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# Convergence Theorems for the McShane Integral in Locally Convex Riesz Spaces

I. Temaj, E. Canhasi Kasemi *

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**Abstract.** In this paper, we establish new convergence theorems for the McShane integral of functions taking values in locally convex Riesz spaces (topological vector lattices whose topology is generated by families of monotone continuous seminorms). By employing techniques based on regulators, (0)-convergence, and (D)-convergence, we extend classical convergence principles to this broader setting. In particular, we prove uniform convergence and monotone convergence theorems for the McShane integral, thereby generalizing the results from Banach spaces and classical Riesz spaces to locally convex Riesz spaces. These results contribute to the systematic development of McShane integration theory in ordered topological vector spaces and highlight the deep interplay between order-theoretic and topological convergence.

**Key Words and Phrases**: Locally convex Riesz space, McShane integral, uniform convergence, monotone convergence, regulators

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## 1. Introduction

The theory of nonabsolute integration has undergone significant development since the introduction of the Henstock–Kurzweil and McShane integrals, which generalize the classical Lebesgue integral while preserving many of its fundamental convergence properties. Comprehensive treatments of modern integration theory can be found in Bartle [1], while further developments of gauge-type integrals in Banach spaces were studied by Gordon [5] and Schwabik and Guoju [13].

The extension of gauge integrals to functions taking values in ordered structures, such as Riesz spaces, has attracted considerable attention in recent decades. Early contributions in this direction include the work of Riečan [10] on Kurzweil

*Corresponding author.

 

integrals in ordered spaces, as well as subsequent developments for Riesz-space-valued functions and related ordered structures (see, for example, Bocuţă and Riečan [2] and Monteiro [9]). The lattice-theoretic framework underlying these developments is described in classical monographs on Riesz spaces and ordered vector spaces (see Luxemburg and Zaanen [6] and Zaanen [17]).

At the same time, several authors have investigated integration of vector-valued functions in locally convex spaces. Important contributions in this direction include Blondia [3] and Marraffa [7], where nonabsolute integrals for functions with values in locally convex spaces were studied. The functional analytic background for locally convex and topological vector spaces can be found in standard references such as Schaefer and Wolff [12] and Meise and Vogt [8].

More recently, convergence properties of McShane-type integrals in ordered settings have been studied in various contexts. In particular, convergence theorems for McShane integrals in Riesz spaces have been investigated in Temaj and Tato [16] and in subsequent works such as Shkembi, Temaj, and Tato [15]. Further developments concerning uniform convergence results for generalized McShane integrals in Riesz spaces were obtained by Shkëmbi, Sila, and Liftaj [14].

Despite these advances, the interaction between the order structure of Riesz spaces and the locally convex topology generated by monotone seminorms has not yet been fully explored in the context of McShane integration. In particular, a systematic treatment of convergence theorems for McShane integrals in locally convex Riesz spaces requires a framework that combines order-theoretic convergence with topological control through families of seminorms.

The present work contributes to the systematic development of McShane integration in ordered topological vector spaces in two main directions.

First, we formulate McShane integrability for functions taking values in locally convex Riesz spaces whose topology is generated by families of monotone continuous seminorms. Within this framework we show that regulators, (0)-sequences and $(D)$-convergence provide an effective mechanism for controlling McShane sums in the combined order–topological setting. A key technical tool is a regulator-mixing argument (Lemma 1), inspired by techniques developed in lattice-ordered groups by Riečan and Vrabelóvá [11].

Second, within this framework we establish several convergence results for the McShane integral. In particular, we prove a uniform convergence theorem and a conditional monotone convergence theorem. The uniform convergence theorem extends earlier convergence results for McShane integrals in Riesz spaces by combining order convergence with convergence with respect to monotone seminorms. The conditional monotone convergence theorem provides a monotone convergence principle under the additional assumption that the limit of the integrals exists a priori, thereby yielding a natural analogue of the classical Beppo–Levi theorem in the setting of locally convex Riesz spaces.

The paper is organized as follows. In Section 2 we recall the basic notions concerning Riesz spaces, regulators, and the concepts of (0)- and $(D)$-convergence that will be used throughout the paper. Section 3 contains the main results of the paper. After establishing the uniqueness of the McShane integral, we prove a result on the (0)-convergence of integrals of uniformly Cauchy sequences and derive the uniform convergence theorem. Finally, we establish a conditional monotone convergence theorem and illustrate the obtained results with a concrete example in a locally convex Riesz space.

## 2. Preliminaries

We begin by recalling some basic notions and results that will be used throughout the paper. A standard reference for locally convex spaces is Schaefer and Wolff [12], and for functional analysis in locally convex settings see also Meise and Vogt [8]. From [12], Theorem 2.36 and [8], Theorem 5.1.3 we recall the following result.

**Theorem 1.** *A topological vector space $L$ is locally convex if and only if its topology can be defined by a family of continuous seminorms.*

Thus, if a Riesz space $L$ admits a topology generated by such a family, it is automatically locally convex. However, not every Riesz space has this property. For example, the Riesz space $L^p([0,1])$ for $0 < p < 1$ is not locally convex, since its topology cannot be generated by seminorms.

Hence, our construction only applies to those Riesz spaces already equipped with a topology generated by a family of continuous seminorms.

Let $L$ be a locally convex Riesz space with topology generated by a family of monotone continuous seminorms $P(L)$ and that each $p \in P(L)$ satisfies monotonicity: $0 \leq x \leq y \implies p(x) \leq p(y)$.
The space $L$ is called an *ordered vector space* if there exists a partial order $\leq$ on $L$ such that:

1. **(Compatibility with addition)** If $x \leq y$, then $x + h \leq y + h$ for every $h \in L$.

2. (**Compatibility with scalar multiplication**) If $x \geq 0$, then $p(k \cdot x) \geq 0$ for every $k \geq 0$, for all $p \in P(L)$.

If, in addition, $L$ is a *lattice* with respect to this partial order (i.e., every pair $x, y \in L$ has a supremum and an infimum), then $L$ is called a *locally convex Riesz* *space.*

We now recall several notions from the theory of Riesz spaces and $(D)$-convergence that will play a central role in what follows. Our terminology is adapted from [17] and [15].

