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Mercer theorem for a limit of Reproducing Kernel Hilbert Spaces

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Abstract. Let $(X_n)_{n\geq 0}$ be an increasing sequence of compact metric spaces, let K_n be a Mercer kernel on X_n for any $n \in \mathbb{N}$ and let H_n be the reproducing kernel Hilbert space associated with K_n for any $n \in \mathbb{N}$. In this paper, we construct a reproducing kernel Hilbert space H_∞, on $X_\infty = \bigcup_{n=0}^{+\infty} X_n$ from the sequence $(H_n)_{n\geq 0}$ and determine its reproducing kernel. We end by establishing a version of Mercer theorem in this non-compact context.

Key Words and Phrases: Mercer kernel, reproducing kernel, reproducing kernel Hilbert spaces, integral operator

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1. Introduction

Mercer's theorem was first established on a closed and bounded interval of $\mathbb{R}$ by J. Mercer in 1909 [7]. It gives a representation of a symmetric positive definite function on a square as a sum of a convergent sequence of product functions. Mercer's theorem is an important theoretical tool in theory of integral equations [13], in the Hilbert space theory of stochastic processes [11], in the theory of reproducing kernel Hilbert spaces [12], and in machine learning applications [8]. Mercer's theorem has been generalized and proved later on a compact metric space by many authors, such as J. Mairal and V. Philippe [8], S. Saitoh and S. Yoshihiro [12].

But the domain of a reproducing kernel Hilbert space is not necessarily compact. So, it would be interesting to study the Hilbert space structure of reproducing kernel Hilbert spaces on a more general domain, namely a non-compact domain. Thus, our purpose in this paper is to prove the Mercer theorem on a non-compact metric space. Recently, H. Sun has proved Mercer's theorem on a non-compact domain which is a union of compact metric spaces [9]. He has obtained his results under certain assumptions. It is then a natural question to know whether it is possible to obtain these results without some of these assumptions. Our main concern in this paper is to consider a non-compact domain which is the union of compact metric spaces and to prove Mercer's theorem by weakening Sun's assumptions. In the next section, we present notations that are useful for the remainder of the paper. In section 3, from a sequence of reproducing kernel Hilbert spaces on compact metric spaces, we construct a reproducing kernel Hilbert space on a non-compact space and determine its reproducing kernel. In section 4, we study first the properties of the integral operator associated to the reproducing kernel obtained in section 3 and thanks to these properties, we prove a version of Mercer's theorem.

2. Notations and Preliminaries

In this section, we fix the notations, give some definitions and review some properties of reproducing kernel Hilbert spaces.

2.1. Notations and Symbols

X	A nonempty set
$\mathcal{H}$	Reproducing Kernel Hilbert Space on X
K	The reproducing kernel of $\mathcal{H}$
$(,)_{\mathcal{H}}$	A scalar product on $\mathcal{H}$
$\|.\|_{\mathcal{H}}$	The norm on $\mathcal{H}$
$RKHS$	Reproducing Kernel Hilbert Space
$H = \bigcup_{n=0}^{+\infty} H_n$	The inductive limit of H_n
$H_\infty = \overline{\bigcup_{n=0}^{\infty} H_n}$	The functional completion of H
K_∞	The kernel of H_∞
$L^2(X,\mu)$	The space of square integrable functions on X
L_K	Operator associated with the kernel K
P	The orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_0$
K^x	The reproducing function at x of $\mathcal{H}$
μ	A Borel measure on X

E_x	The evaluation map at x
δ_x	Dirac measure at x
1_A	The characteristic function of a set A
$X \backslash Y$	Elements of X not in Y
$f \mid Y$	The restriction of the function f to the set Y
$K_{\mathcal{H}}$	Reproducing kernel associated with $\mathcal{H}$
$\mathcal{H}_K$	Reproducing kernel Hilbert space admitting K as reproducing kernel

2.2. Preliminaries

Let X be a nonempty set. A kernel K on X is positive definite if for all $x_1, x_2, \ldots, x_n \in X$ and for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j K(x_i, x_j) \geq 0$$

