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Sufficient conditions for weighted integrability of the κ-Hankel transform

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Abstract. In this paper, we establish new sufficient conditions for the weighted integrability of the κ-Hankel transform. Our results extend and refine earlier Titchmarsh-type theorems by removing restrictive assumptions on the modulus of smoothness, such as the Bary and Δ_2 conditions that appeared in recent related works. Using the κ-Hankel translation operator and higher-order moduli of smoothness associated with this translation, we derive weighted $L^r_\kappa(\mathbb{R})$-integrability criteria for the κ-Hankel transform of functions belonging to $L^p_\kappa(\mathbb{R})$ and Sobolev-type spaces $W^s_{p,\kappa}$. The obtained results generalize several known integrability theorems for Fourier, Hankel, Jacobi–Dunkl, and Hankel–Clifford transforms. As applications, we present integrability results under power-type and logarithmic smoothness assumptions, as well as Boas-type uniqueness theorems. These findings contribute to the harmonic analysis of deformed integral transforms and provide tools applicable to problems with underlying symmetry structures.

Key Words and Phrases: κ-Hankel transform, κ-Hankel translation operator, Generalized Lipschitz spaces, the modulus of smoothness

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1. Introduction

Signal processing, image processing, and radar all extensively use the Fourier transform and its related integral transforms in the field of harmonic analysis. These mathematical tools enable the decomposition of complex signals into their constituent frequencies, facilitating the analysis and interpretation of data across various applications. As a result, researchers and engineers can enhance system performance and develop innovative solutions in communication and imaging technologies. The modulus of smoothness is used to measure the smoothness of a function. In practical algorithm applications, finite differences are used to replace derivatives. These techniques not only offer commentary on the behavior of functions but also aid in optimizing computational efficiency. This optimization is crucial in scenarios where processing speed and accuracy are paramount, such as in real-time signal processing and machine learning algorithms. By using these methods, professionals can get more accurate results while using fewer computer resources. This will lead to improvements in technology and data analysis. By leveraging such approaches, practitioners can improve the accuracy of numerical simulations and enhance their understanding of dynamic systems.

Let's consider a function $f : \mathbb{R} \to \mathbb{C}$ that is integrable in the Lebesgue sense over the real line, denoted as $f \in L^1(\mathbb{R})$. We define the Fourier transform of f by:

$$\widehat{f}(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} f(t) \mathrm{e}^{-itx} \mathrm{d}t, \quad x \in \mathbb{R}.$$

If, in addition, $\hat{f} \in L^1(\mathbb{R})$ and $f \in C(\mathbb{R})$ (f is continuous on $\mathbb{R}$), then the inversion formula

$$f(t) = (2\pi)^{-1/2} \int_{\mathbb{R}} \widehat{f}(x) \mathrm{e}^{itx} \mathrm{d}x,$$

takes place for all $t \in \mathbb{R}$ (see [3, Ch. 5, p. 192]). In this case, we have by the Riemann–Lebesgue lemma $\lim_{x\to\infty} \hat{f}(x) = 0$, that is, $f \in C_0(\mathbb{R})$.

In the case $1 < p \leq 2$, the Fourier transform of a function $f \in L^p(\mathbb{R})$ is defined as a limit of $(2\pi)^{-1/2} \int_a^b f(x)e^{-itx}dx$ in the $L^q(\mathbb{R})$ norm sense, where $\frac{1}{p} + \frac{1}{q} = 1$ and $a \to -\infty$, $b \to +\infty$.

In particular, $f \in L^q(\mathbb{R})$ and the following Hausdorff-Young inequality:

$$\|\widehat{f}\|_q \leq C\|f\|_p = C \left(\int_{\mathbb{R}} \|f(t)\|^p \mathrm{d}t \right)^{1/p}, \quad f \in L^p(\mathbb{R}),\ 1 < p \leq 2, \tag{1}$$

holds. For $p = 2$, the inequality in (1) is substituted by the Plancherel equality. For more about these results, refer to [12, Ch. III and IV] or [3, Ch. 5].

For $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, we consider the modulus of smoothness of order $k \in \mathbb{N}$

$$\omega_k(t, \delta)_p = \sup_{0 \leq h \leq \delta} ||\Delta_h^k f||_p, \quad \Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x + (k - 2j)h/2).$$

If $\omega_1(f, \delta)_p = O(\delta^\alpha)$, $0 < \alpha \leq 1$, then we write $f \in Lip(\alpha, p)$. If $f \in C_0(\mathbb{R})$ or f is uniformly continuous and bounded on $\mathbb{R}$ ($f \in BUC(R)$), then, by definition

$$\omega_k(f, \delta) = \omega_k(f, \delta)_\infty = \sup_{0 \leq h \leq \delta} \sup_{x \in \mathbb{R}} |\Delta_h^k f(x)|.$$

The following result of Titchmarsh is well known (see [2, Ch. 4, Theorem 84]). Let $1 < p \leq 2$, $0 < \alpha \leq 1$, and $f \in Lip(\alpha, p)$. Then $\hat{f}(t) \in L^{\beta}(\mathbb{R})$ for all β satisfying the inequality

$$\frac{p}{p + ap - 1} < \beta \leq q = \frac{p}{p-1}.$$

We denote that a non-negative measurable function $\lambda(t) \in L^1_{loc}(\mathbb{R}_+)$ belongs to the class $A\gamma$, $\gamma \geq 1$, if there exists $C(\gamma) \geq 1$ such that

$$\left(\int_{2^i}^{2^{i+1}} \lambda^{\gamma}(t)\mathrm{d}t\right)^{1/\gamma} \leq C(\gamma)2^{i(1-\gamma)/\gamma} \int_{2^{i-1}}^{2^i} \lambda(t)\mathrm{d}t, \quad i \in \mathbb{Z}. \tag{2}$$

According to the Hölder inequality, it is evident that the set A_{γ_1} is a subset of A_{γ_2} when $1 \leq \gamma_2 < \gamma_1$. The strictness of this embedding has been demonstrated in [12]. It is evident that a function $\lambda(t) \geq 0$ is measurable and possesses the attribute

$$\sup\{\lambda(t) : 2^i \leq t < 2^{i+1}\} \leq C \inf\{\lambda(t) : 2^{i-1} \leq t < 2^i\}, \quad i \in \mathbb{Z}$$

is contained in all classes $\mathcal{A}_\gamma$, $\gamma \geq 1$.