**Definition 1** (Dedekind completeness)**.** *A Riesz space $L$ is said to be Dedekind complete if every nonempty subset of $L$, bounded from above, has supremum in $L$.*

**Definition 2** (Regulator/ (D)-sequence)**.** *A bounded double sequence $(a_{i,j})_{i,j} \subset L_+$ is called regulator or (D)-sequence if $a_{i,j} \geq a_{i,j+1}$ $\forall i, j \in \mathbb{N}$ and for every $p \in P(L)$*

$$p\left(\bigwedge_{j=1}^{\infty} a_{i,j}\right) = 0 \quad \forall i \in \mathbb{N}. \tag{1}$$

*In this case, we write also $a_{i,j} \downarrow 0$.*

**Definition 3** (0-convergence)**.** *A sequence $(b_n)_n \subset L$ is said to be (0)-convergent to $b \in L$, and we write*

$$(0)-\lim_{n\to\infty} b_n = b,$$

*if there exists a nonincreasing sequence $(a_n)_n \subset L_+$ such that:*

$$a_n \downarrow 0 \quad \text{and} \quad p(|b_n - b|) \leq p(a_k), \quad \forall k \in \mathbb{N}, p \in P(L). \tag{2}$$

**Definition 4** ($(D)$-convergence)**.** *A sequence $(r_n)_n \subset L$ is said to $(D)$-converge to $r \in L$ if there exists a regulator $(a_{i,j})_{i,j} \subset L_+$ such that for every $\varphi : \mathbb{N} \to \mathbb{N}$ (denoted by $\varphi \in \mathbb{N}^{\mathbb{N}}$) and every $p \in \mathcal{P}(L)$, there exists an integer $n_0$ such that*

$$p(|r_n - r|) \leq p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right), \quad \forall\, n \geq n_0. \tag{3}$$

*In this case we write*

$$(D)\text{-}\lim_{n\to\infty} r_n = r.$$

**Definition 5** (Weakly $\sigma$-distributive)**.** *A locally convex Riesz space $L$ is said to be weakly $\sigma$-distributive if for every $(D)$-sequence $(a_{i,j})_{i,j}$ we have*

$$p\left(\bigwedge_{\varphi\in\mathbb{N}^{\mathbb{N}}} \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right) = 0, \quad \forall\, p \in \mathcal{P}(L). \tag{4}$$

This condition is standard in the literature on $(D)$-convergence in Riesz spaces and ordered structures; see, for example, [2], [10], and [15].

The next result characterizes $(o)$-convergence in terms of an infimum-supremum representation of limits. It is a direct consequence of Dedekind $\sigma$-completeness and weak $\sigma$-distributivity and is adapted from [6].

**Proposition 1.** *Let $L$ be a Dedekind $\sigma$-complete, weakly $\sigma$-distributive locally convex Riesz space whose topology is generated by a family $\mathcal{P}(L)$ of monotone continuous seminorms. For a sequence $(b_n)_n \subset L$ and $b \in L$, the following are equivalent: $(o)$-$\lim_{n\to\infty} b_n = b$ if and only if $(b_n)_n$ is bounded in $L$ and*

$$b = \inf_m \sup_{i \geq m} b_i = \bigwedge_{m=1}^{\infty} \left( \bigvee_{i=m}^{\infty} b_i \right).$$

*Proof.* Assume first that $(o)$-$\lim_n b_n = b$. By Definition 3, there exists a decreasing sequence $(a_k) \subset L_+$ with $a_k \downarrow 0$ such that

$$p(| \, b_n - b \, |) \leq p(a_k)$$

for all $n \geq k$ and all $p \in \mathcal{P}(L)$. This implies that $(b_n)$ is bounded in the topology generated by $\mathcal{P}(L)$. Using Dedekind $\sigma$-completeness and weak $\sigma$-distributivity, one shows that the order limit of $(b_n)$ can be expressed as

$$b = \inf_m \sup_{i \geq m} b_i = \bigwedge_{m=1}^{\infty} \left( \bigvee_{i=m}^{\infty} b_i \right).$$

Conversely, suppose that $(b_n)$ is bounded and that

$$b = \inf_m \sup_{i \geq m} b_i = \bigwedge_{m=1}^{\infty} \left( \bigvee_{i=m}^{\infty} b_i \right).$$

For each $n$ and each $p \in \mathcal{P}(L)$, define

$$a_n := \sup_{i \geq n} p(| \, b_i - b \, |).$$

Then $(a_n)$ is nonincreasing and $a_n \downarrow 0$ by the assumed infimum-supremum representation of the limit. Moreover, for every $k$ and all $n \geq k$, we have $p(| \, b_n - b \, |) \leq p(a_k)$. Hence Definition 3 is satisfied, and therefore

$$(o)\text{-}\lim_{n\to\infty} b_n = b.$$

Let $T$ be a compact Hausdorff topological space and let $\mathcal{B}$ denote the $\sigma$-algebra of Borel subsets of $T$. We assume that $\mu : \mathcal{B} \to [0, +\infty]$ is a monotone, positive, and finitely additive measure. Here "additive" means that

$$\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)$$

whenever $E, F, E \cup F \in \mathcal{B}$.

We recall the notions of partitions and gauges that underlie the McShane integral.

**Definition 6.** *A partition of $T$ is a finite collection $\Pi = \{(E_i, t_i)_{i=1}^n\}$ of pairs such that:*

*i.* $\bigcup_{i=1}^{n} E_i = T$;

*ii.* $t_i \in E_i$, $E_i \in \mathcal{B}$;

*iii.* $\mu(E_i \cap E_j) = 0 \quad (i \neq j)$.

*A collection $\Pi = \{(E_i, t_i)_{i=1}^n\}$ that satisfies ii) and iii), but not necessarily i), is called a* decomposition *of $T$.*

A gauge on $T$ is a function $\gamma : T \to (0, +\infty)$. Given a gauge $\gamma$, a partition $\Pi = \{(E_i, t_i)\}_{i=1}^n$ is said to be $\gamma$-fine (and we write $\Pi \preceq \gamma$) if

$$t_i \in E_i \subset B(t_i, \gamma(t_i)), \quad i = 1, \ldots, n,$$

where $B(t_i, \gamma(t_i))$ denotes the open ball (or neighbourhood) of radius $\gamma(t_i)$ around $t_i$ in $T$. We denote by $\mathcal{A}_T(\gamma)$ the set of all $\gamma$-fine partitions of $T$.