Let $\mathcal{H}$ be a Hilbert space of real-valued functions on X and the scalar product (resp. the norm) is denoted by $(,)_{\mathcal{H}}$ (resp. $\|.\|_{\mathcal{H}}$). $\mathcal{H}$ is a reproducing kernel Hilbert space (as short RKHS) if for each $x \in X$, the map E_x from $\mathcal{H}$ to $\mathbb{R}$ defined by $E_x(f) = f(x)$ is continuous. E_x is the evaluation map at x. By Riesz representation theorem, for each $x \in X$, there exists a unique vector $K^x \in \mathcal{H}$ such that for every $f \in \mathcal{H}$, $f(x) = (f, K^x)_{\mathcal{H}}$. The function K^x is called the reproducing function at x and the function $K_{\mathcal{H}} : X \times X \to \mathbb{R}$ defined by $K_{\mathcal{H}}(x, y) = K^x_{\mathcal{H}}(y)$ is called the reproducing kernel for $\mathcal{H}$. Nevertheless, if there is no possible confusion, $K_{\mathcal{H}}$ will simply be denoted by K. We record in the following proposition some easily proved facts about RKHS.(Some of which may be found in [11, 2, 3, 4, 6]).

Proposition 1. *Let the notations be as above.*

(i) *The kernel K is positive definite*

(ii) $K(x, y) = K^x(y) = (K^x, K^y)_{\mathcal{H}} = (K^y, K^x)_{\mathcal{H}} = K^y(x) = K(y, x)$

(iii) $\|K^x\|^2_{\mathcal{H}} = (K^x, K^x)_{\mathcal{H}} = K(x, x)$

(iv) $|K(x, y)|^2 \leq K(x, x)K(y, y)$

(v) *The family $(K^x)_{x \in X}$ generates $\mathcal{H}$.*

The result (i) admits a converse due essentially to E.H. Moore [10]; namely, for every positive definite kernel K there corresponds one and only one RKHS

$\mathcal{H}_K$ which admits K as a reproducing kernel. If $\mathcal{H}_0$ is a closed subspace of $\mathcal{H}$, then $\mathcal{H}_0$ is also a reproducing kernel Hilbert space with kernel defined by

$$K_0^x(y) = P(K^x)(y) \quad \text{with } y \in X$$

where P is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_0$.
The projection P is given by

$$P(f)(x) = (K_0^x, f)_{\mathcal{H}} \quad \text{with } f \in \mathcal{H} \quad [6].$$

Now let X be a topological space. A Mercer kernel K is a positive definite kernel which is continuous. In this case, $\mathcal{H}_K$ consists of continuous functions on X [11]. Let (X, d) be a compact metric space, let μ be a finite Borel measure on X and let $L^2(X, \mu)$ be the space of square integrable functions on X. If K is a Mercer kernel then the operator L_K on $L^2(X, \mu)$ defined by

$$L_K f(x) = \int_X K(x, y) f(y) d\mu(y)$$

is compact, positive and symmetric [8]. It has at most countably many positive eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ and corresponding orthonormal eigenfunctions $\{\Phi_i\}_{i=1}^{\infty}$ [8]. Mercer's theorem permits then to characterize the RKHS $\mathcal{H}_K$ and the reproducing kernel K in terms of spectral decomposition of the operator L_K. Precisely, the Mercer's theorem [8] asserts that:

$$K(x, y) = \sum_{i=1}^{\infty} \lambda_i \Phi_i(x) \Phi_i(y)$$

where the convergence is absolute and uniform on $X \times X$. We achieve this section with the notion of functional completion. Let $(H_1, \langle ., . \rangle)$ be a pre-Hilbertian space of real-valued functions defined on X (not necessarily a topological space). We say that H is a functional completion of H_1 if

- H is a vector space such that $H_1 \subset H$
- H is equipped with an inner product $\langle ., . \rangle_H$ such that $(H, \langle ., . \rangle_H)$ is a Hilbert space and for every $x \in X$ the linear functional $\delta_x f = f(x)$ is continuous on H.

The following lemma gives sufficient conditions for a pre-Hilbertian space of functions to have a functional completion.

