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Fractional maximal function in total Morrey-Guliyev spaces for the Dunkl operator on the real line

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Abstract. On the real line, the Dunkl operators D_ν are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. In the paper, in the setting , we study the fractional maximal operator associated with the Dunkl operator $M_{\alpha,\nu}$ in the total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$. We give necessary and sufficient conditions for the boundedness of the operator $M_{\alpha,\nu}$ on total D_ν-Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

Key Words and Phrases: Fractional maximal operator; total D_ν-Morrey-Guliyev space; Dunkl operator.

2010 Mathematics Subject Classifications: 42B20, 42B25, 42B35

1. Introduction

Morrey spaces, introduced by Morrey [26], play an important role in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In harmonic analysis, Morrey spaces are crucial for analyzing the behavior of integral operators and providing conditions for the global existence of solutions to nonlinear PDEs, such as the Schrödinger equation. The total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, introduced by Guliyev [14], extend the Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ by including the second parameter μ, which can be seen as the intermediate spaces between Lebesgue spaces and Morrey spaces. The norm in these spaces is defined by a combination of the norms of $L_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\mu}(\mathbb{R}^n)$, which allows a wider range of behavior. Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. The total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$ are the set of all locally integrable functions f with the finite (quasi-)norm

$$\|f\|_{L_{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \, [1/t]_1^{\frac{\mu}{p}} \, \|f\|_{L_p(B(x,t))},$$

where $B(x,t)$ denotes the ball centered at x with radius $t > 0$. Here the norm in the case $\mu \leq \lambda$ is equal to the maximum of the norms of $L_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\mu}(\mathbb{R}^n)$. Total Morrey-Guliyev spaces can be viewed as generalizations of both classical and modified Morrey spaces. In particular, the case where $\lambda = \mu$ corresponds to classical Morrey space, and the case where $\mu = 0$ corresponds to modified Morrey space $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$, see [1, 3, 4, 6, 7, 8, 9, 10, 18, 24, 31, 32, 33].

On the real line, the Dunkl operators Λ_ν are differential-difference operators introduced in 1989 by Dunkl [15]. For a real parameter $\nu \geq -1/2$, we consider the *Dunkl operator*, associated with the reflection group $_2$ on $\mathbb{R}$:

$$D_\nu(f)(x) := \frac{df(x)}{dx} + (2\nu+1)\frac{f(x)-f(-x)}{2x}, \quad x \in \mathbb{R}.$$

Note that $D_{-1/2} = d/dx$.

Let $\nu > -1/2$ be a fixed number and m_ν be the *weighted Lebesgue measure* on $\mathbb{R}$, given by

$$dm_\nu(x) := \left(2^{\nu+1}\Gamma(\nu+1)\right)^{-1} |x|^{2\nu+1}\, dx, \quad x \in \mathbb{R}.$$

For any $x \in \mathbb{R}$ and $r > 0$, let $B(x,r) := \{y \in \mathbb{R} : |y| \in]\max\{0, |x| - r\}, |x| + r[\ \}$ be a Dunkl-ball in $\mathbb{R}$. Then $B(0,r) =]-r, r[$ and $m_\nu B(0,r) = c_\nu r^{2\nu+2}$, where $c_\nu := \left[2^{\nu+1}(\nu+1)\Gamma(\nu+1)\right]^{-1}$.

The *maximal operator* M_ν associated by Dunkl operator on the real line is given by

$$M_\nu f(x) := \sup_{r>0} \left(m_\nu(B(x,r))\right)^{-1} \int_{B(x,r)} |f(y)|\, dm_\nu(y), \quad x \in \mathbb{R}$$

and *fractional maximal operator* $M_{\alpha,\nu}$, $0 \leq \alpha < 2\nu+2$ associated by Dunkl operator on the real line is given by

$$M_{\alpha,\nu} f(x) := \sup_{r>0} \left(m_\nu B(x,r)\right)^{-1+\frac{\alpha}{2\nu+2}} \int_{B(x,r)} |f(y)|\, dm_\nu(y), \quad x \in \mathbb{R}$$

