac{t^2}{k}+\frac{(3k+4)(k+1)^2}{3k^4}+\frac{(k+1)^2}{k^4}\right\}\\ &\le \frac{\|w''\|_\infty}{2}\frac{(k+1)^2(3k+7)}{3k^4}.\end{aligned}$$

For $k \ge 2$,

$$\frac{(k+1)^2(3k+7)}{6k^4} \le \frac{8}{k},$$

and consequently

$$|D_k^*(w,t)-w(t)| \le \frac{8}{k}\|w''\|_\infty. \tag{2.6}$$

Now, from the definition of D_k^*,

$$D_k(v,t)-v(t) = D_k^*(v,t)-v(t)+v(t+a_k)-v(t).$$

Thus, for any $w \in C^2[-2,2]$,

$$\begin{aligned}|D_k(v,t)-v(t)| \le\ & |D_k^*(v-w,t)-(v-w)(t)|\\ &+|D_k^*(w,t)-w(t)|+|v(t+a_k)-v(t)|.\end{aligned}$$

Furthermore,

$$D_k^*(v-w,t)-(v-w)(t) = D_k(v-w,t)-(v-w)(t+a_k),$$

and hence, by Lemma 2.1,

$$\begin{aligned}|D_k^*(v-w,t)-(v-w)(t)| &\le |D_k(v-w,t)|+|(v-w)(t+a_k)|\\ &\le 2\|v-w\|_\infty.\end{aligned}$$

Also, since

$$a_k = \frac{k+1}{k^2} \leq \frac{2}{k},$$

we have

$$|v(t+a_k) - v(t)| \leq \omega_1\left(v, \frac{2}{k}\right) \leq 2\omega_1\left(v, \frac{1}{k}\right).$$

Using (2.6), we therefore obtain

$$|D_k(v,t) - v(t)| \leq 2\|v-w\|_\infty + \frac{8}{k}\|w''\|_\infty + 2\omega_1\left(v, \frac{1}{k}\right).$$

Since

$$2\|v-w\|_\infty + \frac{8}{k}\|w''\|_\infty \leq 8\left(\|v-w\|_\infty + \frac{1}{k}\|w''\|_\infty\right),$$

we obtain

$$|D_k(v,t) - v(t)| \leq 8\left(\|v-w\|_\infty + \frac{1}{k}\|w''\|_\infty\right) + 2\omega_1\left(v, \frac{1}{k}\right).$$

Taking the infimum over all $w \in C^2[-2,2]$, we get

$$|D_k(v,t) - v(t)| \leq 8K_2\left(v, \frac{1}{k}\right) + 2\omega_1\left(v, \frac{1}{k}\right).$$

Finally, using the standard relation

$$K_2(v,\delta) \leq M\omega_2(v, \sqrt{\delta}),$$

with $\delta = 1/k$, we conclude that

$$|D_k(v,t) - v(t)| \leq 8M\omega_2\left(v, \frac{1}{\sqrt{k}}\right) + 2\omega_1\left(v, \frac{1}{k}\right).$$

This completes the proof.

Theorem 5. *Let $v \in C^2[-2,2]$ and $t \in [-1,1]$. Then*

$$\lim_{k\to\infty} k\left\{D_k(v,t) - v(t)\right\} = v'(t) + \frac{v''(t)}{2}(1-t^2).$$

Proof. Let $v \in C^2[-2,2]$ and fix $t \in [-1,1]$. By Taylor's formula with the Peano remainder, we have

$$v(y) = v(t) + (y-t)v'(t) + \frac{1}{2}(y-t)^2 v''(t) + (y-t)^2\theta(y,t), \tag{2.12}$$

where

$$\theta(y,t)=\begin{cases}\dfrac{v(y)-v(t)-(y-t)v'(t)-\dfrac{1}{2}(y-t)^2v''(t)}{(y-t)^2}, & y\neq t,\\ 0, & y=t.\end{cases}$$

Since $v\in C^2[-2,2]$, we have

$$\lim_{y\to t}\theta(y,t)=0. \tag{2.13}$$

For each fixed $t\in[-1,1]$, the function $y\mapsto\theta(y,t)$ is continuous on $[-2,2]\setminus\{t\}$. Since

$$\lim_{y\to t}\theta(y,t)=\theta(t,t)=0,$$

it extends continuously to $[-2,2]$ and is therefore bounded.