Lemma 1. *([1]) Let $(H_1, \langle ., . \rangle)$ be a pre-Hilbert space of real-valued functions defined on X. If*

- *For every $x \in X$, the linear functional $\delta_x f = f(x)$ is bounded on H_1*
- *For every Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ of elements in H_1, the condition $\lim_{n\to+\infty} f_n(x) = 0$ for all $x \in X$ implies that $\lim\limits_{n\to+\infty} \| f_n \|_{H_1} = 0$,*

then H_1 admits a functional completion.

3. Limit of Reproducing Kernels Hilbert Spaces

In this section, we consider $X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots$ an increasing sequence of metric compact spaces. We set $X_\infty = \bigcup\limits_{n=0}^{+\infty} X_n$ and let μ be a non-degenerate Borel measure on X_∞. For each $n \in \mathbb{N}$, we consider a Mercer kernel K_n on X_n. We assume that for any $n \in \mathbb{N}$, K_n is the restriction of K_{n+1} to $X_n \times X_n$. For all $n \in \mathbb{N}$, let H_n be the reproducing kernel Hilbert space associated with K_n as its reproducing kernel by Moore's theorem [10].

Proposition 2. *For all $n \in \mathbb{N}$, the map S_n from H_n to H_{n+1} defined by*

$$S_n(f)(x) = \begin{cases} f(x) \ \textit{if } x \in X_n \\ 0 \ \textit{otherwise} \end{cases}$$

is an isometric embedding.

Proof. Let $f \in H_n$. Since the family $(K^x)_{x\in X_n}$ generates H_n then $f = \sum\limits_{i=1}^{\infty} \alpha_i K_n^{x_i}$, with $\alpha_i \in \mathbb{R}$, $x_i \in X_n$ for any $i \in \mathbb{N}$. So according to $S_n(f)$'s definition and restriction assumption on K_n's, we have

$$S_n(f) = \sum_{i=1}^{\infty} \alpha_i K_n^{x_i} 1_{X_n} = \sum_{i=1}^{\infty} \alpha_i K_{n+1}^{x_i} 1_{X_n}$$

where 1_A is the characteristic function of a set A. So $S_n(f) \in H_{n+1}$. Also, S_n is clearly linear and injective. Now,

$$\begin{aligned} \|S_n(f)\|^2_{H_{n+1}} &= \langle S_n(f), S_n(f)\rangle_{H_{n+1}} \\ &= \langle \sum_{i=1}^{\infty} \alpha_i K_n^{x_i} 1_{X_n}, \sum_{j=1}^{\infty} \alpha_j K_n^{x_j} 1_{X_n}\rangle_{H_n} \\ &= \langle f, f\rangle_{H_n} = \|f\|^2_{H_n} \end{aligned}$$

So S_n is an isometry. $\square$

Then thanks to proposition 2, we can identify H_n to a closed subspace of H_{n+1} and assume that

$H_n \subset H_{n+1}$ for all $n \in \mathbb{N}$.

Let us designate by $T_\infty = \{f : X_\infty \longrightarrow \mathbb{R}\}$, the set of real-valued functions defined on X_∞.

Remark 1. *For any integer n, we can identify H_n to a subspace of T_∞. In fact, the map: $V_n : H_n \to T_\infty$ defined by*

$$V_n(f)(x) = \begin{cases} f(x) \text{ if } x \in X_n \\ 0 \text{ otherwise} \end{cases}$$

is an embedding.

It follows from the remark 1 that we can assume for all $n \in \mathbb{N}$, $H_n \subset T_\infty$. We set $H = \bigcup_{n=0}^{+\infty} H_n$ and let us consider the map: $\langle , \rangle_H : H \times H \longrightarrow \mathbb{R}$ defined by $\langle f, g\rangle_H = \langle f, g\rangle_{H_n}$ if $f, g \in H_n$. It is clear that $\langle , \rangle_H$ defines a scalar product on H.