Gogoladze and Meskhi [9] presented an analog to formula (2) specifically for sequences. The formula (2) was proposed by Móricz [14] who demonstrated the following outcome: Given $1 < p \leq 2$ and $f \in L^p(\mathbb{R})$, if the equation $\frac{1}{p} + \frac{1}{q} = 1$ holds, where $0 < r < q$, and λ belongs to the set $A_{p/(p-rp+r)}$, then

$$\int_{|t|\geq 2} \lambda(t)|\widehat{f}(t)|^r \mathrm{d}t \leq \int_1^{\infty} \lambda(t)t^{-r/q}\omega^r(f, \pi/t)_p \mathrm{d}t.$$

A more general result and proof of its sharpness may be found in [12].

The main purpose of this article is to obtain a variant of the above proposition by the use of the modulus of smoothness $\omega_m(f,\delta)_{p,\kappa} = \sup_{0\leq h\leq\delta} ||\Delta^m_{h,\kappa} f||_{L^p_\kappa(\mathbb{R})}$. This result generalize a previous one by A. Elgargati, M El Loualid and R Daher see[7]

A recent similar result obtained by S. Volosivets [24] considering a majorant of the modulus of smoothness ω with additional condition on ω^q, that is most be in $B \cap \Delta_2$ where B is the Bary class and Δ_2 -condition which is $\omega(2x) \leq C\omega(x)$ with $f \in H^{m,\omega}_{\kappa,p,*}(\mathbb{R})$ defined by the modulud of smoothness $\omega^*_m(f,\delta)_{\kappa,p} = \|\Delta^m_{h,\kappa} f + \Delta^m_{-h,\kappa} f\|$, In this paper and based on the article [26], we use the $L^p_\kappa(\mathbb{R})$ and the Sobolev spaces $W^s_{p,\kappa}$ instead, see [24] for more details. Similar results are already obtained for Jacobi-Dunkl transform, Bessel transform, first and second Hankel-Clifford transform, see [8, 27, 29].

The subsequent sections of this work are structured as follows:

- In the second section we give the definition and properties of the κ-Hankel transform.
- In the third section, we state and prove several valuable auxiliary results, particularly the estimations of the κ-Hankel kernel.
- In the fourth section we state and prove the main result:

  Theorem 1. *Let* $1 < p \le 2$, $1/p + 1/q = 1$, $f \in L^p_\kappa(\mathbb{R})$, $m \in \mathbb{N}$. *If*

  $$\lambda \in \mathcal{A}_{p/(p-pr+r),k} = \mathcal{A}_{q/(q-r),k}\,,$$

  for some $r \in (0, q)$, $\lambda \in L^{q/(q-r)}_k[-1, 1]$ *and the integral*

  $$\int_1^\infty \lambda(x) x^{-2\kappa r/q} \omega^r_m(f,x^{-1})_{p,k} d\mu_\kappa(x),$$

  converges, then $\lambda(x)|\mathcal{F}_\kappa(f)(x)|^r \in L^1_k(\mathbb{R})$.

  and

  Theorem 2. *Let* $1 < p \le 2$, $1/p + 1/q = 1$, $s \in \mathbb{N}$, $m > 0$ *and* $f \in W^s_{p,\kappa}$. *If* $\lambda \in \mathcal{A}_{q/(q-r),k}$ *for some* $0 < r < q$, $\lambda \in L^{q/(q-r)}_k([-1, 1])$ *and the integral*

  $$\int_1^\infty \lambda(x) x^{-2r\kappa/q-2rs} \omega^r_m(\mathbb{L}^s_\kappa f, x^{-1})_{p,\kappa} \mathrm{d}\mu_\kappa(x),$$

  converges, then $\lambda(x)|\mathcal{F}_\kappa(f)(x)|^r \in L^1_k(\mathbb{R})$.

Finally, we conclude some special cases, and one of them is about the integrability of $\mathcal{F}_\kappa(f)$

2. κ-Hankel transform and generalized translations

In this section, we present the fundamental tools required for the κ-Hankel transform $\mathcal{F}_\kappa$. To obtain additional information, we direct the reader to the references [20, 21].

For $\kappa \geq 1/2$ and $1 \leq p \leq \infty$, let $L^p_\kappa(\mathbb{R})$ be the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{L^p_\kappa(\mathbb{R})} = \Big(\int_{\mathbb{R}} |f(x)|^p d\mu_\kappa(x)\Big)^{\frac{1}{p}} < \infty, \qquad \text{for } 1 \leq p < \infty,$$
$$||f||_{L^\infty_k(\mathbb{R})} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f| < \infty,$$

where

$$d\mu_\kappa(x) = \frac{1}{2\Gamma(2\kappa)}|x|^{2\kappa-1}dx.$$

For $p = 2$, we provide this space with the scalar product

$$\langle f, g\rangle_{L^2_k(\mathbb{R})} = \int_{\mathbb{R}} f(x)\overline{g(x)}d\mu_\kappa(x).$$

The κ-Hankel transform of $f \in L^1_k(\mathbb{R})$ is defined by

$$\mathcal{F}_\kappa(f)(\lambda) = c_\kappa \int_{\mathbb{R}} f(x)B_\kappa(\lambda, x)d\mu_\kappa(x), \qquad \lambda \in \mathbb{R}$$

where

$$c_\kappa := \Big(\int_{\mathbb{R}} e^{-|x|}d\mu_\kappa(x)\Big)^{-1} = 2^{-1}\Gamma(2k)^{-1},$$

and $\mathcal{B}_\kappa(\lambda, x)$ is the κ-Hankel kernel given by

$$B_\kappa(\lambda, x) = B_\kappa(\lambda x) = j_{2\kappa-1}\big(2\sqrt{|\lambda.x|}\big) - \frac{\lambda.x}{2\kappa(2\kappa+1)}j_{2k+1}\big(2\sqrt{|\lambda.x|}\big).$$

Here

$$j_\alpha(u) := \Gamma(\alpha+1)\Big(\frac{u}{2}\Big)^{-\alpha}J_\alpha(u) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(\alpha+m+1)}\Big(\frac{u}{2}\Big)^{2m},$$

denotes the normalized Bessel function of index α.

