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On the dynamics of derivatives of a singularly perturbed fourth-order equation

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Abstract. In this paper, the preliminary integration method based on the first-kind Chebyshev polynomials is applied to the numerical computation of the derivatives of a singularly perturbed fourth-order differential equation. Although numerous numerical methods and approaches have been developed for solving such equations, the issues of their accuracy and computational efficiency remain of considerable interest. The investigation of the behavior of derivatives of different orders is of particular importance for the construction of highly accurate and efficient numerical methods, as well as for the localization of boundary-layer regions. To this end, the preliminary integration method is employed to investigate the behavior of derivatives of different orders up to the highest-order derivative for the considered class of problems. To demonstrate the high accuracy and computational efficiency of the proposed method, it is compared with the spectral collocation method in terms of both the maximum absolute errors and the number of arithmetic operations. Since the number of arithmetic operations is one of the principal measures of the computational efficiency of a numerical method, the obtained numerical results confirm the high accuracy and efficiency of the proposed method.

Key Words and Phrases: Chebyshev polynomials, preliminary integration, dynamics of derivatives, characteristic parameters.

2010 Mathematics Subject Classifications: 65L11, 65L70, 65L20

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1. Introduction

The development and investigation of highly accurate and efficient numerical methods for analyzing the behavior of derivatives of singularly perturbed non-homogeneous differential equations constitute an important problem in applied mathematics. Various numerical approaches, including finite difference methods, finite element methods, spectral methods, and the method of preliminary integration, can be employed to obtain the solution of singularly perturbed differential equations. Among these approaches, spectral methods and the method of preliminary integration not only provide accurate numerical solutions but also make it possible to investigate the behavior of derivatives up to the highest-order derivative of the singularly perturbed differential equation under consideration. The principal difference between the spectral method and the method of preliminary integration is as follows. In the spectral method, the solution of the singularly perturbed differential equation is approximated by a finite expansion in terms of the first-kind Chebyshev polynomials, while derivatives of different orders are obtained by differentiating the resulting polynomial approximation. In general, each differentiation reduces the degree of the approximating Chebyshev polynomial expansion. In contrast, the method of preliminary integration approximates not the solution itself but the highest-order derivative of the singularly perturbed differential equation by a finite expansion in terms of Chebyshev polynomials. All lower-order derivatives are then determined through successive integrations of the finite expansion representing the highest-order derivative. It is evident that successive integration increases the degree of the approximating polynomial. For example, a second-degree polynomial becomes a sixth-degree polynomial after four successive integrations. Conversely, in the spectral method, a sixth-degree polynomial is reduced to a second-degree polynomial after four successive differentiations. For this reason, the application of the method of preliminary integration to the investigation of the behavior of derivatives of different orders in singularly perturbed differential equations provides reliable numerical accuracy.

The proposed method possesses the following advantages. First, the basis functions constructed from the first-kind Chebyshev polynomials are infinitely differentiable throughout the computational domain. Second, the boundary conditions of the problem can be satisfied exactly by an appropriate selection of the basis functions. Third, the derivatives are determined directly from the derivatives of the basis functions without introducing any additional approximation. Fourth, the method exhibits a high rate of convergence. Furthermore, owing to the dual representation of the numerical solution, the proposed method provides deeper insight into the physical processes underlying the problem under investigation. The computed solution can be interpreted both through its behavior in the physical domain and through the relative contributions of the individual basis functions in the spectral domain. Moreover, due to the high accuracy of the method, the same level of accuracy as that achieved by other numerical methods can be obtained using a considerably smaller number of basis functions. In other words, for the same computational cost, the proposed method enables the investigation of problems involving more demanding values of the characteristic parameters, thereby improving computational efficiency. In addition, efficient Fast Fourier Transform (FFT) algorithms are available for Chebyshev polynomial expansions, further enhancing the efficiency of the numerical computations.

In this paper, it is demonstrated that the method of preliminary integration constitutes a universal numerical approach for the numerical modeling of a broad class of singularly perturbed differential equations. The proposed method provides high numerical accuracy together with computational efficiency, measured by the number of arithmetic operations required to investigate the behavior of both the solution itself and its derivatives of various orders.

The first-kind Chebyshev polynomials are widely employed in the numerical modeling of a variety of applied problems. Several representative applications are presented below. In [1], the spectral homotopy analysis method based on the first-kind Chebyshev polynomials is applied to the numerical solution of the Duffing and Van der Pol equations. To demonstrate the practicality and efficiency of the proposed method, the authors compare the approximate and exact solutions of the problem and investigate the behavior of the approximation errors. In [2], linear fractional Fredholm integro-differential equations are solved numerically by employing the first-kind Chebyshev polynomials, where the fractional derivative is defined in the Caputo sense. The objective of the authors is to represent the unknown function as a finite series of linear combinations of Chebyshev polynomials and subsequently reduce the original problem to a system of linear algebraic equations for the unknown coefficients associated with the approximate solution. The resulting system is solved using MATLAB. Several numerical examples are presented to verify the reliability, applicability, and efficiency of the proposed method. The first-kind Chebyshev polynomials have also been employed in the construction of the generalized Chebyshev–Poisson integral [3]. In [4], an algorithm for obtaining approximate solutions of various well-known Lane–Emden-type equations is presented. The proposed algorithm represents the unknown solution in terms of shifted first-kind Chebyshev polynomials. The author claims that the proposed method is considerably simpler than any other existing method for solving this class of initial value problems. In [5], the dynamics of the stock prices of eight Czech companies are analyzed using an expansion based on the first-kind Chebyshev polynomials. The authors demonstrate that the proposed model makes it possible to analyze the dynamics of stock portfolio prices without taking variance and correlation into account. The model is particularly useful when the dynamics of stock prices, owing to certain circumstances, do not follow the normal probability distribution. In [6], a new numerical algorithm is proposed for solving a first-order nonlinear singularly perturbed differential equation with an integral boundary condition. The authors construct the basis function IMSCPFK (Improved Modified Shifted Chebyshev Polynomial Function Kernel) together with the corresponding operational matrix of its derivatives. By employing the spectral collocation method, the original problem is transformed into a system of algebraic equations, from which the numerical solution is obtained. Theoretical analysis and numerical examples confirm the validity, efficiency, and high accuracy of the proposed method. In [7], direct discontinuous Galerkin methods on a Bakhvalov-type mesh are developed for solving a singularly perturbed reaction–diffusion problem. A unified generalized framework for the proposed methods is presented and analyzed. The methods are shown to possess parameter-uniform convergence.