It is well known that maximal and fractional maximal operators play an important role in harmonic analysis (see [36]). Also the fractional maximal function and the fractional integral, associated with D_ν differential-difference Dunkl operators play an important role in Dunkl harmonic analysis, differentiation theory and PDE's. The harmonic analysis of the one-dimensional Dunkl operator and Dunkl transform was developed in [11, 12, 23, 25]. The Dunkl operator and Dunkl transform considered here are the rank-one case of the general Dunkl theory, which is associated with a finite reflection group acting on a Euclidean space. The Dunkl theory provides a useful framework for the study of multivariable analytic structures and has gained considerable interest in various fields of mathematics and in physical applications (see, for example, [16]). The maximal function, the fractional integral and related topics associated with the Dunkl differential-difference operator have been research areas for many mathematicians such as C. Abdelkefi and M. Sifi [2], V.S. Guliyev and Y.Y. Mammadov [11, 12, 13], Y.Y. Mammadov [20], L. Kamoun [17], M.A. Mourou [27], F. Soltani [34, 35], K. Trimeche [37] and others. Moreover, the results on $L_\Phi(\mathbb{R}, dm_\nu)$-boundedness of fractional maximal operator and its commutators associated with D_ν were obtained in [13, 21].

It is well known that maximal operator play an important role in harmonic analysis (see [36]). Harmonic analysis associated to the Dunkl transform and the Dunkl differential-difference operator gives rise to convolutions with a relevant generalized translation. In this paper, in the framework of this analysis in the setting $\mathbb{R}$, we study the boundedness of the fractional maximal operator $M_{\alpha,\nu}$ on total D_ν-Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Preliminaries in the Dunkl setting on $\mathbb{R}$

Definition 1. *Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L_{p,\lambda}(\mathbb{R}, dm_\nu)$ the Morrey space [28] ($\equiv D_\nu$-Morrey space), by $\widetilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)$ the modified Morrey space [28] ($\equiv$ modified D_ν-Morrey space), and by $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ the total Morrey-Guliyev space [29] ($\equiv$ total D_ν-Morrey-Guliyev space), associated with the Dunkl operator the set of all classes of locally integrable functions f with the finite norms*

$$\|f\|_{L_{p,\lambda}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R},\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t),dm_\nu)},$$

$$\|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R},\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t),dm_\nu)},$$

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L_p(B(x,t),dm_\nu)},$$

respectively.

Definition 2. *Let $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak Morrey space $L_{p,\lambda}(\mathbb{R}, dm_\nu)$ [28] ($\equiv$ weak D_ν-Morrey space), the weak modified Morrey space $\widetilde{L}_{p,\lambda}(\mathbb{R}, dm_\nu)$ [28] ($\equiv$ weak modified D_ν-Morrey space), and the weak total Morrey-Guliyev space $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ [29] ($\equiv$ weak total D_ν-Morrey-Guliyev space), associated with the Dunkl operator the set of all classes of locally integrable functions f with the finite norms*

$$\|f\|_{WL_{p,\lambda}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R},\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,t),dm_\nu)},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R},\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,t),dm_\nu)},$$

$$\|f\|_{WL_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{WL_p(B(x,t),dm_\nu)},$$

respectively.