Consider the Dunkl Laplacian, defined by

$$\Lambda_k = \frac{d^2}{dx^2} + \frac{2k}{x}\frac{d}{dx} - \frac{k}{x^2}(1 - S),$$

where $Sf(x) = f(-x)$. The kernel B_κ satisfies the following differential-difference equation:

$$|x|\Lambda_k B_\kappa(\lambda, x) = -|\lambda|B_\kappa(\lambda, x). \tag{3}$$

We point out that

$$\mathcal{B}_\kappa(0,y) = 1, \qquad |\mathcal{B}_\kappa(x,y)| \leq 1,$$

for all $x, y \in \mathbb{R}$. In particular, we deduce that for all f in $L^1_k(\mathbb{R})$,

$$\|\mathcal{F}_\kappa(f)\|_{L^\infty_k(\mathbb{R})} \leq c_\kappa \|f\|_{L^1_k(\mathbb{R})}, \tag{4}$$

The observation is that $\mathcal{F}_\kappa$ serves as a natural extension of the usual Hankel transform.

$$\mathcal{B}^{\text{even}}_\kappa(x,y) = \frac{1}{2}(\mathcal{B}_\kappa(x,y) + \mathcal{B}_\kappa(x,-y)) = j_{2k-1}(2\sqrt{|xy|}).$$

The κ-Hankel transform of an even function on the real line is a specialized form of the Hankel type transform on $\mathbb{R}_+$.

The following statement is a particular case of [20, Theorem 3.39].

Theorem 3. *Assume that $\kappa \geq 1/2$.*

1. *(Plancherel's theorem) $\mathcal{F}_\kappa$ is an isometric isomorphism on $L^2_k(\mathbb{R})$,*

$$\int_{\mathbb{R}} |f(x)|^2 \, d\mu_\kappa(x) = \int_{\mathbb{R}} |\mathcal{F}_\kappa(f)(\lambda)|^2 \, d\mu_\kappa(\lambda). \tag{5}$$

2. *(Parseval's formula) For all f, g in $L^2_k(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} f(x)\overline{g(x)} d\mu_\kappa(x) = \int_{\mathbb{R}} \mathcal{F}_\kappa(f)(\lambda)\overline{\mathcal{F}_\kappa(g)(\lambda)} d\mu_\kappa(\lambda). \tag{6}$$

3. *(Inversion formula) $\mathcal{F}_\kappa$ satisfies*

$$\mathcal{F}^{-1}_\kappa = \mathcal{F}_\kappa.$$

By using the Plancherel formula (5) and the inequality (4), it can be easily concluded that for every function f belonging to the space $L^p_\kappa(\mathbb{R})$ with $1 \leq p \leq 2$, the κ-Hanekl transform $\mathcal{F}_\kappa(f)$ also belongs to the space $L^q_k(\mathbb{R})$, where $\frac{1}{q} + \frac{1}{p} = 1$.

$$||\mathcal{F}_\kappa(f)|||_{L^q_k(\mathbb{R})} \leq c_\kappa^{\frac{2-p}{p}} ||f||_{L^p_\kappa(\mathbb{R})}. \tag{7}$$

Said in [21] introduced a translation operator on the space $L^p_\kappa(\mathbb{R}), 1 \leq p \leq \infty$, using the following formula:

$$\tau^\kappa_x f(y) = \int_{\mathbb{R}} f(z) d\zeta^\kappa_{x,y}(z), \qquad x,\, y \in \mathbb{R}.$$

where

$$d\zeta^{\kappa}_{x,y}(z) = \begin{cases} K_\kappa(x,y,z)|z|^{2\kappa-1}dz, & \text{if } xy \neq 0, \\ d\delta_x(z), & \text{if } y = 0, \\ d\delta_y(z), & \text{if } x = 0, \end{cases}$$

with an explicit kernel $K_\kappa(x,y,z)$ supported on

$$\left(\sqrt{|x|} - \sqrt{|y|}\right)^2 < |z| < \left(\sqrt{|x|} + \sqrt{|y|}\right)^2.$$

For the precise formulation of $K_\kappa(x,y,z)$, we refer to the paper by Ben Saïd [21].

Here, we present a summary of the findings for the κ-Hankel translation operator as documented in [21].

Proposition 1. *Assume that $\kappa \geq 1/2$.*

1. *For all $f \in L^p_\kappa(\mathbb{R})$, $1 \leq p \leq \infty$, and for all $x, y \in \mathbb{R}$, we have*

$$\tau^{\kappa}_x(f)(y) = \tau^{k}_y(f)(x).$$

2. *If $f \in L^2_k(\mathbb{R})$ and $\lambda, x \in \mathbb{R}$, then*

$$\mathcal{F}_\kappa(\tau^\kappa_x f)(\lambda) = \mathcal{B}_\kappa(\lambda, x)\mathcal{F}_\kappa(f)(\lambda). \tag{8}$$

3. *For all f in $L^2_k(\mathbb{R})$, we have*

$$\|\tau^\kappa_x f\|_{L^2_k(\mathbb{R})} \leq \|f\|_{L^2_k(\mathbb{R})}, \qquad \forall x \in \mathbb{R}.$$

4. *For all $f \in L^1_k(\mathbb{R})$ such that $\mathcal{F}_\kappa(f) \in L^1_k(\mathbb{R})$, we have*

$$\tau^\kappa_x f(y) = c_\kappa \int_{\mathbb{R}} \mathcal{B}_\kappa(x,\xi)\mathcal{B}_\kappa(y,\xi)\mathcal{F}_\kappa(f)(\xi)d\mu_\kappa(\xi).$$

The explicit formula of the kernel $K_\kappa(x,y,z)$ implies that the operator $\tau_y f$ is bounded. To be more exact:

Proposition 2. *(See [21]) For all $f \in L^p_\kappa(\mathbb{R})$, $1 \leq p \leq \infty$, there exists a positive constant A_k such that*

$$\|\tau^\kappa_x f\|_{L^p_\kappa(\mathbb{R})} \leq A_\kappa \|f\|_{L^p_\kappa(\mathbb{R})}, \quad \forall x \in \mathbb{R}.$$

Several essential properties of $\tau^\kappa_x f$ for f being an even function are established in [21]. Let $L^p_{\kappa,e}(\mathbb{R})$ be the space of even functions in $L^p_\kappa(\mathbb{R})$.