In [8], a time-dependent singularly perturbed problem with a spatial shift is investigated. The problem is discretized by the ultra-weak discontinuous Galerkin method in space on a Bakhvalov-type mesh and by the Crank–Nicolson scheme in time. Optimal-order convergence estimates are established. According to the authors, the numerical experiments demonstrate the efficiency of the proposed method. The weak Galerkin finite element method for a fourth-order singularly perturbed reaction–diffusion problem is investigated in [9]. On the basis of the solution expansion and the Shishkin mesh, the parameter-uniform convergence of the proposed method is established. Numerical examples are presented to confirm the theoretical results and the accuracy of the error estimates. In [10], the hydrodynamic stability of a two-phase Poiseuille flow is investigated. The problem is described by a system of nonlinear singularly perturbed differential equations.

The numerical modeling of the hydrodynamic stability problem for a two-phase flow in the boundary layer is carried out in [11]. Paper [12] is devoted to the numerical modeling of a nonlinear wave in a dissipative non-dispersive medium governed by the Burgers equation with the corresponding initial and boundary conditions. To solve the problem numerically, the spectral-grid method developed in [13] is employed. The numerical results demonstrate the high accuracy and computational efficiency of the proposed method.

In [14], the numerical solution of the inverse problem of determining the kinematic viscosity in the vorticity equation is investigated. The resulting problem is reduced to a variational problem involving the minimization of a functional. A computational experiment is carried out to illustrate the reconstruction of the kinematic viscosity for various levels of measurement error. The following studies are directly related to the numerical solution of boundary value problems for singularly perturbed differential equations, that is, differential equations containing a small parameter multiplying the highest-order derivative. The numerical solution of a boundary value problem for a second-order singularly perturbed differential equation is investigated in [15]. The application of the spectral method based on the first-kind Chebyshev polynomials to the numerical solution of a boundary value problem for a fourth-order singularly perturbed differential equation is presented in [16]. The convergence analysis of the spectral method for evolutionary equations is presented in [17]. In [18], the method of preliminary integration is applied to the numerical solution and investigation of the behavior of the solution of a boundary value problem for a nonhomogeneous fourth-order differential equation with a small parameter multiplying the highest-order derivative.

The convergence theorems and convergence rate estimates for the method of preliminary integration applied to the Orr–Sommerfeld eigenvalue problem, which is formulated as a boundary value problem for a fourth-order nonlinear ordinary differential equation with a small parameter multiplying the highest-order derivative, are presented in [19].

In [20], an approach to the numerical solution of a boundary value problem for a nonhomogeneous second-order singularly perturbed differential equation is developed based on the continuous version of the method of preliminary integration.

In [21], a discrete version of the method of preliminary integration is proposed for the numerical solution of a nonhomogeneous biharmonic equation. The partial derivatives are represented by finite double series of the first-kind Chebyshev polynomials with unknown expansion coefficients. Numerical experiments performed on a variety of test functions confirm the high accuracy and computational efficiency of the proposed method. The objective of [22] is to develop a highly accurate and computationally efficient numerical method for investigating the behavior of derivatives of various orders of a second-order singularly perturbed differential equation. In the continuous version of the method of preliminary integration employed in that work, the highest-order derivative and the right-hand side of the differential equation are represented by finite expansions in terms of the first-kind Chebyshev polynomials with unknown coefficients. These coefficients are determined by solving a system of algebraic equations, after which they are used to analyze the behavior of derivatives of different orders as well as the solution of the boundary value problem under consideration.

2. Statement of the problem

Consider the problem of numerically investigating the behavior of derivatives of various orders of the nonhomogeneous singularly perturbed differential equation

$$\varepsilon \frac{d^4u}{dy^4} - 2\frac{d^2u}{dy^2} + u(y) = f(y), \qquad y \in (-1,1), \tag{1}$$

subject to the boundary conditions

$$u(\pm 1) = \frac{du}{dy}(\pm 1) = 0, \tag{2}$$

where ε is a small perturbation parameter.

To investigate the behavior of derivatives up to the highest-order derivative of the boundary value problem (1)–(2), we employ the method of preliminary integration, in which the first-kind Chebyshev polynomials are used as basis functions. To verify the convergence and accuracy of an arbitrary numerical method, the method of trial functions is commonly employed [16]. According to this approach, an arbitrary trial function $u(y)$ satisfying the boundary conditions (2) is selected. By substituting this function into Eq. (1), the right-hand side $f(y)$ and the derivatives of different orders of $u(y)$ are determined.

The boundary value problem is then solved by the method of preliminary integration, and the obtained approximate solution together with its derivatives of various orders is compared with the corresponding values of the selected trial function and its derivatives. For the boundary value problem (1)–(2), the following trial function is chosen:

$$u(y) = \left(1 - y^2\right)^2 e^{\varepsilon y}, \tag{3}$$

which satisfies the boundary conditions (2).

