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Uniqueness of recovery of the Dirac system on a segment by two spectra

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Abstract. The article considers the Dirac system under some boundary conditions, one of which linearly contains the spectral parameter. The statement is given, the uniqueness theorem is proved, and an algorithm for solving the inverse problem of recovering boundary value problems from two spectra is constructed.

Key Words and Phrases: Dirac system, eigenvalues, inverse problem, uniqueness theorem, solution algorithm.

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1. Introduction

The development of the theory of inverse problems, starting from the middle of the last century, was stimulated by its numerous applications in natural sciences and various fields of natural science and engineering. One of the important classes of inverse spectral problems consists of problems of reconstruction of systems of differential equations from spectral data. Inverse problems related to the canonical Dirac system have been studied by many authors. The first work devoted to reconstruction of the Dirac system on an interval from two spectra is the article [1]. Here the authors developed a constructive solution based on transformation operators and obtained a characterization of the spectral data. Similar results for the Dirac operator with summable potential, small delay and separated boundary conditions were established in [2-3]. In the case of non-separated (including periodic, antiperiodic, quasiperiodic and generalized periodic) boundary conditions, some versions of inverse problems are completely solved in [4-8], where the recovery of the Dirac operator is mainly carried out from two and three spectra and some sequence of signs. There are several works devoted to the non-self-adjoint case (see [9] and the literature therein). Note that in [10-14],

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direct and inverse problems for systems of differential equations with a spectral parameter in separated and non-separated boundary conditions are investigated.

In this paper, the canonical Dirac system is considered under some boundary conditions, one of which contains a spectral parameter. Some spectral properties of boundary value problems are studied, a statement is given, a uniqueness theorem is proved, an algorithm for solving the inverse problem of reconstructing boundary value problems from spectral data, which are the spectra of two boundary value problems, is compiled.

2. Some spectral properties of the problem

The one-dimensional stationary Dirac system (associated with the behavior of a relativistic electron in an electrostatic field) has the following canonical form (see [1]):

$$BY'(x) + Q(x)\,Y(x) = \lambda Y(x), \tag{1}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix},\ Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}.$$

Assume that the elements $p(x)$ and $q(x)$ matrices $Q(x)$ in (1) are real functions belonging to the space $W_2^1[0,\pi]$. By $W_2^1[0,\pi]$ we denote the space consisting of absolutely continuous functions defined on a segment $[0,\pi]$ that have a derivative that is summable with the square of $[0,\pi]$. Consider a boundary value problem generated on a segment $[0,\pi]$ by the Dirac equation (1) and boundary conditions of the form

$$\begin{gathered} y_1(0) = 0, \\ y_2(0) - \lambda\left[d_1 y_1(\pi) + d_2 y_2(\pi)\right] = 0, \end{gathered} \tag{2}$$

where λ is the spectral parameter, d_1 and d_2 are positive numbers. We will denote this problem by D.

It is clear that the boundary value problem D always has a trivial solution $Y(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Definition. If for some $\lambda = \lambda_0$ there exists a nontrivial solution of equation (1) satisfying the boundary conditions (2), then such a number λ_0 is called an eigenvalue, and the corresponding solution is called an eigenvector function of the boundary value problem D. The set of eigenvalues is called the spectrum D.

Let $S(x,\lambda) = \begin{pmatrix} s_1(x,\lambda) \\ s_2(x,\lambda) \end{pmatrix}$ and $C(x,\lambda) = \begin{pmatrix} c_1(x,\lambda) \\ c_2(x,\lambda) \end{pmatrix}$ be solutions of equation (1) satisfying the initial conditions $S(0,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C(0,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Then it is easily established that the characteristic function, whose zeros are the eigenvalues of the boundary value problem D, has the form

$$\Delta(\lambda) = 1 - \lambda \left[d_1 s_1 (\pi,\, \lambda) + d_2 s_2 (\pi,\, \lambda) \right]. \tag{3}$$

Lemma [13]. *For functions $s_1 (\pi,\, \lambda)$, $s_2 (\pi,\, \lambda)$ the following representations hold:*

$$s_1 (\pi,\, \lambda) = -\sin \lambda\pi + A_1 \frac{\cos \lambda\pi}{\lambda} + B_1 \frac{\sin \lambda\pi}{\lambda} + \frac{\psi_1 (\lambda)}{\lambda},$$

$$s_2 (\pi, \lambda) = \cos \lambda\pi + A_2 \frac{\sin \lambda\pi}{\lambda} + B_2 \frac{\cos \lambda\pi}{\lambda} + \frac{\psi_2 (\lambda)}{\lambda},$$