Lemma 1. [22, 30] *If $0 < p < \infty$, $0 \leq \mu \leq \lambda \leq 2\nu + 2$, then*

$$L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) = L_{p,\lambda}(\mathbb{R}, dm_\nu) \cap L_{p,\mu}(\mathbb{R}, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} = \max\left\{\|f\|_{L_{p,\lambda}(\mathbb{R},dm_\nu)}, \|f\|_{L_{p,\mu}(\mathbb{R},dm_\nu)}\right\}.$$

Lemma 2. [*22, 30*] *If* $0 < p < \infty$, $0 \le \mu \le \lambda \le 2\nu + 2$, *then*

$$WL_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) = WL_{p,\lambda}(\mathbb{R}, dm_\nu) \cap WL_{p,\mu}(\mathbb{R}, dm_\nu)$$

and

$$\|f\|_{WL_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} = \max\left\{\|f\|_{WL_{p,\lambda}(\mathbb{R},dm_\nu)}, \|f\|_{WL_{p,\mu}(\mathbb{R},dm_\nu)}\right\}.$$

Remark 1. *If* $0 < p < \infty$, *and* $\lambda > 2\nu + 2$ *or* $\mu < 0$, *then*

$$L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) = WL_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) = \Theta(\mathbb{R}),$$

where $\Theta \equiv \Theta(\mathbb{R})$ *is the set of all functions equivalent to* 0 *on* $\mathbb{R}$.

Lemma 3. [*22*] *If* $0 < p < \infty$, $0 \le \lambda_2 \le \lambda_1 \le 2\nu + 2$ *and* $0 \le \mu_1 \le \mu_2 \le 2\nu + 2$, *then*

$$L_{p,\lambda_1,\mu_1}(\mathbb{R}, dm_\nu) \subset\!\!\succ L_{p,\lambda_2,\mu_2}(\mathbb{R}, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda_2,\mu_2}(\mathbb{R},dm_\nu)} \le \|f\|_{L_{p,\lambda_1,\mu_1}(\mathbb{R},dm_\nu)}.$$

Lemma 4. [*22*] *If* $0 < p < \infty$, $0 \le \lambda \le 2\nu + 2$ *and* $0 \le \mu \le 2\nu + 2$, *then*

$$L_{p,2\nu+2,\mu}(\mathbb{R}, dm_\nu) \subset\!\!\succ L_\infty(\mathbb{R}, dm_\nu) \subset\!\!\succ L_{p,\lambda,2\nu+2}(\mathbb{R}, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda,2\nu+2}(\mathbb{R},dm_\nu)} \le c_\nu^{1/p}\|f\|_{L_\infty(\mathbb{R},dm_\nu)} \le \|f\|_{L_{p,2\nu+2,\mu}(\mathbb{R},dm_\nu)}.$$

Lemma 5. [*22*] *If* $0 \le \lambda < 2\nu + 2$, $0 \le \mu < 2\nu + 2$, $0 \le \alpha < 2\nu + 2 - \lambda$ *and* $0 \le \beta < 2\nu + 2 - \mu$, *then for* $\frac{2\nu+2-\lambda}{\alpha} \le p \le \frac{2\nu+2-\mu}{\beta}$

$$L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu) \subset\!\!\succ L_{1,2\nu+2-\alpha,2\nu+2-\beta}(\mathbb{R}, dm_\nu)$$

and for $f \in L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$ *the following inequality*

$$\|f\|_{L_{1,2\nu+2-\alpha,2\nu+2-\beta}(\mathbb{R},dm_\nu)} \le c_\nu^{1/p'}\|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)}$$

is valid.

3. Fractional maximal operator $M_{\alpha,\nu}$ in total D_ν-Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$

In this section, we investigate the boundedness of the fractional maximal operator $M_{\alpha,\nu}$ in total D_ν-Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

The following Guliyev type local estimates are valid (see also [5]).

Lemma 6. *Let $0 \le \alpha < 2\nu + 2$, $1 \le p < \frac{2\nu+2}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\nu+2}$ and $B(x,r)$ be any Dunkl-ball in $\mathbb{R}$. If $p > 1$, then the inequality*

$$\|M_{\alpha,\nu} f\|_{L_q(B(x,r),dm_\nu)} \lesssim \|f\|_{L_p(2B,dm_\nu)} + r^{\frac{2\nu+2}{q}} \sup_{t>2r} t^{-2\nu-2+\alpha} \|f\|_{L_1(B(x,t),dm_\nu)} \qquad (1)$$

holds for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}, dm_\nu)$.