Applying the positive linear operator D_k to both sides of (2.12), we obtain

$$\begin{aligned}D_k(v,t)-v(t)=v'(t)&D_k(y-t,t)\\&+\frac{v''(t)}{2}D_k((y-t)^2,t)\\&+D_k\left((y-t)^2\theta(y,t),t\right).\end{aligned}$$

Multiplying both sides by k, we get

$$\begin{aligned}k\{D_k(v,t)-v(t)\}=v'(t)\,&kD_k(y-t,t)\\&+\frac{v''(t)}{2}\,kD_k((y-t)^2,t)\\&+kD_k\left((y-t)^2\theta(y,t),t\right).\end{aligned}$$

By Lemma 1.2,

$$\lim_{k\to\infty}kD_k(y-t,t)=\lim_{k\to\infty}k\theta_1(t)=1$$

and

$$\lim_{k\to\infty}kD_k((y-t)^2,t)=\lim_{k\to\infty}k\theta_2(t)=1-t^2.$$

Therefore, it remains only to prove that

$$\lim_{k\to\infty}kD_k\left((y-t)^2\theta(y,t),t\right)=0.$$

By the Cauchy–Bunyakovsky–Schwarz inequality,

$$\begin{aligned}\left|kD_k\left((y-t)^2\theta(y,t),t\right)\right| &\leq k\sqrt{D_k((y-t)^4,t)}\sqrt{D_k(\theta^2(y,t),t)}\\ &= \sqrt{k^2D_k((y-t)^4,t)}\sqrt{D_k(\theta^2(y,t),t)}.\end{aligned}$$

For fixed $t \in [-1,1]$, the function

$$g_t(y) = \theta^2(y,t)$$

belongs to $C[-2,2]$ and satisfies

$$g_t(t) = \theta^2(t,t) = 0.$$

Hence, by Theorem 2.1,

$$\lim_{k\to\infty} D_k(\theta^2(y,t),t) = \theta^2(t,t) = 0.$$

On the other hand, by Lemma 1.2,

$$\lim_{k\to\infty} k^2D_k((y-t)^4,t) = \lim_{k\to\infty} k^2\theta_4(t) = 3t^4 - 6t^2 + 3.$$

Since

$$3t^4 - 6t^2 + 3 = 3(1-t^2)^2$$

and $t \in [-1,1]$, we have

$$0 \leq 3(1-t^2)^2 \leq 3.$$

Therefore,

$$\left\{k^2D_k((y-t)^4,t)\right\}_{k\geq 1}$$

is bounded for sufficiently large k.

Consequently,

$$\lim_{k\to\infty} kD_k\left((y-t)^2\theta(y,t),t\right) = 0.$$

Combining the above limits, we conclude that

$$\lim_{k\to\infty} k\{D_k(v,t) - v(t)\} = v'(t) + \frac{v''(t)}{2}(1-t^2).$$

This completes the proof.

Example 1. *Let*

$$z(t) = t^4.$$

Then

$$z'(t) = 4t^3, \qquad z''(t) = 12t^2.$$

Therefore,

$$z'(t) + \frac{z''(t)}{2}(1 - t^2) = 4t^3 + 6t^2(1 - t^2),$$

that is,

$$z'(t) + \frac{z''(t)}{2}(1 - t^2) = 4t^3 + 6t^2 - 6t^4.$$

Thus, the example is consistent with the Voronovskaja-type formula.

Graphical results.

1. Graphs of the function $h(t) = \frac{sin(2\pi t)}{1+t^2}$ and the operator D_k.

   Black: h, blue: D_{10}, red: D_{20}, brown: D_{40}.