For the trial function (3), the right-hand side of Eq. (1) takes the form

$$\begin{aligned} f(y) = &\Big[(\varepsilon^5 + 2\varepsilon^2 + 1)y^4 + 16\varepsilon(\varepsilon^3 - 1)y^3 + \left(72\varepsilon^3 - 2\varepsilon^5 + 4\varepsilon^2 - 26\right) y^2 \\ &+ \left(96\varepsilon^2 - 16\varepsilon^4 + 16\varepsilon\right) y + 24\varepsilon + \varepsilon^5 - 24\varepsilon^3 - 2\varepsilon^2 + 9\Big] e^{\varepsilon y}. \end{aligned} \tag{4}$$

Since the primary objective of this paper is to investigate the behavior of derivatives of different orders, the derivatives of the trial function (3) up to the fourth order are computed as follows:

$$\frac{du}{dy} = (y^2 - 1)(\varepsilon y^2 + 4y - \varepsilon)e^{\varepsilon y}, \tag{5}$$

$$\frac{d^2u}{dy^2} = \left[\varepsilon y^3(\varepsilon y + 8) + 2(6 - \varepsilon^2)y^2 - \varepsilon(8y - \varepsilon) - 4\right] e^{\varepsilon y}, \tag{6}$$

$$\frac{d^3u}{dy^3} = \left[\varepsilon^2 y^3(\varepsilon y + 12) + 2\varepsilon(18 - \varepsilon^2)y^2 + 12(2 - \varepsilon^2)y + \varepsilon(\varepsilon^2 - 12)\right] e^{\varepsilon y}, \tag{7}$$

$$\frac{d^4u}{dy^4} = \left[\varepsilon^3(\varepsilon y + 16)y^3 + 2\varepsilon^2(36 - \varepsilon^2)y^2 + 16\varepsilon(6 - \varepsilon^2)y + \varepsilon^2(\varepsilon^2 - 24) + 24\right] e^{\varepsilon y}. \tag{8}$$

Expressions (5)–(8) will be used to compare the exact derivatives with their corresponding approximate values obtained by the method of preliminary integration.

3. Solution method

In the method of preliminary integration, the highest-order derivative and the right-hand side of the boundary value problem (1)–(2) are represented by the following finite expansions:

$$\frac{d^4u}{dy^4} = u^{(4)}(y) = \sum_{i=0}^{N} a_i T_i(y), \qquad f(y) = \sum_{i=0}^{N} b_i T_i(y), \tag{9}$$

where $T_i(y)$ denotes the Chebyshev polynomial of the first kind, and a_i, b_i $(i = 0, 1, 2, \ldots, N)$ are the unknown expansion coefficients.

The main idea of the method of preliminary integration is as follows. The expansion (9) for the fourth-order derivative is integrated four times before solving the boundary value problem (1)–(2). Consequently, all lower-order derivatives and the solution itself are represented as finite expansions in terms of Chebyshev polynomials. Only after these representations have been obtained are they substituted into the differential equation (1). The resulting procedure is described below. Thus, the expressions for the lower-order derivatives are given by

$$u^{(3)}(y) = \sum_{i=0}^{N+1} \sum_{j=0}^{N} f_{ij}^{(3)} a_j T_i + c_1 T_0, \tag{10}$$

$$u^{(2)}(y) = \sum_{i=0}^{N+2} \sum_{j=0}^{N} f_{ij}^{(2)} a_j T_i + c_1 T_1 + c_2 T_0, \tag{11}$$

$$u^{(1)}(y) = \sum_{i=0}^{N+3} \sum_{j=0}^{N} f_{ij}^{(1)} a_j T_i + \frac{c_1}{4} T_2 + c_2 T_1 + c_3 T_0, \tag{12}$$

$$u(y) = \sum_{i=0}^{N+4} \sum_{j=0}^{N} f_{ij}^{(0)} a_j T_i + \frac{c_1}{24} T_3 + \frac{c_2}{4} T_2 + \left(c_3 - \frac{c_1}{8}\right) T_1 + c_4 T_0. \tag{13}$$

Here, c_1, c_2, c_3, and c_4 are unknown constants of integration. These constants are determined from the boundary conditions (2) by employing the property of the Chebyshev polynomials $T_n(\pm 1) = (\pm 1)^n$.

As a result, a system of four equations is obtained for determining the unknown constants c_1, c_2, c_3, and c_4. These equations involve the parameters

$$\zeta_j^{(\beta)} = \sum_{i=0}^{N+4-\beta} f_{ij}^{(\beta)}, \qquad \bar{\zeta}_j^{(\beta)} = \sum_{i=0}^{N+4-\beta} (-1)^i f_{ij}^{(\beta)}, \qquad \beta = 0, 1. \tag{14}$$

The functions $f_{ij}^{(\beta)}$, $\beta = 0, 1, 2, 3$, appearing in Eqs. (10)–(13), are computed according to the following formulas:

$$
\begin{aligned}
f_{ij}^{(3)} &= \delta_{i,j+1}\beta_j^{(3)} + \delta_{i,j-1}\gamma_j^{(3)}, \\
f_{ij}^{(2)} &= \delta_{i,j+2}\beta_j^{(2)} + \delta_{i,j}\gamma_j^{(2)} + \delta_{i,j-2}\delta_j^{(2)}, \\
f_{ij}^{(1)} &= \delta_{i,j+3}\beta_j^{(1)} + \delta_{i,j+1}\gamma_j^{(1)} + \delta_{i,j-1}\delta_j^{(1)} + \delta_{i,j-3}\varepsilon_j^{(1)}, \\
f_{ij}^{(0)} &= \delta_{i,j+4}\beta_j^{(0)} + \delta_{i,j+2}\gamma_j^{(0)} + \delta_{i,j}\delta_j^{(0)} + \delta_{i,j-2}\varepsilon_j^{(0)} + \delta_{i,j-4}\eta_j^{(0)}.
\end{aligned}
\tag{15}
$$