Proposition 3. *1. For all f in* $L^p_{k,e}(\mathbb{R})$, $1 \le p \le \infty$, *we have*

$$\tau^\kappa_x f(y) = \int_0^\pi f(|x| + |y| - 2\sqrt{|xy|}\cos\phi) \left\{1 + \frac{\text{sgn}(xy)}{4k-1}\Big(4k\cos^2\phi - 1\Big)\right\}(\sin\phi)^{4k-2}d\phi.$$

2. For all non-negative f *in* $L^1_{\kappa,e}(\mathbb{R})$, *we have, for all* $x \in \mathbb{R}$,

$$\tau^\kappa_x f \ge 0, \quad \tau^\kappa_x f \in L^1_k(\mathbb{R}),$$

and

$$\int_{\mathbb{R}} \tau^\kappa_x f(y) d\mu_\kappa(y) = \int_{\mathbb{R}} f(y) d\mu_\kappa(y).$$

3. For all f in $L^p_{\kappa,e}(\mathbb{R})$, $1 \le p \le \infty$, *we have*

$$\|\tau^\kappa_x f\|_{L^p_\kappa(\mathbb{R})} \le \|f\|_{L^p_\kappa(\mathbb{R})}, \quad \forall x \in \mathbb{R}. \tag{9}$$

3. Auxiliary propositions and definitions

Denote

$$\mathbb{L}_\kappa(\lambda) = |\lambda|\Lambda_\kappa(\lambda), \tag{10}$$

and $W^n_{p,\kappa}$ be the Sobolev space constructed by $\mathbb{L}^n_\kappa$ as follows:

$$W^n_{p,\kappa} = \{f \in L^p_\kappa(\mathbb{R}) : \mathbb{L}^i_\kappa f \in L^p_\kappa(\mathbb{R}), \quad i = 1, ..., n\},$$

where $\mathbb{L}^i_\kappa f = \mathbb{L}_\kappa(\mathbb{L}^{i-1}_\kappa f)$ and $\mathbb{L}^0_\kappa f = f$.

Denote by Φ the set of continuous and increasing on $\mathbb{R}_+ = [0, \infty)$ functions ω such that $\omega(0) = 0$. If $\omega \in \Phi$ and $\int_0^\delta t^{-1}\omega(t)dt = O(\omega(\delta))$, then ω belongs to the Bary class B; if $\omega \in \Phi$ and $\delta^m \int_\delta^\infty t^{-m-1}\omega(t)dt = O(\omega(\delta))$, $m > 0$, then ω belongs to the Bary-Stechkin class B_m (see [13]). We say that $\omega \in \Phi$ satisfies the $\triangle_2$ -condition, if $\omega(2x) \le C\omega(x)$, $x \in \mathbb{R}_+$.

Let $\omega \in \Phi$. Let us introduce for $m \in \mathbb{N}$, $\omega \in \Phi$ and $\kappa > 1/2$ the spaces of generalized Lipschitz classes :

$$H^{m,\omega}_{p,k}(\mathbb{R}) = \{f \in L^p_\kappa(\mathbb{R}) : ||\Delta^m_{h,\kappa} f||_{L^p_\kappa(\mathbb{R})} = O(\omega(h)),\ h > 0\},$$

and

$$h^{m,\omega}_{p,k}(\mathbb{R}) = \{f \in H^{m,\omega}_{p,k}(\mathbb{R}) : \|\Delta^m_{h,\kappa} f(x)\|_{L^p_\kappa(\mathbb{R})} = o(\omega(h)),\ h \to 0^+\}.$$

The spaces $H^{1,\omega}_{p,k}(\mathbb{R})$ and $h^{1,\omega}_{p,k}(\mathbb{R})$, are called the Lipschitz class $Lip(\kappa)$ and little Lipschitz class $lip(\kappa)$, respectively. The spaces $H^{2,\omega}_{p,k}(\mathbb{R})$ and $h^{2,\omega}_{p,k}(\mathbb{R})$, are called the Zygmund class $Zyg(\kappa)$ and little Zygmund class $zyg(k)$, respectively. Also we consider for $\omega \in \Phi, 1 \le p < \infty$ and $m, r \in \mathbb{N}$ an analogue of Nikol'skii space

$$W^r H^{m,\omega}_{p,k}(\mathbb{R}) = \left\{ f \in W^r_{p,\kappa}(\mathbb{R}) : \mathbb{L}^r_\kappa f \in H^{m,\omega}_{p,k}(\mathbb{R}) \right\}.$$

For $r = 0$ we have by definition $W^r H^{m,\omega}_{p,k}(\mathbb{R}) = H^{m,\omega}_{p,k}(\mathbb{R})$. If $\omega(\delta) = \delta^\beta, \beta > 0$, we denote $W^r H^{m,\omega}_{p,k}(\mathbb{R})$ as $DLip(\beta, p, m, r)$.

Let $\lambda(x)$ be a non-negative measurable function from $L^1_{loc}(\mathbb{R}_+)$, $1 \le \gamma < \infty$, $\mathbb{Z}_+ = \{0, 1, ...\}$. If there exists $C(\gamma) \ge 1$ such that

$$\left(\int_{2^i}^{2^{i+1}} \lambda^\gamma(x) d\mu_\kappa(x) \right)^{1/\gamma} \le C(\gamma) 2^{2\kappa i(1-\gamma)/\gamma} \int_{2^{i-1}}^{2^i} \lambda(x) d\mu_\kappa(x), \quad i \in \mathbb{Z},$$

then $\lambda \in \mathcal{A}_{\gamma,k}$. Also we put $\lambda(x) = \lambda(-x)$ for all $x < 0$.

Using the operator τ^κ_h we define the generalized difference operator of order $m \in \mathbb{N}^*$ with step $h \ge 0$

$$\Delta^m_{h,\kappa} f(x) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} (\tau^\kappa_h) f(x),$$

We consider the modulus of smoothness of order m in $L^p_\kappa(\mathbb{R})$ related to κ-Hankel translation

$$\omega_m(f, \delta)_{p,k} = \sup_{0 \le h \le \delta} ||\Delta^m_{h,\kappa} f||_{L^p_\kappa(\mathbb{R})}.$$

By 2 we see that $\omega_m(f, \delta)_{p,k}$ is well defined for $f \in L^p_\kappa(\mathbb{R})$.

Lemma 1. *Let $n \in \mathbb{N}$,$1 \le p < \infty$ and $f \in W^n_{p,\kappa}$. Then*

$$\mathcal{F}_\kappa(\mathbb{L}^s_\kappa f)(x) = (-1)^s |x|^s \mathcal{F}_\kappa(f)(x),$$

$s = 1, 2, \dots, n$ a.e. on $\mathbb{R}$.

Proof. Using the formulas 3 and 10.