Let $f : T \to L$ be a function. For a partition $\Pi = \{(E_i, t_i)\}_{i=1}^n$ of $T$, we define the corresponding McShane sum by

$$S(f, \Pi) := \sum_{i=1}^{n} f(t_i)\, \mu(E_i) \in L.$$

Let $L$ be a locally convex Riesz space whose topology is generated by a family $\mathcal{P}(L)$ of monotone continuous seminorms, and let $(T, \mathcal{B}, \mu)$ be a finite measure space.

**Definition 7.** *A function $f : T \to L$ is said to be McShane integrable on $T$ (with respect to $\mu$) if there exist an element $\omega \in L$ and a regulator $(a_{i,j})_{i,j} \subset L_+$ such that the following property holds: for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and every $p \in \mathcal{P}(L)$ there exists a gauge $\gamma$ on $T$ such that*

$$p(|S(f,\Pi) - \omega|) \le p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right) \quad \text{for all } \Pi \in \mathcal{A}_T(\gamma). \tag{5}$$

*Any element $\omega \in L$ satisfying this condition is called the McShane integral of $f$ over $T$ and is denoted by*

$$\omega = \int_T f \, d\mu.$$

**Remark 1.** *Definition 7 extends the classical McShane integral to functions taking values in locally convex Riesz spaces. The regulator $(a_{i,j})$ plays the role of a control sequence replacing the scalar error bounds appearing in the Banach space setting.*

**Remark 2.** *In [2], [6], [16], [15], [14], [7], [17], [10], [3], and related works, McShane and Kurzweil–Henstock type integrals are studied mainly in classical Riesz spaces, where either no locally convex topology is assumed or only the order structure is used. In contrast, in the present paper we adapt these notions to locally convex Riesz spaces whose topology is generated by families of monotone continuous seminorms. Within this setting, regulators, $(o)$-convergence, and $(D)$-convergence are used to control McShane sums in a combined order-topological framework. This extension is essential for establishing the convergence theorems for the McShane integral proved in the next section.*

**Lemma 1** (Regulator mixing)**.** *Let $L$ be a Dedekind $\sigma$-complete, weakly $\sigma$-distributive locally convex Riesz space whose topology is generated by a family of monotone continuous seminorms $\mathcal{P}(L)$. Suppose that $(a_{n,i,j})_{n,i,j \in \mathbb{N}} \subset L_+$ is a triple sequence such that, for each $n \in \mathbb{N}$, the double sequence $(a_{n,i,j})_{i,j}$ is a $(D)$-sequence. Then there exists a double sequence $(b_{i,j})_{i,j} \subset L_+$ with the following properties:*

(i) *For every $S \in L_+ \setminus \{0\}$, every $\varphi \in \mathbb{N}^{\mathbb{N}}$, every $k \in \mathbb{N}$, and every $p \in \mathcal{P}(L)$,*

$$p\left(S \wedge \sum_{n=1}^{k} \bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n)}\right) \le p\left(\bigvee_{j=1}^{\infty} \left(S \wedge b_{j,\varphi(j)}\right)\right).$$

(ii) *For every $S \in L_+ \setminus \{0\}$, the double sequence $(S \wedge b_{i,j})_{i,j}$ is a $(D)$-sequence; that is, for each $i \in \mathbb{N}$, the sequence $(S \wedge b_{i,j})_j$ is decreasing in $j$, and*

$$p\left(\bigwedge_{j=1}^{\infty}(S \wedge b_{i,j})\right) = 0, \quad \forall\, p \in \mathcal{P}(L).$$

*Proof.* The argument follows the regulator-mixing ideas developed in [11], [4], and [9] Lemma 3.2, adapted here to the setting of locally convex Riesz spaces whose topology is generated by monotone continuous seminorms.

For each $n \in \mathbb{N}$, since $(a_{n,i,j})_{i,j}$ is a $(D)$-sequence, we have:

- $a_{n,i,j} \geq a_{n,i,j+1} \geq 0$ for all $i, j \in \mathbb{N}$;
- for every seminorm $p \in \mathcal{P}(L)$,

$$p\left(\bigwedge_{j=1}^{\infty} a_{n,i,j}\right) = 0, \quad \forall\, i \in \mathbb{N}.$$

Thus, for each fixed $n$, the rows of the double sequence $(a_{n,i,j})_{i,j}$ are decreasing and converge to zero in the sense of Definition 2.

Let now $S \in L_+ \setminus \{0\}$, $\varphi \in \mathbb{N}^{\mathbb{N}}$, and $k \in \mathbb{N}$. For each $n \in \mathbb{N}$, define the diagonal selection

$$u_n(\varphi) := \bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n)}.$$

Then consider the finite sum

$$x_k(\varphi) := \sum_{n=1}^{k} u_n(\varphi) = \sum_{n=1}^{k}\left(\bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n)}\right).$$

The lattice operations used below are standard: if $0 \leq x \leq y$, then $S \wedge x \leq S \wedge y$, and since each seminorm $p \in \mathcal{P}(L)$ is monotone, it follows that

$$p(S \wedge x) \leq p(S \wedge y).$$

For each fixed $r \in \mathbb{N}$, the double sequence $(a_{r,i,j})_{i,j}$ is a $(D)$-sequence by assumption. Hence the family

$$\{(a_{r,i,j})_{i,j} : r \in \mathbb{N}\}$$

is a countable family of regulators. By weak $\sigma$-distributivity, together with the standard regulator-mixing argument for countable families of $(D)$-sequences (cf.

[11], [4], [9] Lemma 3.2), there exists a double sequence $(b_{i,j})_{i,j} \subset L_+$ such that, for every $S \in L_+ \setminus \{0\}$, every $\varphi \in \mathbb{N}^{\mathbb{N}}$, every $k \in \mathbb{N}$, and every $p \in \mathcal{P}(L)$,

$$p(S \wedge x_k(\varphi)) \leq p\left(\bigvee_{j=1}^{\infty} (S \wedge b_{j,\varphi(j)})\right).$$

Recalling the definition of $x_k(\varphi)$, this becomes

$$p\left(S \wedge \sum_{n=1}^{k} \bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n)}\right) \leq p\left(\bigvee_{j=1}^{\infty} \big(S \wedge b_{j,\varphi(j)}\big)\right).$$

which proves part (i).

