Journal of Contemporary Applied Mathematics
V. 17, No 1, 2027, January-June
ISSN 2222-5498, E-ISSN 3006-3183
https://doi.org/10.62476/jcam171.4

An Age-Structured Model with Delay Applied to Tuberculosis in the Context of Population Mobility

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Abstract. In this article, we investigate an age-structured tuberculosis model with delay in the context of population mobility. The model is formulated as a system of delay differential equations. First, we establish the existence, uniqueness, and positivity of solutions by applying Cauchy's theorem. Next, we analyze the stability of the equilibrium points. The disease-free equilibrium is identified, and the basic reproduction number (R_0) is computed using the next-generation matrix method. The stability of the disease-free equilibrium is established using Lyapunov's method. Both analytical and numerical results confirm its stability. The endemic equilibrium is also derived, and its stability is investigated through numerical simulations implemented in Python. The results indicate that the endemic equilibrium is stable only when vaccination is taken into account.

Key Words and Phrases: VSEIRS model, patches, age-structured, mobility of population, time delay, nonlinear integro-differential system, differential equations, local asymptotic stability, tuberculosis.

2020 Mathematics Subject Classifications: 37N25, 47G20, 47F05, 15A30, 34K20, 65P40

1. Introduction

Faced with the emergence of infectious diseases, increasing bacterial resistance, the appearance of new variants, and the rapid spread of epidemics, the prediction and prevention of diseases have become major concerns for humanity. Understanding the dynamics of infectious disease transmission has therefore become very important. This requires an integrated and multidisciplinary approach. Mathematical modeling is a fundamental tool for understanding the interactions

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among the environment, animals, climate, and infectious diseases. It also remains an important tool for decision-makers.

Several works have been devoted to the study of such models, including [1, 2, 4, 9, 11, 12, 13, 14, 20, 21].

In our previous work, we studied an age-structured model with delay in a single patch [5]. The results showed that, in the absence of time delay, stability is achieved when the vaccination rate increases, whereas in the presence of time delay, a higher vaccination level is required to control the disease.

In the present work, we extend the model by considering two patches and applying it to tuberculosis. This choice is motivated not only by the fact that the model captures important features of tuberculosis transmission, but also by the fact that tuberculosis remains one of the top ten leading causes of death worldwide. Thus, we aim to contribute to the mathematical modeling of tuberculosis and hope that this manuscript will enrich the existing literature and stimulate further research for improved disease management.

As in [5], we divide the population into five compartments: susceptible individuals (S), exposed individuals (E), infected individuals (I), recovered individuals (R), and vaccinated individuals (V). In [5], we studied an age-structured model with delay and movement restriction between patches. In the present work, we consider population mobility between two patches, so that the spread of the disease is both spatial and temporal.

2. Model formulation

We consider Vaccinated (V), Susceptible (S), Exposed (E), Infected (I) and Recovered (R) of two age stages in age-structured epidemiological model with delay. The susceptible (S) are people who are able to develop the disease. The vaccinated are a group of individuals who are extracted among the susceptible and vaccinated. The exposed are those who have been in contact with the infected, but who do not show the clinical signs of the disease. Infected individuals can transmit the disease to susceptible individuals through contact. Recovered (people with temporary immunity againts disease either naturally or after vaccination). The individuals of each subpopulation are divided into two age groups: a first group goes from birth to maturity (from 0 to a_1) and a second group goes from maturity to maximum age A (from a_1 to A). In the first phase, the individual can be born, grow and die before maturity just as he can reach the age a_1 of maturity and begin to procreate [19], [21]. a_1 is considered the minimum age to be mature, it is the age from which individuals enter in the second phase and can begin to procreate. In the second phase, individuals can grow, procreate, die or ... reach the maximum age A.