Lemma 2. *Let $m \in \mathbb{N}$, $s \in \mathbb{N}$, $1 < p \le 2$, $f \in W^s_{p,\kappa}$ and $h > 0$. Then*

$$\mathcal{F}_\kappa(\Delta^m_{h,\kappa} \mathbb{L}^s_\kappa f)(x) = (-1)^s |x|^s (1 - \mathcal{B}_\kappa(x, h))^m \mathcal{F}_\kappa(f)(x).$$

Proof. It is sufficient to study the case $s = 0$. If $f \in L^p_\kappa(\mathbb{R})$, then $\Delta^m_{h,\kappa} f$ exists due to the inequality (9). On the other hand, by (7) from the equality $\lim_{n\to\infty} \|f - f_n\|_{L^p_\kappa(\mathbb{R})} = 0$, $1 < p \leq 2$, we conclude that

$$\lim_{n\to\infty} \|\mathcal{F}_\kappa(f) - \mathcal{F}_\kappa(f_n)\|_{L^q_k(\mathbb{R})} = 0, \quad 1/p + 1/q = 1.$$

Since by (8) $\mathcal{F}_\kappa((\tau^\kappa_h)^j f)(\lambda) = (\mathcal{B}_\kappa(\lambda, h))^j \mathcal{F}_\kappa(f)(\lambda)$ a.e. on $\mathbb{R}$ for $j \in \mathbb{Z}_+ = \{0, 1, \dots\}$, we obtain

$$\mathcal{F}_\kappa(\Delta^m_{h,\kappa} f)(x) = \sum_{i=0}^{m} (-1)^j \binom{m}{j} (\mathcal{B}_\kappa(\lambda, h))^j \mathcal{F}_\kappa(f)(x) = (1 - \mathcal{B}_\kappa(\lambda, h))^m \mathcal{F}_\kappa(f)(x),$$

a.e. on $\mathbb{R}$.

in the next of this section and in order to prove that $C \neq 0$ in part 1 of Lemma 6 see [[15], Lemma 1] witch is not clear, we state and prove the following important results:

Lemma 3. *The Bessel function j_α have following the integral representation:*

$$j_\alpha(\xi) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} \cos(\xi t) dt.$$

Proof. using the integral representation of j_α in [19] and the fact that the function $t \mapsto (1-t^2)^{\alpha-\frac{1}{2}} \sin(\xi t)$ is odd,

$$\begin{aligned}
j_\alpha(\xi) =& \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} e^{i\xi t} dt \\
=& \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} \left(\cos(\xi t) + i \sin(\xi t)\right) dt \\
=& \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \left(\int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} \cos(\xi t) dt + i \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} \sin(\xi t) dt \right) \\
=& \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} \cos(\xi t) dt.
\end{aligned}$$

Lemma 4. *The κ-Hankel kernel $B_\kappa(\xi)$ admit the flowing integral representations,*

- *if $\xi \leq 0$*

$$B_\kappa(\xi) = \frac{\Gamma(2\kappa+1)}{\sqrt{\pi}\Gamma(2\kappa+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{2\kappa-\frac{1}{2}} \cos(2\sqrt{|\xi|} t) dt,$$

- *if* $\xi \geq 0$

$$B_\kappa(\xi) = \frac{\Gamma(2\kappa)}{\sqrt{\pi}\Gamma(2\kappa-\frac{1}{2})}\frac{2\kappa-1}{2\kappa-\frac{1}{2}}\int_{-1}^{1}\left(1+\frac{2\kappa t^2}{2\kappa-1}\right)\left(1-t^2\right)^{2\kappa-\frac{3}{2}}\cos(2t(\sqrt{\xi}))dt.$$

Proof.

- if $\xi \leq 0$ then by the formula in [19]

$$\begin{aligned} B_\kappa(\xi) =& j_{2\kappa}\left(2\sqrt{|\xi|}\right), \\ =& \frac{\Gamma(2\kappa+1)}{\sqrt{\pi}\Gamma(2\kappa+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{2\kappa-\frac{1}{2}}\cos(2\sqrt{|\xi|}t)dt. \end{aligned}$$

- if $\xi \geq 0$ then by the formula in [19] and the fact that the function $t \mapsto \sin\left(2t(\sqrt{\xi})\right)\left(1+\frac{2\kappa t^2}{2\kappa-1}\right)\left(1-t^2\right)^{2\kappa-\frac{3}{2}}$ is odd,

$$\begin{aligned} B_\kappa(\xi) =& 2j_{2\kappa-1}\left(2\sqrt{\xi}\right) - j_{2\kappa}\left(2\sqrt{\xi}\right) \\ =& \frac{\Gamma(2\kappa)}{\sqrt{\pi}\Gamma(2\kappa-\frac{1}{2})}\frac{2\kappa-1}{2\kappa-\frac{1}{2}}\int_{-1}^{1}\exp\left(2it(\sqrt{\xi})\right)\left(1+\frac{2\kappa t^2}{2\kappa-1}\right)\left(1-t^2\right)^{2\kappa-\frac{3}{2}}dt \\ =& \frac{\Gamma(2\kappa)}{\sqrt{\pi}\Gamma(2\kappa-\frac{1}{2})}\frac{2\kappa-1}{2\kappa-\frac{1}{2}}\int_{-1}^{1}\cos\left(2t(\sqrt{\xi})\right)\left(1+\frac{2\kappa t^2}{2\kappa-1}\right)\left(1-t^2\right)^{2\kappa-\frac{3}{2}}dt+ \\ & i\frac{\Gamma(2\kappa)}{\sqrt{\pi}\Gamma(2\kappa-\frac{1}{2})}\frac{2\kappa-1}{2\kappa-\frac{1}{2}}\int_{-1}^{1}\sin\left(2t(\sqrt{\xi})\right)\left(1+\frac{2\kappa t^2}{2\kappa-1}\right)\left(1-t^2\right)^{2\kappa-\frac{3}{2}}dt \\ =& \frac{\Gamma(2\kappa)}{\sqrt{\pi}\Gamma(2\kappa-\frac{1}{2})}\frac{2\kappa-1}{2\kappa-\frac{1}{2}}\int_{-1}^{1}\left(1+\frac{2\kappa t^2}{2\kappa-1}\right)\left(1-t^2\right)^{2\kappa-\frac{3}{2}}\cos\left(2t(\sqrt{\xi})\right)dt. \end{aligned}$$

Lemma 5. *Let $F(t)$ be an even, non-negative and Lebesgue integrable function on $[-1,1]$ satisfy the condition $\int_{-1}^{1}F(t)dt > 0$. Then there exists a constant c such that:*

$$\int_{-1}^{1}F(t)\sin^2(rt)\mathrm{d}t \geq c > 0,$$

for $r > \frac{1}{2}$.