Here, δ_{ij} denotes the Kronecker delta. The nonzero coefficients β_j, γ_j, δ_j, ε_j, and η_j are computed according to the following formulas:

$$
\begin{aligned}
&\beta_j^{(3)} = \frac{c_j}{2(j+1)}, \ \beta_j^{(\beta-1)} = \frac{\beta_j^{(\beta)}}{2(j+5-\beta)}, \ \beta = 3, 2, 1, \quad j \geq 0, \\
&\gamma_j^{(3)} = -\frac{1}{2(j-1)}, \ j \geq 2, \ \gamma_j^{(\beta-1)} = \frac{-\beta_j^{(\beta)} + \gamma_j^{(\beta)}}{2(j+3-\beta)}, \ \beta = 3, 2, 1, \quad j \geq \beta - 2, \\
&\delta_j^{(2)} = -\frac{\gamma_j^{(3)}}{2(j-2)}, \ j \geq 3, \ \delta_j^{(\beta-1)} = \frac{-\gamma_j^{(\beta)} + \delta_j^{(\beta)}}{2(j+1-\beta)}, \ \beta = 3, 2, \quad j \geq \beta, \\
&\varepsilon_j^{(1)} = -\frac{\delta_j^{(2)}}{2(j-3)}, \ j \geq 4, \ \varepsilon_j^{(0)} = \frac{-\delta_j^{(1)} + \varepsilon_j^{(1)}}{2(j-2)}, \ j \geq 3, \\
&\eta_j^{(0)} = -\frac{\varepsilon_j^{(1)}}{2(j-4)}, \ j \geq 5.
\end{aligned}
\tag{16}
$$

Substituting the expressions for the constants c_1, c_2, c_3, and c_4 into Eqs. (10)–(13), the resulting expressions can be written in the following general form:

$$
u^{(\beta)}(y) = \sum_{i=0}^{N+4-\beta} \sum_{j=0}^{N} g_{ij}^{(\beta)} a_j T_i, \qquad \beta = 0, 1, 2, 3. \tag{17}
$$

where the coefficients $g_{ij}^{(\beta)}$, $\beta = 0, 1, 2, 3$, are given by

$$
\begin{aligned}
g_{ij}^{(3)} &= f_{ij}^{(3)} + \delta_{j,0}\frac{3}{2}\left[(\zeta_j^{(0)} - \bar{\zeta}_j^{(0)}) - (\zeta_j^{(1)} + \bar{\zeta}_j^{(1)})\right], \\
g_{ij}^{(2)} &= f_{ij}^{(2)} + \delta_{j,1}\frac{3}{2}\left[(\zeta_j^{(0)} - \bar{\zeta}_j^{(0)}) - (\zeta_j^{(1)} + \bar{\zeta}_j^{(1)})\right] \\
&\quad + \delta_{j,0}\left[-\frac{1}{2}(\zeta_j^{(1)} - \bar{\zeta}_j^{(1)})\right], \\
g_{ij}^{(1)} &= f_{ij}^{(1)} + \delta_{j,2}\frac{3}{8}\left[(\zeta_j^{(0)} - \bar{\zeta}_j^{(0)}) - (\zeta_j^{(1)} + \bar{\zeta}_j^{(1)})\right] \\
&\quad + \delta_{j,1}\left[-\frac{1}{2}(\zeta_j^{(1)} - \bar{\zeta}_j^{(1)})\right] \\
&\quad + \delta_{j,0}\left[\frac{1}{8}\left(3(\zeta_j^{(0)} - \bar{\zeta}_j^{(0)}) + (\zeta_j^{(1)} + \bar{\zeta}_j^{(1)})\right)\right], \\
g_{ij}^{(0)} &= f_{ij}^{(0)} \\
&\quad + \delta_{j,3}\frac{1}{24}\left[\frac{3}{2}\left(\zeta_j^{(0)} - \bar{\zeta}_j^{(0)}\right) - \left(\zeta_j^{(1)} + \bar{\zeta}_j^{(1)}\right)\right] \\
&\quad + \delta_{j,2}\frac{1}{4}\left[-\frac{1}{2}\left(\zeta_j^{(1)} - \bar{\zeta}_j^{(1)}\right)\right] \\
&\quad + \delta_{j,1}\left\{-\frac{1}{8}\left[3\left(\zeta_j^{(0)} - \bar{\zeta}_j^{(0)}\right) + \left(\zeta_j^{(1)} + \bar{\zeta}_j^{(1)}\right)\right]\right. \\
&\qquad \left. -\frac{1}{8}\left[\frac{3}{2}\left(\zeta_j^{(0)} - \bar{\zeta}_j^{(0)}\right) - \left(\zeta_j^{(1)} + \bar{\zeta}_j^{(1)}\right)\right]\right\} \\
&\quad + \delta_{j,0}\left[\frac{1}{2}\left(-\zeta_j^{(0)} + \bar{\zeta}_j^{(0)}\right) + \frac{1}{4}\left(\zeta_j^{(1)} - \bar{\zeta}_j^{(1)}\right)\right].
\end{aligned}
\tag{18}
$$

The unknown coefficients b_j in expansion (9) corresponding to the known right-hand side $f(y)$ given by Eq. (4) are determined by the following inverse transformation:

$$
b_j = \frac{2}{NC_j}\sum_{l=0}^{N}\frac{1}{C_l}f(y_l)T_j(y_l), \qquad j = 0, 1, 2, \ldots, N, \tag{19}
$$

where $C_0 = C_N = 2$, $C_l = 1$, $l \neq 0, N$, and $y_l = \cos\frac{\pi l}{N}$, $l = 0, 1, 2, \ldots, N$, are the collocation nodes associated with the first-kind Chebyshev polynomials.