It remains to verify part (ii). By construction, the double sequence $(b_{i,j})_{i,j}$ may be chosen so that, for each fixed $i \in \mathbb{N}$, the sequence $(b_{i,j})_j$ is decreasing in $j$. Since the map $x \mapsto S \wedge x$ is order-preserving on $L_+$, it follows that, for each fixed $i \in \mathbb{N}$, the sequence $(S \wedge b_{i,j})_j$ is also decreasing.

Moreover, since

$$0 \leq S \wedge b_{i,j} \leq b_{i,j},$$

and each seminorm $p \in \mathcal{P}(L)$ is monotone, we obtain

$$0 \leq p\left(\bigwedge_{j=1}^{\infty} (S \wedge b_{i,j})\right) \leq p\left(\bigwedge_{j=1}^{\infty} b_{i,j}\right).$$

Because $(b_{i,j})_{i,j}$ is a regulator, we have

$$p\left(\bigwedge_{j=1}^{\infty} b_{i,j}\right) = 0.$$

Hence

$$p\left(\bigwedge_{j=1}^{\infty} (S \wedge b_{i,j})\right) = 0, \quad \forall\, p \in \mathcal{P}(L).$$

Therefore, for every $S \in L_+ \setminus \{0\}$, the double sequence $(S \wedge b_{i,j})_{i,j}$ is again a $(D)$-sequence. This proves part (ii).

The proof is complete.

## 3. Main Results

In this section, we establish the main convergence results for the McShane integral of functions taking values in a Dedekind $\sigma$-complete, weakly $\sigma$-distributive locally convex Riesz space. The proofs rely on the notions of $(o)$-convergence and $(D)$-convergence introduced in the previous section, together with the regulator technique developed in Lemma 1.

We first prove the uniqueness of the McShane integral, and then derive several convergence theorems, including a uniform convergence theorem and a conditional monotone convergence theorem.

**Theorem 2** (Uniqueness of the McShane integral)**.** *Let $L$ be a locally convex Riesz space whose topology is generated by a family $\mathcal{P}(L)$ of monotone continuous seminorms, and let $(T, \mathcal{B}, \mu)$ be a finite measure space. If $f : T \to L$ is McShane integrable on $T$, then its McShane integral is unique.*

*Proof.* Assume that $\omega_1, \omega_2 \in L$ both satisfy Definition 7 for the function $f$. Then there exist regulators $(a_{i,j})_{i,j} \subset L_+$ and $(b_{i,j})_{i,j} \subset L_+$ such that, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and every $p \in \mathcal{P}(L)$, there exist gauges $\gamma_1$ and $\gamma_2$ on $T$ satisfying

$$p(|S(f, \Pi_1) - \omega_1|) \leq p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right), \quad \forall\, \Pi_1 \in \mathcal{A}_T(\gamma_1),$$

and

$$p(|S(f, \Pi_2) - \omega_2|) \leq p\left(\bigvee_{i=1}^{\infty} b_{i,\varphi(i)}\right), \quad \forall\, \Pi_2 \in \mathcal{A}_T(\gamma_2).$$

Let $\gamma = \min\{\gamma_1, \gamma_2\}$. Then every $\Pi \in \mathcal{A}_T(\gamma)$ belongs to both $\mathcal{A}_T(\gamma_1)$ and $\mathcal{A}_T(\gamma_2)$. Hence, by the triangle inequality,

$$\begin{aligned} p(|\omega_1 - \omega_2|) &\leq p(|\omega_1 - S(f, \Pi)|) + p(|S(f, \Pi) - \omega_2|) \\ &\leq p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right) + p\left(\bigvee_{i=1}^{\infty} b_{i,\varphi(i)}\right). \end{aligned}$$

Since $(a_{i,j})$ and $(b_{i,j})$ are regulators, the right-hand side can be made arbitrarily small in the sense of Definition 2. Therefore

$$p(|\omega_1 - \omega_2|) = 0, \quad \forall\, p \in \mathcal{P}(L).$$

Because the topology of $L$ is generated by the family $\mathcal{P}(L)$, it follows that $\omega_1 = \omega_2$. Thus the McShane integral is unique.

To prove the Uniform Convergence Theorem, we first establish Proposition 2, following the approach presented in [9].

In order to establish the uniform convergence theorem for the McShane integral in locally convex Riesz spaces, we first introduce the notion of uniformly Cauchy sequences of functions. This concept allows us to control the behaviour of McShane sums through the family of seminorms $\mathcal{P}(L)$.

The next definition will be used to relate the uniform behaviour of a sequence of functions to the $(o)$-convergence of the corresponding sequence of integrals. In particular, it will allow us to apply Proposition 1 to the sequence $\left(\int_T f_n \, d\mu\right)$ and obtain an infimum–supremum representation of its limit.

Using this notion, we prove Proposition 2, which shows that the sequence of McShane integrals of a uniformly Cauchy sequence admits an $(o)$-limit in $L$. This result will then be used to establish the uniform convergence theorem for the McShane integral stated in Theorem 3.

**Definition 8** (Uniformly Cauchy sequence)**.** *A sequence $(f_n)_{n\in\mathbb{N}}$ of functions on $T$ is said to be uniformly Cauchy if there exists a decreasing sequence $(b_k)_k \subset L_+$, $b_k \downarrow 0$, such that for every $k \in \mathbb{N}$, every $p \in \mathcal{P}(L)$, and all $i, j \geq k$,*

$$p(|f_i(t) - f_j(t)|) \leq p(b_k), \quad \forall\, t \in T.$$

**Remark 3.** *The condition $b_k \downarrow 0$ means that the sequence $(b_k)$ is decreasing in $L_+$ and converges to $0$ in the sense of Definition 3, that is,*

$$p(b_k) \to 0, \quad \forall\, p \in \mathcal{P}(L).$$

*This notion allows us to control the uniform behavior of the sequence $(f_n)$ simultaneously with respect to all the seminorms in $\mathcal{P}(L)$.*