where $A_1 = A + Q_1$, $A_2 = A + Q_2$,$A = \frac{1}{2} \int_0^\pi \left[p^2 (x) + q^2 (x) \right] dx$,

$$Q_1 = \frac{q (\pi) - q (0)}{2}, \;\; Q_2 = -\frac{q (\pi) + q (0)}{2},$$

$$B_1 = -\frac{p\ (0) + p\ (\pi)}{2}, \; B_2 = \frac{p\ (0) - p\ (\pi)}{2},$$

$$\psi_j (\lambda) = \int_{-\pi}^{\pi} \tilde{\psi}_j (t)\ e^{i\lambda t} dt, \tilde{\psi}_j (t) \in L_2 \left[-\pi, \pi \right], \;\; j = 1, 2.$$

Theorem 1. *The eigenvalues of the boundary value problem for satisfy the asymptotic formula*

$$\mu_k = k + a + \frac{A}{\pi k} + \frac{(-1)^{k+1} \sqrt{d_1^2 + d_2^2} + Q_1 d_1^2 + Q_2 d_2^2 - p\,(\pi)\, d_1 d_2}{\pi \left(d_1^2 + d_2^2 \right)\ k} + \frac{\gamma_k}{k}, \tag{4}$$

where $a = \frac{1}{\pi} \mathrm{arctg} \frac{d_2}{d_1}$, $A = \frac{1}{2} \int_0^\pi \left[p^2 (x) + q^2 (x) \right] dx$, $\{\gamma_k\} \in l_2$.

Proof. Taking into account the representations of the functions $s_1 (\pi,\, \lambda)$, $s_2 (\pi,\, \lambda)$ in the lemma, the characteristic function (3) can be transformed to the form

$$\begin{aligned} \Delta(\lambda) = 1 + \lambda \left(d_1 \sin \lambda\pi - d_2 \cos \lambda\pi \right) - \left(d_1 A_1 + d_2 B_2 \right) \cos \lambda\pi - \\ - \left(d_1 B_1 + d_2 A_2 \right) \sin \lambda\pi + \psi_3 (\lambda), \end{aligned} \tag{5}$$

where $\psi_3 (\lambda) = \int_{-\pi}^{\pi} \tilde{\psi}_3 (t)\ e^{i\lambda t} dt$, $\tilde{\psi}_3 (t) \in L_2 [-\pi, \pi]$. Using the estimate $\psi (\lambda) = 0 \left(e^{|Im \lambda\pi|} \right)$ (for $|\lambda| \to \infty$) and Rouché's theorem, it is easy to establish that the zeros of function (5) form a sequence of the form

$$\mu_k = k + \frac{1}{\pi} \mathrm{arctg} \frac{d_2}{d_1} + \varepsilon_k, \tag{6}$$

where $\varepsilon_k = o(1)$ at $|k| \to \infty$. Substituting the right side of equality (6) into (5) and taking into account that $\Delta(\mu_k) = 0$, after elementary transformations we obtain

$$\varepsilon_k = \frac{(d_1 A_1 + d_2 B_2)\cos \pi a + (d_1 B_1 + d_2 A_2)\sin \pi a + (-1)^{k+1}}{\pi(d_1 \cos \pi a + d_2 \sin \pi a)k} + \frac{\gamma_k}{k}.$$

Using the equalities

$$\cos \pi a = \cos\left(\operatorname{arctg}\frac{d_2}{d_1}\right) = \frac{1}{\sqrt{1+\left(\frac{d_2}{d_1}\right)^2}} = \frac{d_1}{\sqrt{d_1^2+d_2^2}},$$

$$\sin \pi a = \sin\left(\operatorname{arctg}\frac{d_2}{d_1}\right) = \frac{1}{\sqrt{1+\left(\frac{d_1}{d_2}\right)^2}} = \frac{d_2}{\sqrt{d_1^2+d_2^2}},$$

we have

$$\varepsilon_k = \frac{A}{\pi k} + \frac{(-1)^{k+1}\sqrt{d_1^2+d_2^2} + Q_1 d_1^2 + Q_2 d_2^2 - p(\pi)\, d_1 d_2}{\pi\left(d_1^2+d_2^2\right)\, k} + \frac{\gamma_k}{k}\,.$$

Substituting this asymptotics into (6), we arrive at equality (5). The theorem is proven.

In what follows, we will assume that $p(0) = p(\pi) = q(0) = q(\pi) = 0$.