Theorem 1. *The pre-Hilbert space $(H, \langle , \rangle_H)$ admits a functional completion.*

Proof. Let $x \in X_\infty$ and, let us consider the linear form δ_x defined on H by $\delta_x(f) = f(x)$ for all $f \in H$. Let us show that δ_x is bounded. Since $x \in X_\infty$, then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, $x \in X_n$. Also $f \in H$ implies that there exists $n_1 \in \mathbb{N}$ such that for any $n \geq n_1$, $f \in H_n$. Setting $p = \max\{n_0, n_1\}$, we have $K_p^x \in H_p$, $f \in H_p$ and

$\| K_p^x \|_{H_p} = \sqrt{K_p(x,x)} = \sqrt{K_{n_0}(x,x)} = \| K_{n_0}^x \|_{H_{n_0}}$ since $K_{n_0} = K_p|_{X_{n_0} \times X_{n_0}}$. So

$$\begin{aligned} |\delta_x f| = |f(x)| &= |\langle f, K_p^x\rangle_{H_p}| \\ &\leq \| f \|_{H_p} \| K_p^x \|_{H_p} \\ &\leq \| f \|_{H_p} \| K_{n_0}^x \|_{H_{n_0}} \\ &\leq \| f \|_H \| K_{n_0}^x \|_H \end{aligned}$$

and therefore δ_x is bounded on H .

It follows that we can assume for all $n \in \mathbb{N}$, $H_n \subset T_\infty$.

We set $H = \bigcup_{n=0}^{+\infty} H_n$ and let us consider the map: $\langle , \rangle_H : H \times H \longrightarrow \mathbb{R}$ defined by $\langle f, g\rangle_H = \langle f, g\rangle_{H_n}$ if $f, g \in H_n$. It is clear that $\langle , \rangle_H$ defines a scalar product on H. Suppose that $(f_p)_{p\in\mathbb{N}}$ is a Cauchy sequence of elements of H such that for all $x \in X_\infty$, $\lim\limits_{p\to+\infty} f_p(x) = 0$. For all $(p,q) \in \mathbb{N}^2$, we have:

$$\| f_p \|_H^2 \leq \| f_p - f_q \|_H^2 + 2|\langle f_p, f_q\rangle_H|$$

which implies

$$\lim_{p\to+\infty} \| f_p \|_H \leq \sqrt{\lim_{p\to+\infty}(\| f_p - f_q \|_H^2 + 2|\langle f_p, f_q\rangle_H|)}.$$

Now, $f_p \in H$ and $f_q \in H$ imply that there exists $n_{(p,q)} \in \mathbb{N}$ such that $f_p \in H_{n(p,q)}$ and $f_q \in H_{n(p,q)}$. It comes from proposition 1(v) that
$f_p = \sum\limits_{i=1}^{\infty} \alpha_{i,p} K_{n(p,q)}^{x_{i,p}}$; $f_q = \sum\limits_{j=1}^{\infty} \alpha_{j,q} K_{n(p,q)}^{x_{j,q}}$ with $\alpha_{i,p}$, $\alpha_{j,q} \in \mathbb{R}$; $x_{i,p}, x_{j,q} \in X_{n(p,q)}$.
Thus,

$$\begin{aligned}
\langle f_p, f_q\rangle_H &= \langle \sum_{i=1}^{\infty} \alpha_{i,p} K_{n(p,q)}^{x_{i,p}}, \sum_{j=1}^{\infty} \alpha_{j,q} K_{n(p,q)}^{x_{j,q}} \rangle_{H_{n(p,q)}} \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_{i,p}\alpha_{j,q} \langle K_{n(p,q)}^{x_{i,p}}, K_{n(p,q)}^{x_{j,q}} \rangle_{H_{n(p,q)}} \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_{i,p}\alpha_{j,q} K_{n(p,q)}^{x_{i,p}}(x_{j,q}) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_{i,p}\alpha_{j,q} K_{n(p,q)}^{x_{i,p}}(x_{j,q}) \\
&= \sum_{j=1}^{\infty} \alpha_{j,q} \left(\sum_{i=1}^{\infty} \alpha_{i,p} K_{n(p,q)}^{x_{i,p}}(x_{j,q}) \right) \\
&= \sum_{j=1}^{\infty} \alpha_{j,q} f_p(x_{j,q})
\end{aligned}$$