**Proposition 2.** *Let $L$ be a Dedekind $\sigma$-complete, weakly $\sigma$-distributive locally convex Riesz space whose topology is generated by a family $\mathcal{P}(L)$ of monotone continuous seminorms, and let*

$$(f_n)_{n\in\mathbb{N}} \subset \mathcal{M}(T, \mathcal{P}(L), \mu)$$

*be a sequence of McShane integrable functions on $T$. If $(f_n)$ is uniformly Cauchy on $T$, then the element*

$$\omega := \bigwedge_{m=1}^{\infty} \left( \bigvee_{i=m}^{\infty} \int_T f_i \, d\mu \right)$$

*exists in $L$, and*

$$(o)\text{-}\lim_{n\to\infty} \int_T f_n \, d\mu = \omega.$$

*Proof.* Since $(f_n)$ is uniformly Cauchy in $T$, by Definition 8 there exists a decreasing sequence $(b_k) \subset L_+$, with $b_k \downarrow 0$, such that for every $k \in \mathbb{N}$, every $p \in \mathcal{P}(L)$, and all $i, j \geq k$,

$$p(|f_i(t) - f_j(t)|) \leq p(b_k), \quad \forall\, t \in T. \tag{6}$$

Fix $k \in \mathbb{N}$, $p \in \mathcal{P}(L)$, and $i, j \geq k$. Since $f_i$ and $f_j$ are McShane integrable, we may apply Definition 7 to the integrable function $f_i - f_j$. Using (6), for every partition $\Pi = \{(E_r, t_r)\}_{r=1}^m$,

$$\begin{aligned} p(|S(f_i - f_j, \Pi)|) &= p\left(\left|\sum_{r=1}^m (f_i(t_r) - f_j(t_r))\, \mu(E_r)\right|\right) \\ &\leq \sum_{r=1}^m \mu(E_r)\, p(|f_i(t_r) - f_j(t_r)|) \\ &\leq \sum_{r=1}^m \mu(E_r)\, p(b_k) = \mu(T)\, p(b_k). \end{aligned}$$

Passing to the McShane integral gives

$$p\left(\left|\int_T f_i\, d\mu - \int_T f_j\, d\mu\right|\right) \leq \mu(T)\, p(b_k), \quad i, j \geq k. \tag{7}$$

Thus the sequence

$$\left(\int_T f_n\, d\mu\right)_{n\in\mathbb{N}}$$

is $(o)$-Cauchy in $L$. Since $L$ is Dedekind $\sigma$-complete and weakly $\sigma$-distributive, Proposition 1 implies that the element

$$\omega := \bigwedge_{m=1}^{\infty} \left(\bigvee_{i=m}^{\infty} \int_T f_i\, d\mu\right)$$

exists in $L$, and

$$(o)\text{-}\lim_{n\to\infty} \int_T f_n\, d\mu = \omega.$$

This completes the proof.

**Theorem 3** (Uniform Convergence Theorem)**.** *Let $L$ be a Dedekind $\sigma$-complete, weakly $\sigma$-distributive locally convex Riesz space whose topology is generated by a family $\mathcal{P}(L)$ of monotone continuous seminorms, and let $(T, \mathcal{B}, \mu)$ be a finite measure space.*

*Let*

$$(f_n)_{n\in\mathbb{N}} \subset \mathcal{M}(T,\mathcal{P}(L),\mu)$$

*be a sequence of McShane integrable functions on $T$, and assume that $(f_n)$ is uniformly Cauchy on $T$.*

*Moreover, assume that there exist a function $f : T \to L$ and a regulator $(a_{i,j})_{i,j} \subset L_+$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exist $n_0 = n_0(\varphi) \in \mathbb{N}$ with*

$$p(|f_n(t) - f(t)|) \le p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right), \quad \forall\, n \ge n_0,\ \forall\, t \in T,\ \forall\, p \in \mathcal{P}(L). \tag{8}$$

*Then $f \in \mathcal{M}(T,\mathcal{P}(L),\mu)$. Moreover,*

$$\int_T f\,d\mu = \bigwedge_{m=1}^{\infty}\left(\bigvee_{i=m}^{\infty}\int_T f_i\,d\mu\right),$$

*and*

$$(o)\text{-}\lim_{n\to\infty}\int_T f_n\,d\mu = \int_T f\,d\mu.$$

*Proof.* Since the sequence $(f_n)$ is uniformly Cauchy on $T$, by Definition 8 there exists a decreasing sequence $(b_k)_k \subset L_+$, $b_k \downarrow 0$, such that for every $k \in \mathbb{N}$, every $p \in \mathcal{P}(L)$, and all $i, j \ge k$,

$$p(|f_i(t) - f_j(t)|) \le p(b_k), \quad \forall\, t \in T. \tag{9}$$

Since each $f_n \in \mathcal{M}(T,\mathcal{P}(L),\mu)$, Proposition 2 implies that the element

$$\omega := \bigwedge_{m=1}^{\infty}\left(\bigvee_{i=m}^{\infty}\int_T f_i\,d\mu\right)$$

exists in $L$, and

$$(o)\text{-}\lim_{n\to\infty}\int_T f_n\,d\mu = \omega. \tag{10}$$

Hence, by Definition 3, there exists a decreasing sequence $(u_j)_j \subset L_+$, $u_j \downarrow 0$, such that for every $j \in \mathbb{N}$, every $p \in \mathcal{P}(L)$, and every $n \ge j$,

$$p\left(\left|\int_T f_n\,d\mu - \omega\right|\right) \le p(u_j). \tag{11}$$

Now fix $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $p \in \mathcal{P}(L)$. By assumption, there exists $n_0 = n_0(\varphi) \in \mathbb{N}$ such that

$$p(|f_n(t) - f(t)|) \le p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right), \quad \forall\, n \ge n_0,\ \forall\, t \in T. \tag{12}$$

Choose $n \geq n_0$. Since $f_n \in \mathcal{M}(T, \mathcal{P}(L), \mu)$, by Definition 7 there exists a regulator $(a_{i,j}^{(n)})_{i,j} \subset L_+$ such that for every $\psi \in \mathbb{N}^{\mathbb{N}}$ there exists a gauge $\gamma_n$ on $T$ satisfying

$$p\left(\left|S(f_n, \Pi) - \int_T f_n \, d\mu\right|\right) \leq p\left(\bigvee_{i=1}^{\infty} a_{i,\psi(i)}^{(n)}\right), \quad \forall \, \Pi \in \mathcal{A}_T(\gamma_n). \tag{13}$$