Proof. by the Lemma 3.3 in [17] and the fact that $\int_{-1}^{1}F(t)\sin^2(rt)\mathrm{d}t = 2\int_{0}^{1}F(t)\sin^2(rt)\mathrm{d}t$

Proposition 4. *we have* $B_\kappa(\xi) \neq 1$ *for* $\xi > \frac{1}{2}$

Proof. Suppose that $\xi > \frac{1}{2}$ we have

$$\begin{aligned}1 - B_\kappa(\xi) =& B_\kappa(0) - B_\kappa(\xi)\\ =& \frac{\Gamma(2\kappa)}{\sqrt{\pi}\Gamma(2\kappa - \frac{1}{2})} \frac{2\kappa - 1}{2\kappa - \frac{1}{2}} \int_{-1}^{1} \left(1 + \frac{2\kappa t^2}{2\kappa - 1}\right) \left(1 - t^2\right)^{2\kappa - \frac{3}{2}} \left(1 - \cos(2t(\sqrt{\xi}))\right) dt\\ =& \frac{2\Gamma(2\kappa)}{\sqrt{\pi}\Gamma(2\kappa - \frac{1}{2})} \frac{2\kappa - 1}{2\kappa - \frac{1}{2}} \int_{-1}^{1} \left(1 + \frac{2\kappa t^2}{2\kappa - 1}\right) \left(1 - t^2\right)^{2\kappa - \frac{3}{2}} \sin^2(\sqrt{\xi}t) dt,\end{aligned}$$

if we put $F(t) = \left(1 + \frac{2\kappa t^2}{2\kappa - 1}\right) \left(1 - t^2\right)^{2\kappa - \frac{3}{2}}$ and $r = \sqrt{\xi}$ then we have $\int_{-1}^{1} F(t)dt > 0$, by the application of Lemma 5 we conclude that

$$|1 - B_\kappa(\xi)| \geq c > 0.$$

Lemma 6 (see [15])**.** .

1. *Let* $\kappa > 1/2$ *and* $|\lambda x| \geq 1$*. Then we obtain the following inequality:*

$$|1 - B_\kappa(\lambda, x)| \geq C,$$

where C *is positive constant.*

2. *Furthermore the behaviour in* 0 *of the kernel* $B_\kappa(\lambda, x)$ *could be expressed as follows:*

$$|B_\kappa(\lambda, x) - 1|^2 = O(|\lambda x|^2_.),$$

3. *There exists a positive constant* C *and* $\eta > 0$ *such that:*

$$|\lambda x| \leq \eta \Longrightarrow |\mathcal{B}_\kappa(\lambda, x) - 1| \geq C|\lambda x|.$$

Proof. we refer to the proof of [[15], Lemma 1], except of the problem about that $m = \min_{|\lambda x| \in [1,A]} |\mathbf{1} - B_\kappa(\lambda, x)|$ may be equal to 0, by the application of 4 we are sure that $m \neq 0$

4. Weighted integrability of κ-Hankel transform

Theorem 4. *Let* $1 < p \leq 2$, $1/p + 1/q = 1$, $f \in L^p_\kappa(\mathbb{R})$, $m \in \mathbb{N}$. *If*

$$\lambda \in \mathcal{A}_{p/(p-pr+r),k} = \mathcal{A}_{q/(q-r),k}\,,$$

for some $r \in (0,q)$, $\lambda \in L^{q/(q-r)}_k[-1,1]$ *and the integral*

$$\int_1^\infty \lambda(x) x^{-2\kappa r/q} \omega^r_m(f,x^{-1})_{p,k} d\mu_\kappa(x),$$

converges, then $\lambda(x)|\mathcal{F}_\kappa(f)(x)|^r \in L^1_k(\mathbb{R})$.

Proof. By the use of the Lemma (2) and Hausdorff-Young type inequality (2)

$$\int_{\mathbb{R}} |\mathcal{F}_\kappa(f)(x)|^q |1-\mathcal{B}_\kappa(x,h)|^{mq} d\mu_\kappa(x) \leq C_1 ||\Delta^m_{h,\kappa} f||^q_{L^p_\kappa(\mathbb{R})} \leq C_1 \omega_m(f,h)_{p,k}.$$

Let $M_i = \left\{x \in \mathbb{R} : 2^{i-1} \leq |x| < 2^i\right\}$, $i \in \mathbb{N}$ and $h = 2^{-i}$.we have by (6)(3) $1 - \mathcal{B}_\kappa(x,\lambda) \neq 0$ for $x\lambda \neq 0$. Since $1 - \mathcal{B}_\kappa(x,h)$ is continuous on $\mathbb{R}$ we obtain the inequality $C_2 = \min\left\{|1-\mathcal{B}_\kappa(x,\lambda)| : |x\lambda| \in [\frac{1}{2},1]\right\} > 0$ and

$$\begin{aligned} C_2 \int_{M_i} |\mathcal{F}_\kappa(f)(x)|^q \mathrm{d}\mu_\kappa(x) &\leq \int_{M_i} |\mathcal{F}_\kappa(f)(x)|^q |1-\mathcal{B}_\kappa(x,2^{-i})|^{mq} \mathrm{d}\mu_\kappa(x) \\ &\leq \int_{\mathbb{R}} |\mathcal{F}_\kappa(f)(x)|^q |1-\mathcal{B}_\kappa(x,2^{-i})|^{mq} \mathrm{d}\mu_\kappa(x) \leq C_1 \omega^q_m(f,2^{-i})_{p,k}. \end{aligned}$$