Theorem 2. *The assignment of the spectrum* $\{\mu_k\}$ *(*$k = \pm 0, \pm 1, \pm 2, ...$*) uniquely determines the characteristic function* $\Delta(\lambda)$ *of the boundary value problem* D *according to the formula*

$$\Delta(\lambda) = \pi\sqrt{d_1^2+d_2^2}\,(\mu_{-0}-\lambda)\,(\mu_{+0}-\lambda) \prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\mu_k - \lambda}{k}, \tag{7}$$

where

$$\sqrt{d_1^2+d_2^2} = \frac{1}{\pi \lim\limits_{k\to\infty} k\,(\mu_{2k+1} - \mu_{2k} - 1)}.$$

This theorem is proved similarly to Theorem 2 of [15].

Along with the boundary value problem D, the boundary value problem $\tilde{D}$ generated by the same equation (1) and boundary conditions is also considered

$$\begin{gathered} y_1(0) = 0, \\ y_2(0) - \lambda\left[\tilde{d}_1 y_1(\pi) + \tilde{d}_2 y_2(\pi)\right] = 0. \end{gathered} \tag{8}$$

In view of (3), the characteristic function of the problem $\tilde{D}$ will have the form

$$\Delta(\lambda) = 1 - \lambda \left[\tilde{d}_1 s_1 (\pi,\, \lambda) + \tilde{d}_2 s_2 (\pi,\, \lambda)\right]. \tag{9}$$

The spectrum of this problem will be denoted by $\{\tilde{\mu}_k\}$.

3. Statement of the inverse problem. Uniqueness theorem. Solution algorithm

Consider the following inverse problem.

Inverse problem. Given the spectra of boundary value problems D and $\tilde{D}$ construct a matrix function $Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}$ in the Dirac equation (1) and coefficients $d_1,\ d_2,\ \tilde{d}_1$ and $\tilde{d}_2$ in the boundary conditions (2) and (8).

The following uniqueness theorem is true.

Theorem 3. *If* $\lim\limits_{k\to\infty} (\mu_k - \tilde{\mu}_k) \neq 0$, *then the assignment of spectra* $\{\mu_k\}$, $\{\tilde{\mu}_k\}$ *uniquely determines the boundary value problems* D *and* $\tilde{D}$.

Proof. According to Theorem 1, the eigenvalues μ_k and $\tilde{\mu}_k (k = \pm 0, \pm 1, \pm 2, ...)$ of the boundary value problems D and $\tilde{D}$at $|k| \to \infty$ satisfy the asymptotic formulas

$$\mu_k = k + a + \frac{A}{\pi k} + \frac{(-1)^{k+1}}{\sqrt{d_1^2 + d_2^2}\pi\, k} + \frac{\gamma_k}{k}, \tag{10}$$

$$\tilde{\mu}_k = k + \tilde{a} + \frac{A}{\pi k} + \frac{(-1)^{k+1}}{\sqrt{\tilde{d}_1^2 + \tilde{d}_2^2}\pi\, k} + \frac{\tilde{\gamma}_k}{k}, \tag{11}$$

where $\tilde{a} = \frac{1}{\pi}\mathrm{arctg}\frac{\tilde{d}_1}{\tilde{d}_2}$, $\{\tilde{\gamma}_k\} \in l_2$ (since $p\,(\pi) = Q_1 = Q_2 = 0$). Then

$$\mu_{2k+1} = 2k + 1 + a + \frac{A}{\pi(2k+1)} + \frac{1}{\sqrt{d_1^2 + d_2^2}\pi\,(2k+1)} + \frac{\gamma_{2k+1}}{2k+1},$$

$$\mu_{2k} = 2k + a + \frac{A}{2\pi k} - \frac{1}{2\sqrt{d_1^2 + d_2^2}\pi\, k} + \frac{\gamma_{2k}}{2k},$$

$$\tilde{\mu}_{2k+1} = 2k + 1 + \tilde{a} + \frac{A}{(2k+1)\pi} + \frac{1}{\sqrt{\tilde{d}_1^2 + \tilde{d}_2^2}\pi\,(2k+1)} + \frac{\tilde{\gamma}_{2k+1}}{2k+1},$$

$$\tilde{\mu}_{2k} = 2k + \tilde{a} + \frac{A}{2\pi k} - \frac{1}{2\sqrt{\tilde{d}_1^2 + \tilde{d}_2^2}\pi\, k} + \frac{\tilde{\gamma}_{2k}}{2k}.$$

From here

$$\sqrt{d_1^2+d_2^2}=\frac{1}{\pi\lim\limits_{k\to\infty}k\,(\mu_{2k+1}-\mu_{2k}-1)},\ \sqrt{\tilde d_1^2+\tilde d_2^2}=\frac{1}{\pi\lim\limits_{k\to\infty}k\,(\tilde\mu_{2k+1}-\tilde\mu_{2k}-1)}. \tag{12}$$