Substituting expansions (9) and (17) into differential equation (1) and equating the coefficients of the corresponding Chebyshev polynomials yields the following system of algebraic equations for determining the unknown expansion coefficients $a_0, a_1, \ldots, a_N$:

$$\sum_{k=0}^{N} \left[\varepsilon\delta_{jk} - 2g_{jk}^{(2)} + g_{jk}^{(0)} \right] a_k = b_j, \qquad j = 0, 1, \ldots, N. \tag{20}$$

By solving system (20), the unknown expansion coefficients are obtained. Subsequently, using Eqs. (9) and (17) with $\beta = 3, 2, 1, 0$, the numerical solution and its derivatives of different orders, up to the highest-order derivative of the boundary value problem (1)–(2), are computed by the method of preliminary integration.

The implementation procedure (flowchart) of the method of preliminary integration is summarized as follows:

1. The perturbation parameter ε and the number N of approximating Chebyshev polynomials are specified.

2. The coefficients β, γ, δ, ε, and η are computed according to Eqs. (16).

3. The coefficients $f_{ij}^{(3)}$, $f_{ij}^{(2)}$, $f_{ij}^{(1)}$, and $f_{ij}^{(0)}$ are computed using Eqs. (15).

4. The coefficients required for determining the approximate solution and its derivatives up to the third order are computed according to Eqs. (18).

5. The expansion for the fourth-order derivative given by Eq. (9), the corresponding expansions in Eq. (17), and the expansion of the right-hand side given by Eq. (19) are substituted into differential equation (1). Equating the coefficients of the corresponding Chebyshev polynomials yields a system of algebraic equations for determining the unknown expansion coefficients $a_0, a_1, \ldots, a_N$. This system is solved by the Gaussian elimination method.

6. The computed coefficients $a_0, a_1, \ldots, a_N$ are substituted into Eqs. (9) and (17) to obtain the numerical solution of boundary value problem (1)-(2) together with its derivatives up to the fourth order at the Chebyshev collocation nodes $y_l = \cos\frac{\pi l}{N}, \quad l = 0, 1, \ldots, N$.

7. The exact solution (3) and its derivatives are calculated using Eqs. (4) at the Chebyshev polynomial collocation nodes $y_l = \cos\frac{\pi l}{N}, \quad l = 0, 1, \ldots, N$.

8. Finally, tabular and graphical results are generated to investigate the behavior of the numerical solution and its derivatives up to the fourth order.

4. Convergence of the Method

In this section, we investigate the convergence of the method of preliminary integration for boundary value problem (1)–(2). To this end, we rewrite the problem in the following equivalent form:

$$\frac{d^4u}{dy^4} - \frac{1}{\varepsilon}\left(2\frac{d^2u}{dy^2} - u(y)\right) = \frac{1}{\varepsilon}f(y), \qquad y \in (-1,1), \tag{21}$$

subject to the boundary conditions

$$u(\pm 1) = \frac{du}{dy}(\pm 1) = 0.$$

Assume that the homogeneous form of problem (21), corresponding to $f(y) = 0$, admits only the trivial solution. Under this assumption, the Green's function for the considered boundary value problem exists.

To establish the convergence of the method, both the continuous problem (21) and its discrete counterpart obtained by the method of preliminary integration are reduced to the following second-kind operator equation:

$$X + TX = F, \tag{22}$$

where T is a compact operator acting in the Banach space E. Therefore, the general convergence theorems for projection methods [19] can be applied to the above operator equation.

First, we reduce the differential problem (21) to the operator equation (22). For this purpose, we introduce the function

$$x(y) = \frac{d^4u(y)}{dy^4}, \tag{23}$$

where $u(y)$ is the solution of boundary value problem (1)–(2). Then,

$$u(y) = \int_{-1}^{1} G(y,\xi)x(\xi)\,d\xi, \tag{24}$$

where $G(y,\xi)$ is the Green's function associated with problem (21). Under these definitions, differential problem (21) can be written in the form

$$x(y) - \frac{1}{\varepsilon}\int_{-1}^{1}\left[2\frac{d^2G(y,\xi)}{dy^2} - G(y,\xi)\right]x(\xi)\,d\xi = \frac{1}{\varepsilon}\int_{-1}^{1} G(y,\xi)f(\xi)\,d\xi.$$

Introducing the notation

$$T(y,\xi) = -\frac{1}{\varepsilon}\left[2\frac{d^2G(y,\xi)}{dy^2} - G(y,\xi)\right],$$

the above equation can be rewritten in the form

$$x(y) + \int_{-1}^{1} T(y,\xi)x(\xi)\,d\xi = \frac{1}{\varepsilon}\int_{-1}^{1} G(y,\xi)f(\xi)\,d\xi. \tag{25}$$

From the properties of the Green's function, it follows that $T(y,\xi)$ is continuous on $[-1,1]\times[-1,1]$ except along the diagonal $y=\xi$, where it has a finite jump discontinuity. Introducing the operator

$$Tx = \int_{-1}^{1} T(y,\xi)x(\xi)\,d\xi,$$

and the function

$$F(y) = \frac{1}{\varepsilon}\int_{-1}^{1} G(y,\xi)f(\xi)\,d\xi,$$

Eq. (25) can be written in the operator form

$$x + Tx = F. \tag{26}$$

Next, we reduce the discrete problem obtained by the method of preliminary integration to the second-kind operator equation of the form (22).