Let $V(a,t)$, $S(a,t)$, $E(a,t)$, $I(a,t)$ and $R(a,t)$ be the respective densities the Vaccinated denoted V, of the susceptible denoted S, of the exposed denoted E, infected denoted I and recovered denoted R of age a and at time t in the domain $\Omega = \{(a,t)/0 \leqslant t \leqslant T; 0 \leqslant a \leqslant A\}$. The number of Vaccinated, Susceptible, Exposed, Infected, and Recovered individuals is respectively defined as follows:

$$N_s(t) = \int_0^A S(a,t)\, da,$$

$$N_e(t) = \int_0^A E(a,t) da,$$

$$N_i(t) = \int_0^A I(a,t) da,$$

$$N_r(t) = \int_0^A R(a,t) da,$$

$$N_v(t) = \int_0^A V(a,t) da.$$

Note that $S = S_1 + S_2$, $E = E_1 + E_2$, $I = I_1 + I_2$, $R = R_1 + R_2$, $V = V_1 + V_2$. The sum

$$N_s(t) + N_v(t) + N_e(t) + N_i(t) + N_r(t)$$

represents the total number of individuals and is denoted N(t). In constructing the mathematical model, it is assumed that infection in the population does not occur directly in the susceptible after exposure, but at some point with a delay. This slight delay or incubation time is denoted τ. Moreover, as the total number of the initial population is assumed to be finite, then the quantities $V(a,t)$, $S(a,t)$, $E(a,t)$; $I(a,t)$ and $R(a,t)$ belong to $L^1(\Omega)$.
The used parameters are:

- $\hat{\alpha}$ is the natural mortality rate of each subpopulation and is expressed in percentage per unit of time (years^{-1}).
- $\hat{\alpha}_{ad}$ is the death rate due to the disease and is expressed in percentage per unit of time (years^{-1}).
- δ is the rate of vaccination coverage against the disease and it is expressed in percentage.

- v is the rate of vaccinated people who did not have the expected effect of the vaccine and who became susceptible again at the same time. It is expressed in percentage.

- $$\hat{\theta}_i = \begin{cases} 1, & \text{if } a \in [a_1, A], \\ 0, & \text{otherwise.} \end{cases}$$

- γ_2 is the transmission coefficient from the exposed population to the infected population. It is expressed in percentage per unit of time (years^{-1}).

- γ_3 is the recovery rate and is expressed in percentage per unit of time (years^{-1}).

- ρ is the conversion rate of the recovered population losing immunity to the susceptible population. It is expressed in percentage per unit of time (years^{-1}).

- $\mu_i\,(i = 1, 2, 3, 4, 5)$ is the reproduction rate of each subpopulation at the stage of procreation. It is expressed in percentage per unit of time (years^{-1}).

- p is the proportion of exposed individuals that give birth to exposed newborns. It is expressed in percentage per unit of time (years^{-1}).

- Γ_{ji} is the migration rate of the population leaving zone j with status $i \in \{S, E, I, R, V\}$. It is expressed in percentage per unit of time (years^{-1}).

These different processes can be summarized in the following diagram (1):
Model with two patches and free mobility of all individuals

For reasons of simplicity, let:

$$\begin{aligned} A_0 &= \mu_1\hat{\theta}_1 S + \mu_2\hat{\theta}_2(1-p)E + \mu_3\hat{\theta}_3 I + \mu_4\hat{\theta}_4 R + \mu_5\hat{\theta}_5 V, \\ B_0 &= \mu_2\hat{\theta}_2 pE, \quad F_1 = (\alpha + \alpha_{ad})I. \end{aligned}$$

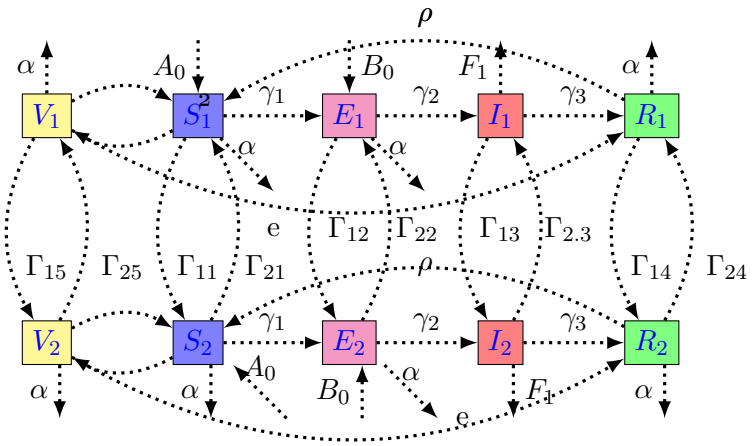

Figure 1: Diagram of mobility between two pacthes

The model is given by the following partial differential equations:

$$
\begin{cases}
\frac{\partial S_1}{\partial t} + \frac{\partial S_1}{\partial a} = -(\hat{\alpha} + \Gamma_{11} - \mu_1\hat{\theta}_1 + \delta)S_1(a,t) - \\
\gamma_1(a,t-\tau)\int_0^A \beta(a,a',t-\tau)I_1(a',t-\tau)da'S_1(a,t) + \mu_2\hat{\theta}_2(1-p)(a,t)E_1(a,t) \\
+\mu_3\hat{\theta}_3 I_1 + (\mu_4\hat{\theta}_4 + \rho)R_1(a,t) + (\mu_5\theta_5 + v)V_1(a,t) + \Gamma_{21}S_2(a,t) \\
\frac{\partial E_1}{\partial t} + \frac{\partial E_1}{\partial a} = -\left(\hat{\alpha} + \Gamma_{12} + \gamma_2 - \mu_2\theta_2 p\right)E_1(a,t) + \\
\gamma_1(a,t-\tau)\int_0^A \beta(a,a',t-\tau)I(a',t-\tau)da'S_1(a,t) + \Gamma_{22}E_2(a,t) \\
\frac{\partial I_1}{\partial t} + \frac{\partial I_1}{\partial a} = -\left(\hat{\alpha} + \gamma_3 + \alpha_{ad} + \Gamma_{13})\right)I_1(a,t) + \gamma_2(a,t)E_1(a,t) + \Gamma_{23}I_2(a,t) \\
\frac{\partial R_1}{\partial t} + \frac{\partial R_1}{\partial a} = -\left(\hat{\alpha} + \Gamma_{14} + \rho\right)R_1(a,t) + \gamma_3(a,t)I(a,t) + eV_1(a,t) + \Gamma_{24}R_2(a,t) \\
\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial a} = -\left(\hat{\alpha} + \Gamma_{15} + e + v\right)V_1(a,t) + \delta S_1(a,t) + \Gamma_{25}V_1(a,t) \\
\frac{\partial S_2}{\partial t} + \frac{\partial S_2}{\partial a} = -(\hat{\alpha} + \Gamma_{21} - \mu_1\hat{\theta}_1 + \delta)S_2(a,t) - \\
\gamma_1(a,t-\tau)\int_0^A \beta(a,a',t-\tau)I_2(a',t-\tau)da'S_2(a,t) + \mu_2\hat{\theta}_2(1-p)(a,t)E_2(a,t) + \\
\mu_3\hat{\theta}_3 I_2 + (\mu_4\hat{\theta}_4 + \rho)R_2(a,t) + (\mu_5\theta_5 + v)V_2(a,t) + \Gamma_{11}S_1(a,t) \\
\frac{\partial E_2}{\partial t} + \frac{\partial E_2}{\partial a} = -\left(\hat{\alpha} + \Gamma_{22}\gamma_2 - \mu_2\theta_2 p\right)E_1(a,t) + \\
\gamma_1(a,t-\tau)\int_0^A \beta(a,a',t-\tau)I(a',t-\tau)da'S_2(a,t) + \Gamma_{12}E_1(a,t) \\
\frac{\partial I_2}{\partial t} + \frac{\partial I_2}{\partial a} = -\left(\hat{\alpha} + \gamma_3 + \alpha_{ad} + \Gamma_{13}\right)I_2(a,t) + \gamma_2 E_2(a,t) + \Gamma_{13}I_1(a,t) \\
\frac{\partial R_2}{\partial t} + \frac{\partial R_2}{\partial a} = -\left(\hat{\alpha} + \Gamma_{24} + \rho\right)R_2(a,t) + \gamma_3(a,t)I_2(a,t) + eV_2(a,t) + \Gamma_{14}R_1(a,t) \\
\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial a} = -\left(\hat{\alpha} + \Gamma_{25} + e + v\right)V_2(a,t) + \delta S_2(a,t) + \Gamma_{15}V_2(a,t)
\end{cases}
\tag{1}
$$

with initial conditions:

$$\begin{cases} S_1(a,0) = S_0(a), & a \in [0,A], \\ E_1(a,0) = E_0(a), & a \in [0,A], \\ I_1(a,0) = I_0(a), & a \in [0,A], \\ R_1(a,0) = 0, & a \in [0,A], \\ V_1(a,0) = 0, & a \in [0,A], \\ S_2(a,0) = S_0(a), & a \in [0,A], \\ E_2(a,0) = E_0(a), & a \in [0,A], \\ I_2(a,0) = I_0(a), & a \in [0,A], \\ R_2(a,0) = 0, & a \in [0,A], \\ V_2(a,0) = 0, & a \in [0,A]. \end{cases} \tag{2}$$