It follows that

$$\begin{aligned}
\lim_{p\to+\infty} \langle f_p, f_q\rangle_H &= \lim_{p\to+\infty} \sum_{j=1}^{\infty} \alpha_{j,q} f_p(x_{j,q}) \\
&= \sum_{j=1}^{\infty} \alpha_{j,q} \lim_{p\to+\infty} f_p(x_{j,q}) = 0
\end{aligned}$$

We also have $\lim_{\substack{p\to+\infty\\ q\to+\infty}} \| f_p - f_q \|_H = 0$ because $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and subsequently, $\lim_{p\to+\infty} \| f_p \|_H = 0$. Thanks to lemma 1., we deduce that H admits a functional completion. $\square$

We denote by H_∞ the functional completion of H. The first part of previous proof shows that H_∞ is a reproducing kernel Hilbert space.
Consider the map: $K_\infty : X_\infty \times X_\infty \to \mathbb{R}$ defined by $K_\infty(x,y) = K_n(x,y)$ if $x, y \in X_n$.

Theorem 2. *We have*

(i) K_∞ *is the reproducing kernel of* H_∞

(ii) K_∞ *is a Mercer's kernel*

Proof.

(i) H_∞ is a reproducing kernel Hilbert space and denote by K its reproducing kernel. Let $x \in X_{n_0}$ and let $f \in H_{n_0}$ for a fixed $n_0 \in \mathbb{N}$. We have $f(x) = \langle f, K^x_{n_0}\rangle_{H_{n_0}} = \langle f, K^x_{n_0}\rangle_{H_\infty}$
But, since $X_{n_0} \subset X_\infty$ and $f \in H_\infty$, we have $f(x) = \langle f, K^x\rangle_{H_\infty}$.
So by the uniqueness of the reproducing function at one point, we deduce that $K^x = K^x_{n_0}$. Now if $x, y \in X_\infty$, then there exists $p \in \mathbb{N}$ such that $x, y \in X_p$ and we have

$$K(x,y) = K^x(y) = K^x_p(y) = K_p(x,y) = K_\infty(x,y).$$

(ii) Since K_∞ is the reproducing kernel of H_∞ then K_∞ is positive definite(see Proposition 1.(i)). It is also clear that K_∞ is continuous for the inductive limit topology.

$\square$

4. Mercer Theorem

Let μ be a non-degenerate bounded measure on X_∞ and let $L^2(X_\infty, \mu)$ be the space of complex-valued square integrable functions on X_∞. In this section, we will assume that for all $x \in X_\infty$, $\| K^x_\infty \|_{L^2(X_\infty)} < \infty$.
Thus, for any $f \in L^2(X_\infty, \mu)$, we have

$$\int_{X_\infty} |K_\infty(x,y) f(y)| d\mu(y) \leq \left(\int_{X_\infty} |K_\infty(x,y)|^2 d\mu(y)\right)^{1/2} \left(\int_{X_\infty} |f(y)|^2 d\mu(y)\right)^{1/2}$$

$$\leq \| f \|_{L^2(X_\infty,\mu)} \left(\int_{X_\infty} |K^x_\infty(y)|^2 d\mu(y) \right)^{1/2} < \infty$$

For any $f \in L^2(X_\infty, \mu)$ and for any $x \in X_\infty$, we set

$$L_{K_\infty} f(x) = \int_{X_\infty} K_\infty(x,y) f(y) d\mu(y)$$

We consider the sequence $(Y_n)_{n\geq 0}$ defined by $Y_0 = X_0$ and for all $n \geq 1$, $Y_n = X_n \setminus X_{n-1}$.
It is well-known that $(Y_n)_{n\in\mathbb{N}}$ is a disjoint sequence and for all $n \in \mathbb{N}$

$$\bigcup_{k=0}^{n} Y_k = \bigcup_{k=0}^{n} X_k = X_n \quad \text{and } X_\infty = \bigcup_{n=0}^{\infty} Y_n = \bigcup_{n=0}^{\infty} X_n.$$