Consider the differential operator defined by Eq. (17). Introduce the following notation:

$$x^{(N)}(y) = \frac{d^4u^{(N)}(y)}{dy^4} = \sum_{i=0}^{N} a_i T_i(y). \tag{27}$$

Then,

$$u^{(N)}(y) = \int_{-1}^{1} G(y,\xi)x^{(N)}(\xi)\,d\xi. \tag{28}$$

Taking into account Eqs. (27) and (28), together with the fact that $u^{(N)}(y)$ satisfies the boundary conditions in (21), differential equation (21) can be written in the form

$$x^{(N)}(y) - \int_{-1}^{1}\frac{1}{\varepsilon}\left[2G^{(2)}(y,\xi) - G(y,\xi)\right]x^{(N)}(\xi)\,d\xi = \frac{1}{\varepsilon}\int_{-1}^{1} G(y,\xi)f(\xi)\,d\xi, \tag{29}$$

where

$$G^{(2)}(y,\xi) = \frac{d^2G(y,\xi)}{dy^2}.$$

Now, define the projection operator $P : L_{2,\rho} \to L_{2,\rho}$ by the following rule. If $x(y) = \sum_{k=0}^{\infty} e_k T_k(\tilde{y})$, then $Px(y) = \sum_{k=0}^{N} e_k T_k(\tilde{y}), \quad -1 \le \tilde{y} \le 1$. Introducing the notation

$$\overline{T}(y,\xi) = -\frac{1}{\varepsilon}\left(2G^{(2)}(y,\xi) - G(y,\xi)\right),$$

$$\widehat{T}x^{(N)}(y) = \int_{-1}^{1} \overline{T}(y,\xi)x^{(N)}(\xi)\,d\xi,$$

and

$$\overline{F}(y) = \int_{-1}^{1} G(y,\xi)f(\xi)\,d\xi,$$

Eq. (29) can be rewritten in the form

$$x^{(N)}(y) + P\widehat{T}x^{(N)}(y) = P\overline{F}(y). \tag{30}$$

The above analysis makes it possible to establish several convergence results concerning the approximation $x^{(N)}(y) \to x(y), \quad N \to \infty$.

Consequently, in view of Eqs. (24) and (28), it follows that $u^{(N)}(y) \to u(y), \quad N \to \infty$.

The following convergence theorems for the method of preliminary integration can now be established. Let the solutions of the operator equations (26) and (30) be denoted by $x_0(y)$ and $x^{(N)}(y)$, respectively.

Theorem 1. Assume that the right-hand side $f(y)$ of Eq. (21) is a continuous function. Then, for sufficiently large values of N, the discrete operator equation (30) has a unique solution $x^{(N)}$, and the following estimate holds:

$$\left\| x^{(N)} - x_0 \right\|_{L^N_{2,\rho}} \le C_0 \left\| x_0 - x_0^N \right\|_{L^N_{2,\rho}}, \tag{31}$$

where x_0^N denotes the truncated Chebyshev–Fourier series of degree N corresponding to the function x_0, namely, $x_0^{(N)}(y) = \sum_{j=0}^{N} \widetilde{C}_j T_j(\widetilde{y}), \quad \widetilde{C}_j = \int_{-1}^{1} x_0(y)T_j(\widetilde{y})\rho(\widetilde{y})\,dy$, with $\|x_0\|^2_{L^N_{2,\rho}} = \sum_{i=0}^{\infty} \widetilde{C}_i^2, \quad \left\| x_0 - x_0^{(N)} \right\|^2_{L^N_{2,\rho}} = \sum_{i=N+1}^{\infty} \widetilde{C}_i^2.$

Here, C_0 is a positive constant depending on the norm of the inverse operator $(E+T)^{-1}$, where E denotes the identity operator, and $\rho(\widetilde{y}) = \dfrac{1}{\sqrt{1-\widetilde{y}^2}}$.

Theorem 2. Suppose that the right-hand side $f(y)$ of Eq. (21) satisfies $C^{s+\alpha}[-1,1]$. Then the approximate solution $u^{(N)}(y)$, obtained by the method of preliminary integration and defined by Eq. (28), converges to the exact solution $u(y)$ of boundary value problem (21). Moreover, the following estimate holds:

$$\left\| u^{(N)}(y) - u(y) \right\|^2_{L^N_{2,\rho}(-1,1)} \leq C \left(\frac{2}{N} \right)^{2(s+\alpha)},$$

where C is a positive constant, N is the number of approximating Chebyshev polynomials, and the parameters s and α are determined by the condition

$$\left| f^{(s)}(x_1) - f^{(s)}(x_2) \right| \leq M|x_1 - x_2|^{\alpha}, \qquad s = 0,1,2,3.$$

The proofs of Theorems 1 and 2 are analogous to the proofs of the corresponding theorems for the eigenvalue problem presented in [19].

5. Discussion of the Results

We present the numerical results obtained by the method of preliminary integration for investigating the dynamics of derivatives of different orders of the singularly perturbed fourth-order differential equation (1) subject to the boundary conditions (2). The computations were carried out for the small parameter values $\varepsilon = 10^{-2}, \ldots, 10^{-5}$, and for the numbers of Chebyshev basis polynomials $N = 30, \ldots, 50$.

The tables below report the maximum absolute errors computed at the Chebyshev collocation nodes $y_l = \cos\dfrac{\pi l}{N}$, $\quad l = 1,2,\ldots,N$, using the following formula:

$$\Delta_k = \max_{1\leq l\leq N} \left| u_e^{(k)} - u_a^{(k)} \right|, \qquad k = 0,1,2,3,4.$$

Here, k denotes the order of the derivative, $u_e = u_{\text{exact}}$ represents the exact solution (3) and its derivatives given by Eqs. (5)–(8), whereas $u_a = u_{\text{approximate}}$ denotes the approximate solution obtained from Eq. (9) together with its derivatives computed by Eqs. (10)–(13) using the method of preliminary integration.

The numerical results were obtained at the Chebyshev collocation nodes $y_l = \cos\dfrac{\pi l}{N}$, $\quad l = 0,1,\ldots,N$. The exact and approximate values of the solution derivatives of different orders were compared at these collocation points.

Table 1 presents the maximum absolute errors of the approximate solution and its derivatives of different orders obtained by the spectral collocation method.

Table 1: Maximum absolute errors obtained by the spectral collocation method for different values of the small parameter ε and the number of Chebyshev basis polynomials N.