Let $\Pi = \{(E_r, t_r)\}_{r=1}^q \in \mathcal{A}_T(\gamma_n)$. Then

$$S(f, \Pi) - \omega = \big(S(f, \Pi) - S(f_n, \Pi)\big) + \left(S(f_n, \Pi) - \int_T f_n \, d\mu\right) + \left(\int_T f_n \, d\mu - \omega\right).$$

Therefore, by the triangle inequality,

$$\begin{aligned} p(|S(f, \Pi) - \omega|) \leq p(|S(f, \Pi) - S(f_n, \Pi)|) + p\left(\left|S(f_n, \Pi) - \int_T f_n \, d\mu\right|\right) \\ + p\left(\left|\int_T f_n \, d\mu - \omega\right|\right). \end{aligned} \tag{14}$$

For the first term, using subadditivity and monotonicity of $p$, we get

$$\begin{aligned} p(|S(f, \Pi) - S(f_n, \Pi)|) &= p\left(\left|\sum_{r=1}^{q} (f(t_r) - f_n(t_r)) \, \mu(E_r)\right|\right) \\ &\leq \sum_{r=1}^{q} \mu(E_r) \, p(|f(t_r) - f_n(t_r)|) \\ &\leq \sum_{r=1}^{q} \mu(E_r) \, p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right) \\ &= \mu(T) \, p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right). \end{aligned} \tag{15}$$

Moreover, by (13),

$$p\left(\left|S(f_n, \Pi) - \int_T f_n \, d\mu\right|\right) \leq p\left(\bigvee_{i=1}^{\infty} a_{i,\psi(i)}^{(n)}\right), \tag{16}$$

and by (11), for every $j \leq n$,

$$p\left(\left|\int_T f_n \, d\mu - \omega\right|\right) \leq p(u_j). \tag{17}$$

Combining (14), (15), (16), and (17), we obtain

$$p(|S(f,\Pi)-\omega|) \le \mu(T)\, p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right) + p\left(\bigvee_{i=1}^{\infty} a^{(n)}_{i,\psi(i)}\right) + p(u_j). \tag{18}$$

We now define the triple sequence $(c_{n,i,j})_{n,i,j} \subset L_+$ by

$$c_{1,i,j} := \mu(T)\, a_{i,j}, \quad c_{2,i,j} := a^{(n)}_{i,j}, \quad c_{3,i,j} := u_{i+j},$$

and

$$c_{n,i,j} := 0, \quad n \ge 4. \tag{19}$$

For each fixed $n \in \mathbb{N}$, the double sequence $(c_{n,i,j})_{i,j}$ is a $(D)$-sequence. Indeed:

- $(c_{1,i,j})_{i,j}$ is a $(D)$-sequence, since $(a_{i,j})_{i,j}$ is a regulator and multiplication by the positive scalar $\mu(T)$ preserves Definition 2;
- $(c_{2,i,j})_{i,j} = (a^{(n)}_{i,j})_{i,j}$ is a $(D)$-sequence. Indeed, by Definition 7 there exists a regulator $(a^{(n)}_{i,j})$ associated with $f_n$, and every regulator is a $(D)$-sequence by Definition 2;
- $(c_{3,i,j})_{i,j} = (u_{i+j})_{i,j}$ is a $(D)$-sequence, because $(u_j) \downarrow 0$;
- for $n \ge 4$, $(c_{n,i,j})_{i,j} = 0$ is trivially a $(D)$-sequence.

Hence all assumptions of Lemma 1 are satisfied.

Applying Lemma 1 to the triple sequence $(c_{n,i,j})$, with $k = 3$, we obtain a double sequence $(b_{i,j})_{i,j} \subset L_+$ such that for every $S \in L_+ \setminus \{0\}$, every $\theta \in \mathbb{N}^{\mathbb{N}}$, and every $p \in \mathcal{P}(L)$,

$$p\left(S \wedge \sum_{n=1}^{3} \bigvee_{i=1}^{\infty} c_{n,i,\theta(i+n)}\right) \le p\left(\bigvee_{j=1}^{\infty} \big(S \wedge b_{j,\theta(j)}\big)\right). \tag{20}$$

Thus the three terms on the right-hand side of (18) are controlled by a single regulator. Consequently, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and every $p \in \mathcal{P}(L)$, there exists a gauge $\gamma_n$ such that for every $\Pi \in \mathcal{A}_T(\gamma_n)$,

$$p(|S(f,\Pi)-\omega|) \le p\left(\bigvee_{j=1}^{\infty} b_{j,\varphi(j)}\right). \tag{21}$$

By Definition 7, it follows that $f \in \mathcal{M}(T, \mathcal{P}(L), \mu)$ and

$$\int_T f\, d\mu = \omega.$$

Finally, by (10),

$$(o)\text{-}\lim_{n\to\infty}\int_T f_n\,d\mu = \omega = \int_T f\,d\mu.$$

This completes the proof.

**Remark 4.** *Theorem 3 extends the classical uniform convergence theorem for the Lebesgue and McShane integrals to the setting of Dedekind $\sigma$-complete, weakly $\sigma$-distributive locally convex Riesz spaces. In the classical theory of Banach-space-valued integration, uniform convergence of integrable functions implies convergence of the corresponding integrals.*

*In the present framework, the proof requires the use of regulators, $(o)$-convergence, and the weak $\sigma$-distributivity property in order to control McShane sums in the topology generated by the family of seminorms $\mathcal{P}(L)$. The regulator-mixing technique introduced in Lemma 1 plays a crucial role in combining the control coming from the integrability of $f_n$ with the uniform convergence of the sequence.*

**Theorem 4** (Conditional Monotone Convergence)**.** *Let $L$ be a Dedekind $\sigma$-complete, weakly $\sigma$-distributive locally convex Riesz space whose topology is generated by a family $\mathcal{P}(L)$ of monotone continuous seminorms, and let $(T, \mathcal{B}, \mu)$ be a finite measure space.*