By the Hölder inequality and the condition $\lambda \in \mathcal{A}_{q/(q-r),k}$ we have for $0 < r < q$

$$\begin{aligned} &\int_{M_i} \lambda(x) |\mathcal{F}_\kappa(f)(x)|^r \mathrm{d}\mu_\kappa(x) \\ &\leq \left(\int_{M_i} |\lambda(x)|^{q/(q-r)} \mathrm{d}\mu_\kappa(x)\right)^{1-r/q} \left(\int_{M_i} |\mathcal{F}_\kappa(f)(x)|^q \mathrm{d}\mu_\kappa(x)\right)^{r/q} \\ &\leq \mathrm{C}_3 \omega^r_m(f,2^{-i})_{p,k} 2^{-2\kappa i r/q} \int_{M_{i-1}} \lambda(x) \mathrm{d}\mu_\kappa(x) \\ &\leq C_3 \int_{M_{i-1}} \lambda(x) \omega^r_m(f,x^{-1})_{p,k} (1/x)^{2\kappa r/q} \mathrm{d}\mu_\kappa(x) \qquad (11) \\ &= 2C_3 \int_{2^{i-2}}^{2^{i-1}} \lambda(x) \omega^r_m(f,x^{-1})_{p,k} (1/x)^{2\kappa r/q} \mathrm{d}\mu_\kappa(x). \qquad (12) \end{aligned}$$

By summing up the inequalities (11) over $i \in \mathbb{N}^*$ we find that

$$\int_{|x|\geq 1} \lambda(x) |\mathcal{F}_\kappa(f)(x)|^r \mathrm{d}\mu_\kappa(x) \leq 2C_3 \int_{1/2}^\infty \lambda(x) \omega^r_m(f,x^{-1})_{p,k} x^{-2\kappa r/q} \mathrm{d}\mu_\kappa(x). \qquad (13)$$

Note that $\omega_m(f,x^{-1})_{p,k} \le C_4 \|f\|_{L^p_\kappa(\mathbb{R})}$ for all $x>0$. By the use of (2), Hölder inequality and (7) we have

$$\int_{1/2}^{1} \lambda(x)\omega_m^r(f,x^{-1})_{p,k}x^{-2\kappa r/q}\mathrm{d}\mu_\kappa(x) \le 2^{2\kappa r/q}C_4^r\|f\|^r_{L^p_\kappa(\mathbb{R})}\int_{1/2}^{1}\lambda(x)\mathrm{d}\mu_\kappa(x)$$
$$\le C_5\left(\int_{-1}^{1}|\lambda(x)|^{q/(q-r)}\mathrm{d}\mu_\kappa(x)\right)^{1-r/q}\left(\int_0^1 t^{2k-1}\mathrm{d}x\right)^{r/q} < \infty.$$

Thus, the right-hand side of (13) is finite. On the other hand, by (7) and the condition $\lambda \in L_k^{q/(q-r)}[-1,1]$ we obtain

$$\int_{-1}^{1}\lambda(x)|\mathcal{F}_\kappa(f)(x)|^r d\mu_\kappa(x)$$
$$\le\left(\int_{-1}^{1}|\mathcal{F}_\kappa(f)(x)|^q\mathrm{d}\mu_\kappa(x)\right)^{r/q}\left(\int_{-1}^{1}|\lambda(x)|^{q/(q-r)}\mathrm{d}\mu_\kappa(x)\right)^{1-r/q}$$
$$<\infty. \tag{14}$$

From (13) and (14) we deduce the statement of (4).

Theorem 5. *Let* $1<p\le 2$, $1/p+1/q=1$, $s\in\mathbb{N}$, $m>0$ *and* $f\in W^s_{p,\kappa}$. *If* $\lambda \in \mathcal{A}_{q/(q-r),k}$ *for some* $0<r<q$, $\lambda\in L_k^{q/(q-r)}([-1,1])$ *and the integral*

$$\int_1^\infty \lambda(x)x^{-2r\kappa/q-2rs}\omega_m^r(\mathbb{L}^s_\kappa f,x^{-1})_{p,\kappa}\mathrm{d}\mu_\kappa(x),$$

converges, then $\lambda(x)|\mathcal{F}_\kappa(f)(x)|^r \in L^1_k(\mathbb{R})$.

Proof. It is easy to see that if $\lambda(x)$ belongs to the class $A_{\alpha,k}$ with the constant $C_1=C_1(\alpha)$, then $\lambda_1(x)=\lambda(x)x^{-2rs}$ also belongs to $\mathcal{A}_{\alpha,k}$ with the same constant C_1. Substituting λ_1 instead of λ and $f_1=\mathbb{L}^s_\kappa f$ instead of f into (13) and applying Lemma (1) we obtain

$$\int_{|x|\ge1}\lambda_1(x)|\mathcal{F}_\kappa(f_1)(x)|^r\mathrm{d}\mu_\kappa(x)=\int_{|x|\ge1}\lambda(x)\frac{|\mathcal{F}_\kappa(\mathbb{L}^s_\kappa f)(x)|^r}{|x|^{sr}}\mathrm{d}\mu_\kappa(x)$$
$$=\int_{|x|\ge1}\lambda(x)|x|^{-sr}|\mathcal{F}_\kappa(\mathbb{L}^s_\kappa f)(x)|^r\mathrm{d}\mu_\kappa(x)$$
$$\le C_2\int_{1/2}^{\infty}\lambda(x)|x|^{-r(2\kappa/q+s)}\omega_m^r(\mathbb{L}^s_\kappa f,|x|^{-1})_{p,k}\mathrm{d}\mu_\kappa(x). \tag{15}$$

As in the proof of Theorem (4) we show that the right-hand side of (15) is finite. By (14) we have $\lambda(t)|\mathcal{F}_\kappa(f)(t)|^r\in L^1_k([-1,1])$. Theorem is proved.

Theorem 6. *Let* $1 < p \leq 2, 1/p + 1/q = 1, m, s \in \mathbb{N}$ *and* $f \in W^s_{p,\kappa}(\mathbb{R})$. *If* $\omega \in \Phi, lim_{\delta\to 0}\omega(\delta)/\delta^m = 0$ *and* $f \in W^s H^{m,\omega}_{p,k}(\mathbb{R})$, *then* $f(x) = 0$ *a.e. on* $\mathbb{R}$.