Small parameter ε	N	Solution	Derivatives			
			First	Second	Third	Fourth
10^{-2}	10	$2.2 \cdot 10^{-16}$	$6.6 \cdot 10^{-16}$	$5.3 \cdot 10^{-15}$	$2.0 \cdot 10^{-4}$	$2.0 \cdot 10^{-2}$
	20	$2.2 \cdot 10^{-16}$	$6.6 \cdot 10^{-16}$	$3.5 \cdot 10^{-15}$	$2.0 \cdot 10^{-4}$	$2.0 \cdot 10^{-2}$
	50	$1.1 \cdot 10^{-15}$	$8.8 \cdot 10^{-16}$	$1.9 \cdot 10^{-14}$	$2.0 \cdot 10^{-3}$	$2.0 \cdot 10^{-2}$
10^{-3}	10	$2.7 \cdot 10^{-15}$	$8.8 \cdot 10^{-15}$	$1.1 \cdot 10^{-13}$	$2.0 \cdot 10^{-6}$	$1.8 \cdot 10^{-4}$
	20	$9.9 \cdot 10^{-16}$	$2.2 \cdot 10^{-15}$	$4.2 \cdot 10^{-14}$	$2.0 \cdot 10^{-6}$	$1.8 \cdot 10^{-4}$
	50	$5.5 \cdot 10^{-16}$	$1.5 \cdot 10^{-15}$	$4.4 \cdot 10^{-14}$	$2.0 \cdot 10^{-6}$	$2.6 \cdot 10^{-4}$
10^{-5}	10	$1.4 \cdot 10^{-15}$	$1.1 \cdot 10^{-15}$	$8.8 \cdot 10^{-15}$	$2.0 \cdot 10^{-10}$	$1.8 \cdot 10^{-8}$
	20	$9.9 \cdot 10^{-16}$	$1.4 \cdot 10^{-14}$	$3.9 \cdot 10^{-13}$	$3.1 \cdot 10^{-10}$	$4.5 \cdot 10^{-6}$
	50	$6.9 \cdot 10^{-16}$	$2.5 \cdot 10^{-15}$	$5.9 \cdot 10^{-13}$	$2.1 \cdot 10^{-10}$	$1.7 \cdot 10^{-3}$

It can be observed that the absolute errors of the third- and fourth-order derivatives remain relatively large and exhibit little sensitivity to increasing the number of Chebyshev basis polynomials from $N = 10$ to $N = 50$.

Table 2 presents the maximum absolute errors obtained by the proposed preliminary integration method for the solution and its derivatives under the same values of the characteristic parameters, namely $\varepsilon = 10^{-2} \div 10^{-5}$ and $N = 5 \div 20$. In this experiment, the number of Chebyshev basis polynomials is limited to $N = 20$, since even for $N = 10$ all absolute errors are already sufficiently small, being of the order of 10^{-10}.

Table 2: Maximum absolute errors obtained by the preliminary integration method for different values of the small parameter ε and the number of Chebyshev basis polynomials N.

Small parameter ε	N	Solution	Derivatives			
			First	Second	Third	Fourth
10^{-2}	5	$4.1 \cdot 10^{-8}$	$1.6 \cdot 10^{-7}$	$1.0 \cdot 10^{-6}$	$1.2 \cdot 10^{-5}$	$5.7 \cdot 10^{-5}$
	10	$6.6 \cdot 10^{-16}$	$1.3 \cdot 10^{-15}$	$4.4 \cdot 10^{-15}$	$3.2 \cdot 10^{-14}$	$5.1 \cdot 10^{-13}$
	20	$5.5 \cdot 10^{-16}$	$1.3 \cdot 10^{-15}$	$7.1 \cdot 10^{-15}$	$1.3 \cdot 10^{-13}$	$6.8 \cdot 10^{-12}$
10^{-3}	5	$1.6 \cdot 10^{-10}$	$8.9 \cdot 10^{-9}$	$4.8 \cdot 10^{-8}$	$1.6 \cdot 10^{-6}$	$1.9 \cdot 10^{-5}$
	10	$4.4 \cdot 10^{-16}$	$1.1 \cdot 10^{-15}$	$1.6 \cdot 10^{-14}$	$7.9 \cdot 10^{-13}$	$2.4 \cdot 10^{-11}$
	20	$5.5 \cdot 10^{-16}$	$8.8 \cdot 10^{-16}$	$3.7 \cdot 10^{-14}$	$2.1 \cdot 10^{-12}$	$8.6 \cdot 10^{-11}$
10^{-5}	5	$2.4 \cdot 10^{-14}$	$4.9 \cdot 10^{-13}$	$1.2 \cdot 10^{-12}$	$6.1 \cdot 10^{-11}$	$7.9 \cdot 10^{-10}$
	10	$7.7 \cdot 10^{-16}$	$2.8 \cdot 10^{-15}$	$3.2 \cdot 10^{-14}$	$4.2 \cdot 10^{-12}$	$1.6 \cdot 10^{-10}$
	20	$2.1 \cdot 10^{-15}$	$9.0 \cdot 10^{-14}$	$3.2 \cdot 10^{-12}$	$1.1 \cdot 10^{-9}$	$1.6 \cdot 10^{-7}$

From the results presented in Table 2, it can be seen that the proposed preliminary integration method provides highly accurate approximations of the solution and its derivatives up to the fourth order for the singularly perturbed fourth-order boundary value problem (1)–(2), even when using a significantly smaller number of Chebyshev basis polynomials, namely $N = 5 \div 10$. In contrast, a comparable level of accuracy cannot be achieved by the spectral collocation method even when $N = 50$.

To further demonstrate the computational efficiency of the proposed preliminary integration method (PIM), we compare it with the spectral collocation method (SCM) in terms of the number of arithmetic operations. Both methods require the solution of the system of linear algebraic equations (20) to determine the unknown expansion coefficients $a_0, a_1, \ldots, a_N$. Since the coefficient matrix of this system is dense, it can be solved efficiently using Gaussian elimination. The computational complexity of Gaussian elimination for solving system (20) is $Q = \frac{2}{3}N^3$ arithmetic operations. Table 3 presents a comparison illustrating the computational efficiency of the proposed preliminary integration method.