*Let*

$$(f_n)_{n\in\mathbb{N}} \subset \mathcal{M}(T, \mathcal{P}(L), \mu)$$

*be a sequence of McShane integrable functions on $T$. Assume that*

$$f_n(t) \le f_{n+1}(t), \quad \forall\, t\in T,\ \forall\, n\in\mathbb{N},$$

*and that the limit*

$$\omega := (o)\text{-}\lim_{n\to\infty}\int_T f_n\,d\mu$$

*exists in $L$.*

*For each $t\in T$, define*

$$f(t) := \bigvee_{n=1}^{\infty} f_n(t).$$

*Assume moreover that there exists a decreasing sequence $(u_n)\subset L_+$, $u_n \downarrow 0$, such that*

$$0 \le f(t) - f_n(t) \le u_n, \quad \forall\, t\in T,\ \forall\, n\in\mathbb{N}.$$

*Then $f \in \mathcal{M}(T, \mathcal{P}(L), \mu)$ and*

$$\int_T f\,d\mu = (o)\text{-}\lim_{n\to\infty}\int_T f_n\,d\mu.$$

*Proof.* For each $t \in T$, define

$$f(t) = \bigvee_{n=1}^{\infty} f_n(t).$$

Since the sequence $(f_n(t))_n$ is increasing in the Dedekind $\sigma$-complete Riesz space $L$, the supremum exists for every $t \in T$. Hence $f : T \to L$ is well defined.

Let

$$\omega := (o)\text{-}\lim_{n\to\infty} \int_T f_n \, d\mu.$$

We prove that $f$ is McShane integrable and that

$$\int_T f \, d\mu = \omega.$$

Since $f_n \in \mathcal{M}(T, \mathcal{P}(L), \mu)$, by Definition 7 there exists for each $n$ a regulator $(a_{i,j}^{(n)})_{i,j} \subset L_+$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and every $p \in \mathcal{P}(L)$ there exists a gauge $\gamma_n$ satisfying

$$p\left(\left|S(f_n, \Pi) - \int_T f_n \, d\mu\right|\right) \le p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^{(n)}\right), \quad \Pi \in \mathcal{A}_T(\gamma_n). \tag{22}$$

For every partition $\Pi \in \mathcal{A}_T(\gamma_n)$,

$$S(f,\Pi) - \omega = \big(S(f,\Pi) - S(f_n,\Pi)\big) + \left(S(f_n,\Pi) - \int_T f_n \, d\mu\right) + \left(\int_T f_n \, d\mu - \omega\right).$$

Hence

$$\begin{aligned} p(|S(f,\Pi) - \omega|) \le p(|S(f,\Pi) - S(f_n,\Pi)|) + p\left(\left|S(f_n,\Pi) - \int_T f_n \, d\mu\right|\right) \\ + p\left(\left|\int_T f_n \, d\mu - \omega\right|\right). \end{aligned} \tag{23}$$

From

$$0 \le f(t) - f_n(t) \le u_n,$$

we obtain for every partition $\Pi = \{(E_i, t_i)\}_{i=1}^m$,

$$0 \le S(f,\Pi) - S(f_n,\Pi) \le u_n \, \mu(T).$$

Since each seminorm $p \in \mathcal{P}(L)$ is monotone,

$$p(|S(f,\Pi) - S(f_n,\Pi)|) \le \mu(T) \, p(u_n). \tag{24}$$

From (22),

$$p\left(\left|S(f_n,\Pi)-\int_T f_n\,d\mu\right|\right)\le p\left(\bigvee_{i=1}^{\infty} a^{(n)}_{i,\varphi(i)}\right). \tag{25}$$

Since $(o)$-$\lim_{n\to\infty}\int_T f_n\,d\mu=\omega$, there exists a decreasing sequence $(v_n)\subset L_+$ with $v_n\downarrow 0$ such that

$$p\left(\left|\int_T f_n\,d\mu-\omega\right|\right)\le p(v_n). \tag{26}$$

Combining (23), (24), (25), and (26),

$$p(|S(f,\Pi)-\omega|)\le \mu(T)\,p(u_n)+p\left(\bigvee_{i=1}^{\infty} a^{(n)}_{i,\varphi(i)}\right)+p(v_n). \tag{27}$$

Define the triple sequence $(c_{n,i,j})_{n,i,j}\subset L_+$ by

$$c_{1,i,j}:=\mu(T)\,u_j,\quad c_{2,i,j}:=a^{(n)}_{i,j},\quad c_{3,i,j}:=v_j,$$

and

$$c_{n,i,j}:=0,\quad n\ge 4. \tag{28}$$

For each $n$, the double sequence $(c_{n,i,j})_{i,j}$ is a $(D)$-sequence. Hence the assumptions of Lemma 1 are satisfied.

Applying Lemma 1, we obtain a regulator $(b_{i,j})_{i,j}\subset L_+$ such that

$$\mu(T)\,p(u_n)+p\left(\bigvee_{i=1}^{\infty} a^{(n)}_{i,\varphi(i)}\right)+p(v_n)\le p\left(\bigvee_{j=1}^{\infty} b_{j,\varphi(j)}\right).$$

Therefore

$$p(|S(f,\Pi)-\omega|)\le p\left(\bigvee_{j=1}^{\infty} b_{j,\varphi(j)}\right).$$

Thus for every $\varphi\in\mathbb{N}^{\mathbb{N}}$ and every $p\in\mathcal{P}(L)$ there exists a gauge $\gamma$ such that

$$p(|S(f,\Pi)-\omega|)\le p\left(\bigvee_{j=1}^{\infty} b_{j,\varphi(j)}\right),\quad \Pi\in\mathcal{A}_T(\gamma).$$

By Definition 7, $f\in\mathcal{M}(T,\mathcal{P}(L),\mu)$ and

$$\int_T f\,d\mu=\omega.$$

Finally,

$$\int_T f\,d\mu = (o)\text{-}\lim_{n\to\infty}\int_T f_n\,d\mu.$$

This completes the proof.