By virtue of Theorem 2, given sequences $\{\mu_k\}$ and $\{\tilde\mu_k\}$, it is possible to reconstruct the characteristic functions $\Delta(\lambda)$ and $\tilde\Delta(\lambda)$ of boundary value problems D and $\tilde D$ in the form of an infinite product using formulas (7) and

$$\tilde\Delta(\lambda)=\pi\sqrt{\tilde d_1^2+\tilde d_2^2}\,(\tilde\mu_{-0}-\lambda)\,(\tilde\mu_{+0}-\lambda)\prod_{\substack{k=-\infty\\ k\neq 0}}^{\infty}\frac{\tilde\mu_k-\lambda}{k}. \tag{13}$$

According to the presentation (5)

$$\Delta(2k)=1-2kd_2-Ad_1+\psi_3(2k),$$

$$\Delta\left(2k+\frac{1}{2}\right)=1+\left(2k+\frac{1}{2}\right)d_1-Ad_2+\psi_3\left(2k+\frac{1}{2}\right).$$

Hence

$$d_1=2\lim_{k\to\infty}\frac{\Delta\left(2k+\frac{1}{2}\right)}{4k+1},\ d_2=-\lim_{k\to\infty}\frac{\Delta(2k)}{2k}, \tag{14}$$

since by virtue of the Riemann-Lebesgue lemma $\lim\limits_{k\to\infty}\psi_3(2k)=\lim\limits_{k\to\infty}\psi_3\left(2k+\frac{1}{2}\right)=0$. Similarly

$$\tilde d_1=-\lim_{k\to\infty}\frac{\tilde\Delta(2k)}{2k},\ \tilde d_2=2\lim_{k\to\infty}\frac{\tilde\Delta\left(2k+\frac{1}{2}\right)}{4k+1}. \tag{15}$$

It follows from the condition of the theorem that $a-\tilde a\neq 0$ or $\operatorname{arctg}\frac{d_1}{d_2}\neq\operatorname{arctg}\frac{\tilde d_1}{\tilde d_2}$. So, $d_1\tilde d_2-\tilde d_1 d_2\neq 0$. This means that the determinant of the system of equations

$$\begin{cases} d_1 s_1(\pi,\lambda)+d_2 s_2(\pi,\lambda)=\frac{1-\Delta(\lambda)}{\lambda},\\ \\ \tilde d_1 s_1(\pi,\lambda)+\tilde d_2 s_2(\pi,\lambda)=\frac{1-\tilde\Delta(\lambda)}{\lambda}\end{cases} \tag{16}$$

(obtained from relations (3) and (9)) with respect to the unknowns $s_1(\pi,\lambda)$ and $s_2(\pi,\lambda)$ are not equal to zero, that is, this system has a unique solution. Solving system (16), we uniquely find the functions $s_1(\pi,\lambda)$ and $s_2(\pi,\lambda)$. It is easy to see that these functions are characteristic functions of the boundary value problems generated by equation (1) and the boundary conditions $y_1(0) = y_1(\pi) = 0$ and $y_1(0) = y_2(\pi) = 0$. It is known [1, 4] that the coefficient $Q(x)$ of the Dirac equation (1) is uniquely determined from the sequences of zeros of these functions.

Thus, from the given spectra $\{\mu_k\}$ and $\{\tilde{\mu}_k\}$ the boundary value problems D and $\tilde{D}$ are completely reconstructed. The theorem is proved.

According to the proof of Theorem 3, the solution to the inverse problem can be obtained using the following algorithm.

Algorithm. Given are sequences $\{\mu_k\}$ and $\{\tilde{\mu}_k\}$– spectra of boundary value problems D and $\tilde{D}$.

Step 1. Using (10) and (11), we calculate the quantities $\sqrt{d_1^2 + d_2^2}$, $\sqrt{\tilde{d}_1^2 + \tilde{d}_2^2}$ by formulas (12).

Step 2. From the sequences $\{\mu_k\}$ and $\{\tilde{\mu}_k\}$, we construct the characteristic functions $\Delta(\lambda)$ and $\tilde{\Delta}(\lambda)$ in the form of an infinite product (7) and (13).

Step 3. We determine the coefficients d_1, d_2, $\tilde{d}_1$ and $\tilde{d}_2$ in the boundary conditions by formulas (14) and (15).

Step 4. Solving system (16), we uniquely find the functions $s_1(\pi, \lambda)$ and $s_2(\pi, \lambda)$.

Step 5. From the sequences of zeros of the functions $s_1(\pi, \lambda)$ and $s_2(\pi, \lambda)$, we construct the coefficient $Q(x)$ of the Dirac equation (1) using the well-known procedure (see, for example, [1, 4]).

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