Moreover if $p = 1$, then the inequality

$$\|M_{\alpha,\nu} f\|_{WL_q(B(x,r),dm_\nu)} \lesssim \|f\|_{L_1(2B,dm_\nu)} + r^{\frac{2\nu+2}{q}} \sup_{t>2r} t^{-2\nu-2+\alpha} \|f\|_{L_1(B(x,t),dm_\nu)} \qquad (2)$$

holds for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}, dm_\nu)$.

Proof. Let $0 \le \alpha < 2\nu + 2$, $1 \le p < \frac{2\nu+2}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\nu+2}$. For arbitrary Dunkl-ball $B = B(x,r)$ let $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2B)}$.

$$\|M_{\alpha,\nu} f\|_{L_q(B,dm_\nu)} \le \|M_{\alpha,\nu} f_1\|_{L_q(B,dm_\nu)} + \|M_{\alpha,\nu} f_2\|_{L_q(B,dm_\nu)}.$$

By the continuity of the operator $M_{\alpha,\nu} : L_p(\mathbb{R}, dm_\nu) \to L_q(\mathbb{R}, dm_\nu)$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\nu+2}$ (see, for example, [28]) we have

$$\|M_{\alpha,\nu} f_1\|_{L_q(B,dm_\nu)} \lesssim \|f\|_{L_p(2B,dm_\nu)}.$$

Let y be an arbitrary point from B. If $B(y,\tau) \cap {}^{\complement}(2B) \ne \emptyset$, then $\tau > r$. Indeed, if $z \in B(y,\tau) \cap {}^{\complement}(2B)$, then $\tau > |y - z| \ge |x - z| - |x - y| > 2r - r = r$.

On the other hand, $B(y,\tau) \cap {}^{\complement}(2B) \subset B(x, 2\tau)$. Indeed, $z \in B(y,\tau) \cap {}^{\complement}(2B)$, then we get $|x - z| \le |y - z| + |x - y| < \tau + r < 2\tau$.

Hence

$$\begin{aligned} M_{\alpha,\nu} f_2(y) &= \sup_{\tau>0} \frac{1}{m_\nu(B(y,\tau))^{1-\frac{\alpha}{2\nu+2}}} \int_{B(y,\tau)\cap {}^{\complement}(2B)} |f(z)|\, dm_\nu(z) \\ &\le 2^{2\nu+2-\alpha} \sup_{\tau>r} \frac{1}{m_\nu(B(x,2\tau))^{1-\frac{\alpha}{2\nu+2}}} \int_{B(x,2\tau)} |f(z)|\, dm_\nu(z) \\ &= 2^{2\nu+2-\alpha} \sup_{\tau>2r} \frac{1}{m_\nu(B(x,\tau))^{1-\frac{\alpha}{2\nu+2}}} \int_{B(x,\tau)} |f(z)|\, dm_\nu(z). \end{aligned}$$

Therefore, for all $y \in B$ we have

$$M_{\alpha,\nu} f_2(y) \le 2^{2\nu+2-\alpha} \sup_{\tau>2r} \frac{1}{m_\nu(B(x,\tau))} \int_{B(x,\tau)} |f(z)|\, dm_\nu(z). \qquad (3)$$

Applying Hölder's inequality, we get

$$M_{\alpha,\nu} f_2(y) \lesssim \sup_{\tau>2r} \frac{1}{m_\nu(B(x,\tau))^{\frac{1}{p}-\frac{\alpha}{2\nu+2}}} \int_{B(x,\tau)} |f(z)|^p\, dm_\nu(z). \qquad (4)$$

Thus

$$\|M_{\alpha,\nu} f\|_{L_q(B,dm_\nu)} \lesssim \|f\|_{L_p(2B,dm_\nu)} + m_\nu(B(x,\tau))^{\frac{1}{q}} \Big(\sup_{\tau>2r} \frac{1}{m_\nu(B(x,\tau))^{\frac{1}{p}-\frac{\alpha}{2\nu+2}}} \int_{B(x,\tau)} |f(z)|\, dm_\nu(z) \Big).$$