**Remark 5** (Conditional character)**.** *Theorem 4 should be interpreted as a conditional monotone convergence theorem. In contrast with the classical Beppo–Levi theorem for the Lebesgue integral, the convergence of the integrals*

$$\int_T f_n\,d\mu$$

*is not derived solely from the monotonicity $f_n(t)\uparrow f(t)$. Instead, the existence of the limit*

$$(o)\text{-}\lim_{n\to\infty}\int_T f_n\,d\mu$$

*is assumed a priori.*

*This assumption is natural in the present framework. Indeed, in locally convex Riesz spaces the classical Beppo–Levi theorem may fail without additional structural conditions such as order continuity of the topology or stronger completeness properties.*

**Example 1.** *Let $L=\mathbb{R}$, endowed with the usual order and the seminorm $p(x)=|x|$. Then $L$ is a Dedekind $\sigma$-complete, weakly $\sigma$-distributive locally convex Riesz space.*

*Let $T=[0,1]$ be equipped with the Lebesgue measure $\mu$, and define $f:T\to L$ by*

$$f(t)=t,\quad t\in[0,1].$$

***McShane integrability.*** *For a partition $\Pi=\{(E_i,t_i)\}_{i=1}^n$ of $T$, the McShane sum is*

$$S(f,\Pi)=\sum_{i=1}^n t_i\,\mu(E_i).$$

*Since $\int_0^1 t\,d\mu(t)=\frac{1}{2}$, we set $\omega=\frac{1}{2}$. Then*

$$p(S(f,\Pi)-\omega)=\left|\sum_{i=1}^n t_i\,\mu(E_i)-\frac{1}{2}\right|\to 0$$

*as the gauge becomes finer. Hence $f\in M(T,P(L),\mu)$ and $\int_T f\,d\mu=\frac{1}{2}$.*

***Application of Theorem 3.*** *Define*

$$f_n(t) = t\left(1 - \frac{1}{n}\right), \quad t \in [0,1].$$

*Each $f_n$ is continuous, hence McShane integrable. Moreover, for all $i, j \geq k$ and all $t \in T$,*

$$|f_i(t) - f_j(t)| = t\left|\frac{1}{j} - \frac{1}{i}\right| \leq \frac{1}{k}.$$

*Setting $b_k = \dfrac{1}{k}$, we obtain $b_k \downarrow 0$ and*

$$p(|f_i(t) - f_j(t)|) \leq p(b_k),$$

*so $(f_n)$ is uniformly Cauchy on $T$ in the sense of Definition 8.*

*Define the regulator*

$$a_{i,j} = \frac{1}{2^{i+j}}, \quad i, j \in \mathbb{N}.$$

*We verify that $(a_{i,j})$ is a $(D)$-sequence in the sense of Definition 2. First, $a_{i,j} \geq a_{i,j+1}$ since $2^{i+j} \leq 2^{i+j+1}$. Second,*

$$p\left(\bigvee_{j=1}^{\infty} a_{i,j}\right) = \bigvee_{j=1}^{\infty} \frac{1}{2^{i+j}} = \frac{1}{2^{i+1}} \to 0 \quad \text{as } i \to \infty.$$

*Hence $(a_{i,j})$ is a regulator.*

*For every $t \in T$ and every $n \in \mathbb{N}$, we estimate:*

$$p(|f_n(t) - f(t)|) = \frac{t}{n} \leq \frac{1}{n}.$$

*For every $\varphi \in \mathbb{N}^{\mathbb{N}}$, we choose $n_0 = n_0(\varphi) \in \mathbb{N}$ large enough so that*

$$\frac{1}{n_0} \leq p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right).$$

*Then for all $n \geq n_0$ and all $t \in T$,*

$$p(|f_n(t) - f(t)|) \leq \frac{1}{n} \leq p\left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\right),$$

*so condition (8) of Theorem 3 is satisfied. Therefore $f \in M(T, P(L), \mu)$ and*

$$(o)\text{-}\lim_{n\to\infty} \int_T f_n \, d\mu = \int_T f \, d\mu.$$

*In particular,*

$$\int_T f_n \, d\mu = \frac{1}{2}\left(1 - \frac{1}{n}\right) \to \frac{1}{2}.$$

***Application of Theorem 4.*** *Define*

$$g_n(t) = \max\left\{t - \frac{1}{n},\ 0\right\}, \quad t \in [0,1].$$

*Each $g_n$ is continuous, hence McShane integrable. For every $t \in T$,*

$$g_n(t) \le g_{n+1}(t) \le f(t), \qquad g_n(t) \uparrow f(t).$$

*We verify that $f(t) = \bigvee_{n=1}^{\infty} g_n(t)$ for every $t \in T$. For $t > 0$,*

$$\bigvee_{n=1}^{\infty} g_n(t) = \sup_{n \ge 1} \max\left\{t - \frac{1}{n}, 0\right\} = \lim_{n\to\infty}\left(t - \frac{1}{n}\right) = t = f(t).$$

*For $t = 0$, $g_n(0) = 0$ for every $n$, so $\bigvee_{n=1}^{\infty} g_n(0) = 0 = f(0)$. Hence $f(t) = \bigvee_{n=1}^{\infty} g_n(t)$ for all $t \in T$.*

*Setting $u_n = \dfrac{1}{n}$, we have $u_n \downarrow 0$ and*

$$0 \le f(t) - g_n(t) \le u_n, \quad \forall\, t \in T.$$

*Hence all assumptions of Theorem 4 are satisfied. A direct computation gives*

$$\int_T g_n \, d\mu = \int_{1/n}^{1} \left(t - \frac{1}{n}\right) dt = \frac{1}{2} - \frac{1}{n} + \frac{1}{2n^2} \to \frac{1}{2}.$$

*Therefore*

$$(o)\text{-}\lim_{n\to\infty} \int_T g_n \, d\mu = \int_T f \, d\mu = \frac{1}{2}.$$

## Conclusion

The example illustrates the convergence results established in the paper. In particular, the uniform convergence theorem (Theorem 3) and the conditional monotone convergence theorem (Theorem 4) extend classical convergence principles of the Lebesgue and McShane integrals to the framework of locally convex Riesz spaces.

The order structure of the space, together with the locally convex topology generated by monotone seminorms, allows one to combine order-theoretic arguments with topological control through regulators.

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Ismet Temaj
*Faculty of Education, University of Prizren "Ukshin Hoti", Rruga e Shkronjave, nr. 1, 20000 Prizren, Kosovo*
*E-mail:* ismet.temaj@uni-prizren.com
ORCID: 0000-0002-5413-9097

Esma Canhasi Kasemi
*Faculty of Education, University of Prizren "Ukshin Hoti", Rruga e Shkronjave, nr. 1, 20000 Prizren, Kosovo*
*E-mail:* esma.canhasi.kasemi@uni-prizren.com
ORCID: 0000-0001-6885-6057

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