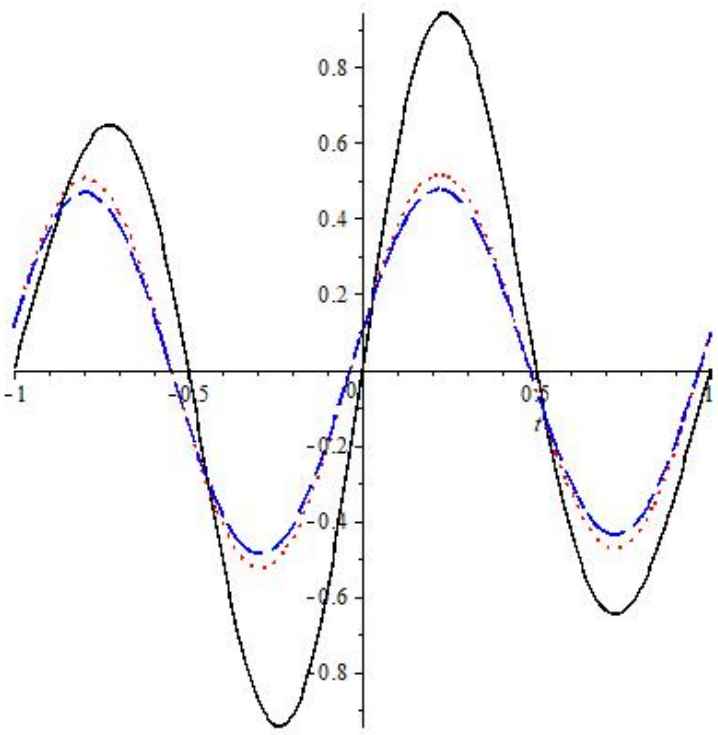


Figure 1: Graphs of h and the operator D_k.

2. Graphical comparison of the operators C_k^* and D_k.

Black: h, red: C_{30}^*, blue: D_{30}.

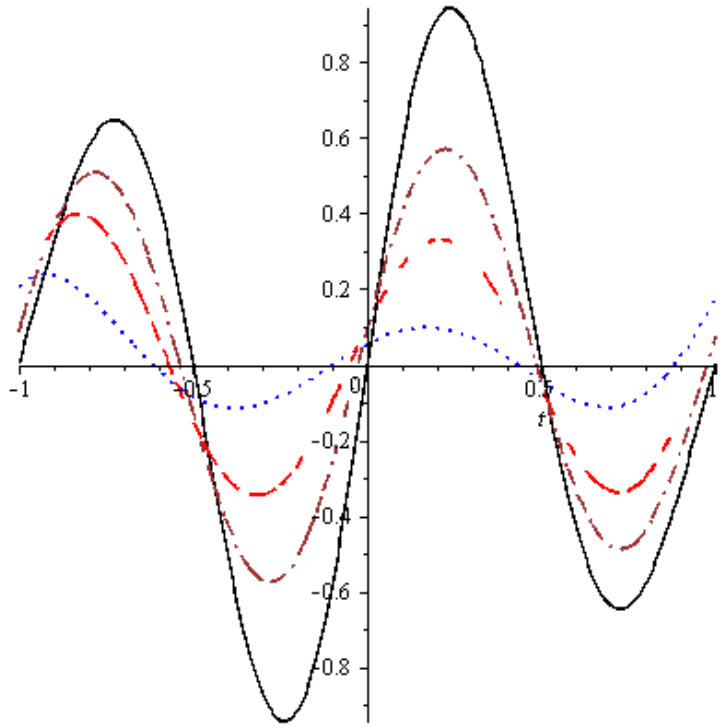


Figure 2: Graphical comparison of the operators C_k^* and D_k.

In addition to the graphical illustrations, numerical approximation errors of the operators D_k and C_k^* are presented in Tables 1 and 2. The numerical results show that the approximation errors decrease as k increases. For the numerical comparison, we consider the test function

$$h(t) = \frac{\sin(2\pi t)}{1+t^2}.$$

Tables 1 and 2 report the absolute approximation errors of D_k and C_k^* at $t = -0.75$ and $t = 0.25$, respectively.

Table 1: Approximation errors of D_k and C_k^* for $t = -0.75$.

k	$\lvert D_k(h,t) - h(t)\rvert$	$\lvert C_k^*(h,t) - h(t)\rvert$
10	0.5069284020	0.4010766989
10^2	0.0522179807	0.0439459046
10^3	0.0048768383	0.0042174279
10^4	0.0004838083	0.0004196138
10^5	0.0000483417	0.0000419398
10^6	0.0000048323	0.0000041923
10^7	0.0000004889	0.0000004250
10^8	0.0000001019	0.0000000912

Table 2: Approximation errors of D_k and C_k^* for $t = 0.25$.

k	$\lvert D_k(h,t) - h(t)\rvert$	$\lvert C_k^*(h,t) - h(t)\rvert$
10	0.8534796555	0.8018392411
10^2	0.1702562149	0.1593759762
10^3	0.0183409602	0.0173799679
10^4	0.0018479437	0.0017536221
10^5	0.0001849336	0.0001755198
10^6	0.0000184962	0.0000175550
10^7	0.0000018523	0.0000017583
10^8	0.0000001235	0.0000017583

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