Let $p = 1$. It is obvious that for any ball $B = B(x,r)$

$$\|M_{\alpha,\nu} f\|_{WL_q(B,dm_\nu)} \le \|M_{\alpha,\nu} f_1\|_{WL_q(B,dm_\nu)} + \|M_{\alpha,\nu} f_2\|_{WL_q(B,dm_\nu)}.$$

By the continuity of the operator $M_\nu : L_1(\mathbb{R}, dm_\nu) \to WL_q(\mathbb{R}, dm_\nu)$ we have

$$\|M_{\alpha,\nu} f_1\|_{WL_q(B,dm_\nu)} \lesssim \|f\|_{L_1(2B,dm_\nu)}.$$

Then by (4) we get the inequality (2).

Lemma 7. *Let* $0 \le \alpha < 2\nu + 2$, $1 \le p < \frac{2\nu+2}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\nu+2}$ *and* $B(x,r)$ *be any Dunkl-ball in* $\mathbb{R}$. *If* $p > 1$, *then the inequality*

$$\|M_{\alpha,\nu} f\|_{L_q(B(x,r),dm_\nu)} \lesssim r^{\frac{2\nu+2}{q}} \sup_{t>2r} t^{-\frac{2\nu+2}{q}} \|f\|_{L_p(B(x,t),dm_\nu)} \tag{5}$$

holds for all $f \in L_p^{\rm loc}(\mathbb{R}, dm_\nu)$.

Moreover if $p = 1$, *then the inequality*

$$\|M_{\alpha,\nu} f\|_{WL_q(B(x,r),dm_\nu)} \lesssim r^{\frac{2\nu+2}{q}} \sup_{t>2r} t^{-\frac{2\nu+2}{q}} \|f\|_{L_1(B(x,t),dm_\nu)} \tag{6}$$

holds for all $f \in L_1^{\rm loc}(\mathbb{R}, dm_\nu)$.

Proof. $0 \le \alpha < 2\nu + 2$, $1 \le p < \frac{2\nu+2}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\nu+2}$. Denote

$$A_1 = m_\nu(B(x,\tau))^{\frac{1}{q}} \Big(\sup_{\tau>2r} \frac{1}{m_\nu(B(x,\tau))^{\frac{1}{p}-\frac{\alpha}{2\nu+2}}} \int_{B(x,\tau)} |f(z)|\, dm_\nu(z) \Big),$$

$$A_2 = \|f\|_{L_p(2B,dm_\nu)}.$$

Applying Hölder's inequality, we get

$$A_1 \lesssim m_\nu(B(x,\tau))^{\frac{1}{q}} \Big(\sup_{\tau>2r} \frac{1}{m_\nu(B(x,\tau))^{\frac{1}{q}}} \int_{B(x,\tau)} |f(z)|^p\, dm_\nu(z) \Big)^{\frac{1}{p}}.$$

On the other hand,

$$m_\nu(B(x,\tau))^{\frac{1}{q}} \Big(\sup_{\tau>2r} \frac{1}{m_\nu(B(x,\tau))^{\frac{1}{q}}} \int_{B(x,\tau)} |f(z)|^p\, dm_\nu(z) \Big)^{\frac{1}{p}}$$

$$\gtrsim m_\nu(B(x,\tau))^{\frac{1}{q}} \Big(\sup_{\tau>2r} \frac{1}{m_\nu(B(x,\tau))^{\frac{1}{q}}} \Big) \|f\|_{L_p(2B,dm_\nu)}.$$

Since by Lemma 6

$$\|M_{\alpha,\nu} f\|_{L_q(B,dm_\nu)} \lesssim A_1 + A_2,$$

we arrive at (5).