Theorem 3. *For any $f \in L^2(X_\infty, \mu)$, $L_{K_\infty}(f) \in H_\infty$.*

Proof. Let $x \in X_\infty$ and $f \in L^2(X_\infty, \mu)$, we have

$$\begin{aligned}
L_{K_\infty} f(x) &= \int_{X_\infty} K_\infty(x,y) f(y) d\mu(y) \\
&= \int_{\bigcup_{n=0}^{\infty} Y_n} K_\infty(x,y) f(y) d\mu(y) \\
&= \sum_{n=0}^{\infty} \int_{Y_n} K_\infty(x,y) f(y) d\mu(y) \\
&= \lim_{m\to+\infty} \sum_{n=0}^{m} \int_{Y_n} K_\infty(x,y) f(y) d\mu(y) \\
&= \lim_{m\to+\infty} \int_{\bigcup_{n=0}^{m} Y_n} K_\infty(x,y) f(y) d\mu(y) \\
&= \lim_{m\to+\infty} \int_{X_m} K_\infty(x,y) f(y) d\mu(y)
\end{aligned}$$

where $f_m = f|X_m$ is the restriction of f to X_m.
For any $x \in X_\infty$ and $m \in \mathbb{N}$, we define ϕ_m by $\phi_m(x) = \displaystyle\int_{X_m} K_\infty(x,y) f(y) d\mu(y)$ and set $m_x = inf\{m \in \mathbb{N} : x \in X_m\}$.

Since $(X_n)_{n\geq 0}$ is an increasing sequence then for any $m \geq m_x$, we have $x \in X_m$ and therefore

$$\begin{aligned}\phi_m(x) &= \int_{X_m} K_m(x,y)f(y)d\mu(y).\\ &= L_{K_m} f_m(x)\end{aligned}$$

So for $m \geq m_x$, we have $\phi_m \in H_m$. Since $H_m \subset H$ for any m, then $\lim_{m\to+\infty} \phi_m = L_{K_\infty}(f) \in \overline{H} = H_\infty$ □

Let us consider for any $m \in \mathbb{N}$, the map $\Gamma_m : L^2(X_m, \mu_m) \to L^2(X_\infty, \mu)$ defined by

$$\Gamma_m(\phi)(x) = \begin{cases}\phi(x) \text{ if } x \in X_m\\ 0 \text{ otherwise}\end{cases}$$

where μ_m is the restriction of μ to X_m.

Theorem 4. *For any $m \in \mathbb{N}$, Γ_m is an isometric embedding.*

Proof. Γ_m is clearly linear and injective. Let $\phi \in L^2(X_m, \mu_m)$.

$$\begin{aligned}\|\Gamma_m(\phi)\|^2_{L^2(X_\infty,\mu)} &= \int_{X_\infty} |\Gamma_m(\phi)(x)|^2 d\mu(x).\\ &= \int_{X_m} |\Gamma_m(\phi)(x)|^2 d\mu(x) + \int_{X_\infty\setminus X_m} |\Gamma_m(\phi)(x)|^2 d\mu(x)\\ &= \int_{X_m} |\phi(x)|^2 d\mu(x)\\ &= \|\phi\|^2_{L^2(X_m,\mu_m)}\end{aligned}$$

So Γ_m is an isometry. □

For sufficiently large m, ϕ_m as defined in the proof of theorem 3. lies in $L^2(X_m, \mu_m)$ and using the theorem 4., we can consider for sufficiently large m, ϕ_m as an element of $L^2(X_\infty, \mu)$.

Theorem 5. $L_{K_\infty}(f) \in L^2(X_\infty, \mu)$ *for any* $f \in L^2(X_\infty, \mu)$.