Proof. using Lemma 1, Lemma 2 and Hausdorff-Young type inequality (7) one has

$$\left(\int_{\mathbb{R}} |B_\kappa(\lambda, x) - 1|^{qm}|x|^{qs}\mathcal{F}_\kappa(f)(x)|^q \mathrm{d}\mu_\kappa(x)\right)^{1/q} \leq C_1||\Delta^m_{h,d}\mathbb{L}^s_\kappa f||_{p,\nu} \leq C_2\omega(\lambda),$$

for $\lambda > 0$. By the condition $\lim_{\delta\to 0} \frac{\omega(\delta)}{\delta^m} = 0$ we obtain that

$$\lim_{\lambda\to 0} \lambda^{-qm} \int_{\mathbb{R}} \frac{|B_\kappa(\lambda, x) - 1|^{qm}}{|x\lambda|^{qm}}|x|^{qs}|x\lambda|^{qm}|\mathcal{F}_\kappa(f)(x)|^q d\gamma_k(x) = 0.$$

The behavior in 0 of the kernel $B_\kappa(\lambda, x)$ could be expressed as follows (see Lemma 1 in [15])

$$B_\kappa(\lambda, x) = 1 - \frac{1}{2k}|\lambda x| - \frac{\lambda x}{2k(2k+1)} + \frac{sgn(\lambda x)}{2k(2k+1)(2k+2)}|\lambda x|^2 + +o(|\lambda x|^2).$$

Since $\lim_{|\lambda x|\to 0} \frac{|B_\kappa(\lambda,x)-1|}{|\lambda x|} = \frac{2k+1+sgn(\lambda x)}{2k(2k+1)} > 0$ (see Introduction), we see that

$$\int_{\mathbb{R}} |x|^{q(m+s)}|\mathcal{F}_\kappa(f)(x)|^q \mathrm{d}\mu_\kappa(x) = 0.$$

Then $\mathcal{F}_\kappa(f)(x) = 0$ a.e. on $\mathbb{R}$ and by the Plancherel equation (5) we have $f(x) = 0$, a.e. on $\mathbb{R}$.

We will write $A(i) \asymp B(i)$ if $A(i) = O(B(i))$ and $B(i) = O(A(i))$.

Corollary 1. *Let* $1 < p \leq 2$, $1/p + 1/q = 1$, $f \in L^p_\kappa(\mathbb{R})$, $m \in \mathbb{N}$, $r \in (0, q)$. *If* $\alpha > 2(\frac{r}{q} - 1)k$ *and the integral*

$$\int_1^\infty x^{\alpha-2r\kappa/q}\omega^r_m(f, x^{-1})_{p,\kappa}d\mu_\kappa(x), \tag{16}$$

converges, then $|x|^\alpha|\mathcal{F}_\kappa(f)(x)|^r \in L^1_k(\mathbb{R})$.

Proof. Note that

$$\int_{-1}^1 |x|^{\alpha q/(q-r)}d\mu_\kappa(x) = \frac{1}{\Gamma(2k)}\int_0^1 x^{2k-1+q\alpha/(q-r)}dx < \infty,$$

if $2k-1+q\alpha/(q-r)>-1$ or $\alpha>2(r/q-1)k$. On the other hand,

$$I_1=\left(\int_{M_i}|x|^{\alpha q/(q-r)}d\mu_\kappa(x)\right)^{1-r/q}\asymp 2^{i(2\kappa-1+q\alpha/(q-r))(1-r/q)}\asymp 2^{i(2\kappa(1-r/q)+\alpha)},$$

and

$$I_2=\int_{M_{i-1}}|x|^\alpha d\mu_\kappa(x)\asymp 2^{i(2\kappa+\alpha)},$$

whence $I_1\le C_1 2^{-2\kappa ir/q}I_2$, $i\in\mathbb{Z}$, and $|x|^\alpha\in\mathcal{A}_{q/(q-r),\kappa}$ for all $\alpha\in\mathbb{R}$. Using Theorem 4 we obtain the statement of Corollary.

Corollary 2. *Let* $1<p\le 2$, $1/p+1/q=1$, $f\in L^p_\kappa(\mathbb{R})$, $m\in\mathbb{N}$, $r\in(0,q)$. *If* $\alpha>2(r/q-1)k$ *and* $f\in DLip(\beta,p,m,0)$ *and*

$$q>r>\frac{\alpha q+2\kappa q}{2\kappa+\beta q},$$

then $|x|^\alpha|\mathcal{F}_\kappa(f)(x)|^r\in L^1_k(\mathbb{R})$.

Proof. is clear that under conditions of Corollary the convergence of integral $\int_1^\infty x^{\alpha-2r\kappa/q-r\beta}x^{2\kappa-1}dt$ is sufficient for the convergence of (16). The condition:

$$\alpha-2r\kappa/q-r\beta+2k-1<-1,$$

is equivalent to $r(\beta+2\kappa/q)>\alpha+2\kappa$ and by Corollary 1 the result of present Corollary follows.

Corollary 3. *Let* $1<p\le 2$,$1/p+1/q=1$, $f\in L^p_\kappa(\mathbb{R})$, $m\in\mathbb{N}$, $r\in(0,q)$, $f\in L^p_\kappa(\mathbb{R})$ *and for some* $\beta,\gamma>0$ *the relation* $\omega_m(f,\delta)_{p,\kappa}=O(\delta^\beta/(ln(1/\delta))^\gamma), 0<\delta<1$. *If*

$$\frac{2\kappa q}{\beta q+2\kappa}<r<q \quad or \quad r=\frac{2\kappa q}{\beta q+2\kappa},\quad r>1/\gamma,$$

then $\mathcal{F}_\kappa(f)\in L^r_\kappa(\mathbb{R})$.

Proof. It is clear that $\lambda(x)=1$ satisfies all conditions of Theorem 4. By this Theorem if the integral

$$\frac{1}{2\Gamma(2k)}\int_1^\infty x^{-2\kappa r/q}x^{-r\beta}(\ln x)^{-r\gamma}x^{2k-1}dx, \tag{17}$$

converges, then $\mathcal{F}_\kappa(f)\in L^r_k(\mathbb{R})$. It is clear that the conditions $2\kappa<2r\kappa/q+r\beta$ or $2\kappa=2r\kappa/q+r\beta$, $r\gamma>1$ are sufficient for the convergence of (17).

5. Conclusion

Harmonic analysis is one of the most active fields in math due to its strong ties to other areas. This field is strongly related to signal processing, image processing, and artificial intelligence, particularly convolutional neural networks and quantum mechanics. These connections allow researchers to develop advanced algorithms that can analyze and manipulate data in innovative ways. As technology continues to evolve, the applications of harmonic analysis are likely to expand further, influencing various scientific and engineering disciplines. In this work, we studied the κ-Hankel transform, a deformation of the classical Hankel transform that is closely related to problems in physics involving circular symmetries. By studying the κ-Hankel transform, we expand the range of problems that can be analyzed to include circular symmetries with a deformation. Further works may be extended to the linear canonical κ-Hankel transform, which is a larger class of integral transforms used to analyze signals.

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