Let $p = 1$. Inequality (6) directly follows from (2).

The following Spanne's type result completely characterizes the boundedness of $M_{\alpha,\nu}$ on total D_ν-Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

Theorem 1. *Let* $0 \le \alpha < 2\nu + 2$*,* $0 \le \lambda, \mu < 2\nu + 2$*,* $1 \le p < \min\{\frac{2\nu+2-\lambda}{\alpha}, \frac{2\nu+2-\mu}{\alpha}\}$*, and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\nu+2}$.

1. If $f \in L_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)$*, then* $M_{\alpha,\nu} f \in WL_{q,\lambda,\mu}(\mathbb{R}, dm_\nu)$ *and*

$$\|M_{\alpha,\nu} f\|_{WL_{q,\lambda q,\mu q}(\mathbb{R},dm_\nu)} \le C_{1,\lambda,\mu} \, \|f\|_{L_{1,\lambda,\mu}(\mathbb{R},dm_\nu)}, \tag{7}$$

where $C_{q,\lambda,\mu}$ *is independent of* f*.*

2. If $f \in L_{p,\lambda,\mu}(\mathbb{R}, dm_\nu)$*,* $1 < p < \infty$*, then* $M_\nu f \in L_{q,\lambda,\mu}(\mathbb{R}, dm_\nu)$ *and*

$$\|M_{\alpha,\nu} f\|_{L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R},dm_\nu)} \le C_{p,q,\lambda,\mu} \, \|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)}, \tag{8}$$

where $C_{p,\lambda,\mu}$ *depends only on* p*,*λ*,*μ *and* ν*.*

Proof. Let $p = 1$. From the inequality (6) we get

$$\begin{aligned}
&\|M_{\alpha,\nu} f\|_{WL_{q,\lambda q,\mu q}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\lambda} \, [1/t]_1^{\mu} \, \|M_{\alpha,\nu} f\|_{WL_q(B(x,t),dm_\nu)} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\lambda} \, [1/t]_1^{\mu} \, t^{\frac{2\nu+2}{q}} \sup_{\tau>2t} \tau^{-\frac{2\nu+2}{q}} \|f\|_{L_1(B(x,\tau),dm_\nu)} \\
&\lesssim \|f\|_{L_{1,\lambda,\mu}(\mathbb{R},dm_\nu)} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\lambda} \, [1/t]_1^{\mu} \, t^{-\alpha+2\nu+2} \sup_{\tau>t} \tau^{\alpha-2\nu+2} \, [\tau]_1^{\lambda} \, [1/\tau]_1^{-\mu} \\
&= \|f\|_{L_{1,\lambda,\mu}(\mathbb{R},dm_\nu)} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\alpha+2\nu+2-\lambda} \, [1/t]_1^{\alpha+\mu-2\nu-2} \sup_{\tau>t} [\tau]_1^{\alpha+\lambda-2\nu-2} \, [1/\tau]_1^{-\alpha+2\nu+2-\mu} \\
&\lesssim \|f\|_{L_{1,\lambda,\mu}(\mathbb{R},dm_\nu)}
\end{aligned}$$

which implies that the operator M_ν is bounded from $L_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)$ to $WL_{1,\lambda,\mu}(\mathbb{R}, dm_\nu)$.

Let $1 < p < \min\{\frac{2\nu+2-\lambda}{\alpha}, \frac{2\nu+2-\mu}{\alpha}\}$. From the inequality (1) we get