Proof. For $p, q \in \mathbb{N}$
$\phi_p = \displaystyle\int_{X_p} K^y_\infty f(y)d\mu(y)$ and $\phi_q = \displaystyle\int_{X_q} K^t_\infty f(t)d\mu(t)$ as defined in the proof of theorem 3. For p and q sufficiently large, we have

$$\lim_{p\to+\infty} \langle \phi_p, \phi_p\rangle_{L^2(X_\infty,\mu)} = \lim_{p\to+\infty} \langle \int_{X_p} K^y_\infty f(y)d\mu(y), \int_{X_p} K^t_\infty f(t)d\mu(t)\rangle_{L^2(X_\infty,\mu)}$$

$$\begin{aligned}
&= \langle \lim_{p\to+\infty} \int_{X_p} K_\infty^y f(y) d\mu(y), \lim_{p\to+\infty} \int_{X_p} K_\infty^t f(t) d\mu(t) \rangle_{L^2(X_\infty,\mu)} \\
&= \langle L_{K_\infty}(f), L_{K_\infty}(f) \rangle_{L^2(X_\infty,\mu)} \\
&= \| L_{K_\infty}(f) \|^2_{L^2(X_\infty,\mu)}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{p,q\to+\infty} \langle \phi_p, \phi_q \rangle_{L^2(X_\infty,\mu)} &= \lim_{p,q\to+\infty} \langle \int_{X_p} K_\infty^y f(y) d\mu(y), \int_{X_q} K_\infty^t f(t) d\mu(t) \rangle_{L^2(X_\infty,\mu)} \\
&= \langle \lim_{p\to+\infty} \int_{X_p} K_\infty^y f(y) d\mu(y), \lim_{q\to+\infty} \int_{X_q} K_\infty^t f(t) d\mu(t) \rangle_{L^2(X_\infty,\mu)} \\
&= \langle L_{K_\infty}(f), L_{K_\infty}(f) \rangle_{L^2(X_\infty,\mu)} \\
&= \| L_{K_\infty}(f) \|^2_{L^2(X_\infty,\mu)}
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{p,q\to+\infty} \| \phi_p - \phi_q \|^2_{L^2(X_\infty,\mu)} &= \lim_{p,q\to+\infty} (\langle \phi_p, \phi_P \rangle_{L^2(X_\infty,\mu)} - 2\langle \phi_p, \phi_q \rangle_{L^2(X_\infty,\mu)} + \langle \phi_q, \phi_q \rangle_{L^2(X_\infty,\mu)}) \\
&= \| L_{K_\infty}(f) \|^2_{L^2(X_\infty,\mu)} - 2\| L_{K_\infty,\mu}(f) \|^2_{L^2(X_\infty,\mu)} + \| L_{K_\infty}(f) \|^2_{L^2(X_\infty,\mu)} = 0
\end{aligned}$$

So $(\phi_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^2(X_\infty,\mu)$ which converges to $L_{K_\infty} f$ (see the proof of theorem 3.). Consequently, $L_{K_\infty} f \in L^2(X_\infty,\mu)$. □

Theorem 6. *For any $f \in L^2(X_\infty,\mu)$ and for any $h \in L^2(X_\infty,\mu) \cap H_\infty$ we have $\langle L_{K_\infty}(f), h \rangle_{H_\infty} = \langle f, h \rangle_{L^2(X_\infty,\mu)}$*

Proof. Let $h \in L^2(X_\infty,\mu) \cap H_\infty$

$$\begin{aligned}
\langle L_{K_\infty}(f), h \rangle_{H_\infty} &= \langle \int_{X_\infty} K_\infty^y f(y) d\mu(y), h \rangle_{H_\infty} \\
&= \int_{X_\infty} f(y) \langle K_\infty^y, h \rangle_{H_\infty} d\mu(y) \\
&= \int_{X_\infty} f(y) h(y) d\mu(y) \\
&= \langle f, h \rangle_{L^2(X_\infty,\mu)}
\end{aligned}$$

□

Theorem 7. *L_{K_∞} is a positive and self-adjoint operator .*

Proof.

$$\begin{aligned}
\langle L_{K_\infty}(f), g\rangle_{L^2(X_\infty,\mu)} &= \int_{X_\infty} L_{K_\infty}(f)(x)g(x)d\mu(x) \\
&= \int_{X_\infty}\left(\int_{X_\infty} K_\infty(x,y)f(y)d\mu(y)\right)g(x)d\mu(x) \\
&= \int_{X_\infty}\left(\int_{X_\infty} K_\infty(y,x)g(x)d\mu(x)\right)f(y)d\mu(y) \\
&= \int_{X_\infty} L_{K_\infty}(g)(y)f(y)d\mu(y) \\
&= \langle f, L_{K_\infty}(g)\rangle_{L^2(X_\infty,\mu)}
\end{aligned}$$

then L_{K_∞} is self-adjoint. Let $f \in L^2(X_\infty,\mu)$.