Table 3: Comparison of the preliminary integration method and the spectral collocation method.

Method	Small parameter ε	Number of polynomials	Number of operations Q	Δ_k $(k = 4)$
PIM	10^{-5}	10	667	$1.69 \cdot 10^{-10}$
SCM		50	83333	$1.7 \cdot 10^{-3}$

From the results presented in Table 3, it follows that the ratio of the numbers of arithmetic operations required by the two methods is

$$M = \frac{Q_{\text{SCM}}}{Q_{\text{PIM}}} = \frac{83333}{667} = 125.$$

This result indicates that the proposed preliminary integration method requires 125 times fewer arithmetic operations than the spectral collocation method. Moreover, in the considered example, the maximum absolute error of the fourth-order derivative obtained by the preliminary integration method is more than three orders of magnitude smaller ($1.69 \cdot 10^{-10}$) than that obtained by the spectral collocation method ($1.7 \cdot 10^{-3}$). These results demonstrate that the proposed preliminary integration method is both computationally efficient and highly accurate. It should be noted that, in the comparison of the two methods, the preliminary integration method was implemented using only ten Chebyshev basis polynomials ($N = 10$), since this number is sufficient to achieve high accuracy for the solution of boundary value problem (1)-(2) and its derivatives. As shown in Table 3, the order of the coefficient matrix of the linear algebraic system (20) is

$10 \times 10 = 100$ for the preliminary integration method, whereas it is $50 \times 50 = 2500$ for the spectral collocation method. Consequently, while maintaining high accuracy and computational efficiency, the proposed preliminary integration method requires 25 times less computer memory than the spectral collocation method. The computed solution and its derivatives up to the fourth order are illustrated in Figures 1-4 for the parameter values $\varepsilon = 10^{-5}$ and $N = 10$.

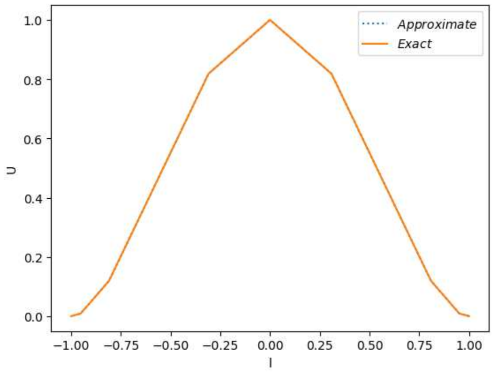

Fig. 1. First derivative.

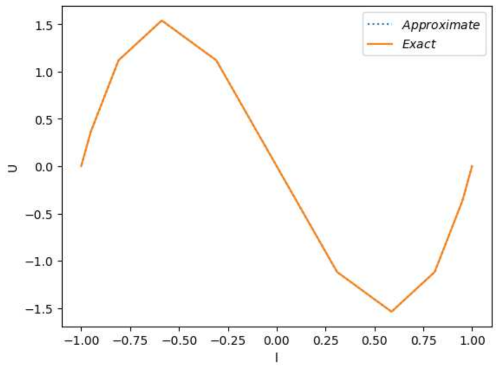

Fig. 2. Second derivative.

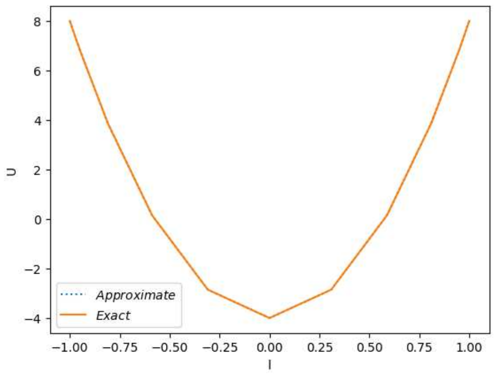

Fig. 3. Third derivative.

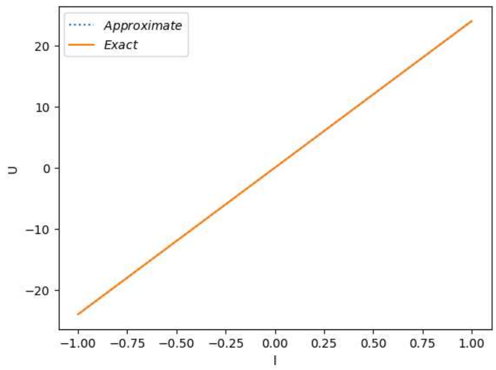

Fig. 4. Fourth derivative.

The numerical results presented in Tables 2 and 3 and Figures 1–4 for different values of the small parameter ε and the number of Chebyshev basis polynomials N demonstrate that the proposed preliminary integration method provides highly accurate approximations of the solution to problem (1)-(2) and its derivatives while requiring relatively low computational time and memory resources.

6. Conclusions

The main results of this study can be summarized as follows:

1. A highly accurate and efficient preliminary integration method has been proposed for investigating the dynamics of the solution and its derivatives of different orders for a singularly perturbed fourth-order boundary value problem.

2. An algorithm and a flowchart of the proposed preliminary integration method have been developed for computing the solution and its derivatives of different orders.

3. The proposed preliminary integration method has been compared with the spectral collocation method in terms of both accuracy and computational complexity. The numerical results demonstrate that the proposed method requires approximately 125 times fewer arithmetic operations while providing more than three orders of magnitude higher accuracy.

4. Extensive numerical experiments have been carried out to demonstrate the high accuracy and computational efficiency of the proposed method for various values of the small perturbation parameter and different numbers of Chebyshev basis polynomials.

5. The numerical results have been presented in both tabular and graphical forms, illustrating the effectiveness and reliability of the proposed preliminary integration method.

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