$$\begin{aligned}
&\|M_{\alpha,\nu} f\|_{L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R},dm_\nu)} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \, [1/t]_1^{\frac{\mu}{p}} \, \|M_{\alpha,\nu} f\|_{L_q(B(x,t),dm_\nu)} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \, [1/t]_1^{\frac{\mu}{p}} \, t^{\frac{2\nu+2}{q}} \sup_{\tau>2t} \tau^{-\frac{2\nu+2}{q}} \|f\|_{L_p(B(x,\tau))} \\
&\lesssim \|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \, [1/t]_1^{\frac{\mu}{p}} \, t^{\frac{2\nu+2}{q}} \sup_{\tau>t} \tau^{-\frac{2\nu+2}{q}} \, [\tau]_1^{\frac{\lambda}{p}} \, [1/\tau]_1^{-\frac{\mu}{p}} \\
&= \|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)} \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{\frac{2\nu+2-\lambda}{p}} \, [1/t]_1^{\frac{\mu-2\nu+2}{p}} \sup_{\tau>t} [\tau]_1^{\frac{\lambda-2\nu+2}{p}} \, [1/\tau]_1^{\frac{2\nu+2-\mu}{p}}
\end{aligned}$$

$\lesssim \|f\|_{L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)}$

which implies that the operator $M_{\alpha,\nu}$ is bounded from $L_{p,\lambda,\mu}(\mathbb{R},dm_\nu)$ to $L_{q,\lambda,\mu}(\mathbb{R},dm_\nu)$.

From Theorem 1 in the case $\lambda=\mu$ or $\mu=0$ we get the following corollaries.

Corollary 1. [2, 19, 34] *Let $0\le\alpha<2\nu+2$, $0\le\lambda<2\nu+2$, $1\le p<\frac{2\nu+2-\lambda}{\alpha}$, and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\nu+2}$.*

1. If $f\in L_{1,\lambda}(\mathbb{R},dm_\nu)$, then $M_{\alpha,\nu}f\in WL_{q,\lambda}(\mathbb{R},dm_\nu)$ and

$$\|M_{\alpha,\nu}f\|_{WL_{q,\lambda}(\mathbb{R},dm_\nu)}\le C_{q,\lambda}\,\|f\|_{L_{1,\lambda}(\mathbb{R},dm_\nu)},$$

where $C_{q,\lambda}$ is independent of f.

2. If $f\in L_{p,\lambda}(\mathbb{R},dm_\nu)$, $p>1$, then $M_{\alpha,\nu}f\in L_{q,\lambda}(\mathbb{R},dm_\nu)$ and

$$\|M_{\alpha,\nu}f\|_{L_{q,\lambda}(\mathbb{R},dm_\nu)}\le C_{p,q,\lambda}\,\|f\|_{L_{p,\lambda}(\mathbb{R},dm_\nu)},$$

where $C_{p,q,\lambda}$ depends only on p, q, λ and ν.

Corollary 2. [20] *Let $0\le\alpha<2\nu+2$, $0\le\lambda<2\nu+2$, $1\le p<\frac{2\nu+2-\lambda}{\alpha}$, and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\nu+2}$.*

1. If $f\in \widetilde{L}_{1,\lambda}(\mathbb{R},dm_\nu)$, then $M_{\alpha,\nu}f\in W\widetilde{L}_{q,\lambda}(\mathbb{R},dm_\nu)$ and

$$\|M_{\alpha,\nu}f\|_{W\widetilde{L}_{q,\lambda}(\mathbb{R},dm_\nu)}\le C_{q,\lambda}\,\|f\|_{\widetilde{L}_{1,\lambda}(\mathbb{R},dm_\nu)},$$

where $C_{1,\lambda}$ is independent of f. , 2. If $f\in\widetilde{L}_{p,\lambda}(\mathbb{R},dm_\nu)$, $p>1$, then $M_{\alpha,\nu}f\in \widetilde{L}_{q,\lambda}(\mathbb{R},dm_\nu)$ and

$$\|M_{\alpha,\nu}f\|_{\widetilde{L}_{q,\lambda}(\mathbb{R},dm_\nu)}\le C_{p,q,\lambda}\,\|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R},dm_\nu)},$$

where $C_{p,q,\lambda}$ depends only on p, q, λ and ν.

Remark 2. *Note that in the case of the multidimensional Dunkl setting, the main results of this paper were proved in [23].*

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