$$\begin{aligned}
\langle L_{K_\infty} f, f\rangle_{L^2(X_\infty,\mu)} &= \int_{X_\infty} L_{K_\infty}(f)(x)f(x)d\mu(x) \\
&= \int_{X_\infty}\left(\int_{X_\infty} K_\infty(x,y)f(y)d\mu(y)\right)f(x)d\mu(x) \\
&= \int_{X_\infty}\int_{X_\infty} K_\infty(x,y)f(y)f(x)d\mu(y)d\mu(x) \geq 0
\end{aligned}$$

since K_∞ is a positive-type function. We deduce that L_{K_∞} is positive. □

L_{K_∞} is a self-adjoint and positive operator, so L_{K_∞} has positive eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ and the associated eigenfunctions $(\psi_k)_{k\in\mathbb{N}}$ form an orthogonal system. By dividing each vector by its norm, we can assume that the system $(\psi_k)_{k\in\mathbb{N}}$ is orthonormal.

Theorem 8. *(Mercer's type theorem)*
For all $x, y \in X_\infty$ there exists $n \in \mathbb{N}$ such that

$$K_\infty(x,y) = \sum_{k\geq 1} \lambda_k P_n\psi_k(x)P_n\psi_k(y)$$

where the convergence is absolute and uniform on $X_n \times X_n$ and P_n is the orthogonal projection from H_∞ onto H_n.

Proof. According to remark 1., for each $n \in \mathbb{N}$, H_n is a subspace of H_∞. Recall that P_n is defined by (see [6])

$$P_n f(x) = \langle K_n^x, f\rangle_{H_\infty}$$

for $f \in H_\infty$ and $x \in X_\infty$. We have on one hand

$$P_n(L_{K_\infty}\psi_k) = P_n(\lambda_k\psi_k) = \lambda_k P_n(\psi_k)$$

and on the other hand for $x \in X_n$

$$\begin{aligned} P_n(L_{K_\infty}\psi_k)(x) &= \langle K_n^x, L_{K_\infty}\psi_k\rangle_{H_\infty} \\ &= \langle K_n^x, \psi_k\rangle_{L^2(X_\infty)} \\ &= \langle P_n(K_n^x), \psi_k\rangle_{L^2(X_\infty)} \\ &= \langle K_n^x, P_n\psi_k\rangle_{L^2(X_\infty)} \\ &= L_{K_n}(P_n\psi_k)(x) \end{aligned}$$

where the second equality is due to theorem 6. and $L_{K_\infty}\psi_k \in H_\infty$ according to theorem 3. It follows that $L_{K_n}(P_n\psi_k)(x) = \lambda_k P_n(\psi_k)(x)$ for any $x \in X_n$ that is $P_n\psi_k$ is an eigenvector for the operator L_{K_n}. It is also clear that an eigenvector Φ of L_{K_n} is an eigenvector of L_{K_∞} as a function in $L^2(X_\infty, \mu)$ and $P_n\Phi = \Phi$. So, $(P_n\psi_k)_{k\in\mathbb{N}}$ is a family of eigenvectors for the operator L_{K_n}. Using the classical Mercer's theorem on compact domain we have

$$K_n(x,y) = \sum_{k\geq 1} \lambda_k P_n\psi_k(x) P_n\psi_k(y),$$

where the convergence is absolute and uniform on $X_n \times X_n$. Now for $x, y \in X_\infty$ there exists $n \in \mathbb{N}$ such that $x, y \in X_n$ and $K_\infty(x,y) = K_n(x,y)$ and consequently we have

$$K_\infty(x,y) = \sum_{k\geq 1} \lambda_k P_n\psi_k(x) P_n\psi_k(y)$$

□

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