3. Existence and Uniqueness of Solution

To study the existence and uniqueness of the solution of the problem, we transform it into an integro-differential problem using the method of characteristics [21, 16, 20].

$$
\begin{cases}
\frac{dS_1}{dt} = -(\hat{\alpha} + \Gamma_{11} - \mu_1\hat{\theta}_1 + \delta)S_1(u+t,t) - \\
\gamma_1(a,t-\tau)\int_0^A \beta(a,a',t-\tau)I_1(a',t-\tau)da'S_1(u+t,t) + \mu_2\hat{\theta}_2(1-p)(a,t)E_1(u+t,t) \\
+\mu_3\hat{\theta}_3 I_1 + (\mu_4\hat{\theta}_4 + \rho)R_1(a,t) + (\mu_5\theta_5 + v)V_1(u+t,t) + \Gamma_{21}S_2(u+t,t) \\
\frac{dE_1}{dt} = -\left(\hat{\alpha} + \Gamma_{12} + \gamma_2 - \mu_2\theta_2 p\right)E_1(u+t,t) + \\
\gamma_1(a,t-\tau)\int_0^A \beta(a,a',t-\tau)I(a',t-\tau)da'S_1(u+t,t) + \Gamma_{22}E_2(u+t,t) \\
\frac{\partial I_1}{\partial t} + \frac{\partial I_1}{\partial a} = -\left(\hat{\alpha} + \gamma_3 + \alpha_{ad} + \Gamma_{13}\right)I_1(u+t,t) + \gamma_2(a,t)E_1(u+t,t) + \Gamma_{23}I_2(u+t,t) \\
\frac{dR_1}{dt} = -\left(\hat{\alpha} + \Gamma_{14} + \rho\right)R_1(u+t,t) + \gamma_3(a,t)I(u+t,t) + eV_1(u+t,t) + \Gamma_{24}R_2(u+t,t) \\
\frac{dV_1}{dt} = -\left(\hat{\alpha} + \Gamma_{15} + e + v\right)V_1(u+t,t) + \delta S_1(u+t,t) + \Gamma_{25}V_1(u+t,t) \\
\frac{dS_2}{dt} = -(\hat{\alpha} + \Gamma_{21} - \mu_1\hat{\theta}_1 + \delta)S_2(u+t,t) - \\
\gamma_1(a,t-\tau)\int_0^A \beta(a,a',t-\tau)I_2(a',t-\tau)da'S_2(u+t,t) + \mu_2\hat{\theta}_2(1-p)(a,t)E_2(u+t,t) + \\
\mu_3\hat{\theta}_3 I_2 + (\mu_4\hat{\theta}_4 + \rho)R_2(a,t) + (\mu_5\theta_5 + v)V_2(u+t,t) + \Gamma_{11}S_1(u+t,t) \\
\frac{dE_2}{dt} = -\left(\hat{\alpha} + \Gamma_{22}\gamma_2 - \mu_2\theta_2 p\right)E_1(u+t,t) + \\
\gamma_1(a,t-\tau)\int_0^A \beta(a,a',t-\tau)I(a',t-\tau)da'S_2(u+t,t) + \Gamma_{12}E_1(u+t,t) \\
\frac{dI_2}{dt} = -\left(\hat{\alpha} + \gamma_3 + \alpha_{ad} + \Gamma_{13}\right)I_2(u+t,t) + \gamma_2 E_2(u+t,t) + \Gamma_{13}I_1(u+t,t) \\
\frac{dR_2}{dt} = -\left(\hat{\alpha} + \Gamma_{24} + \rho\right)R_2(u+t,t) + \gamma_3 I_2(u+t,t) + eV_2(u+t,t) + \Gamma_{14}R_1(u+t,t) \\
\frac{dV_2}{dt} = -\left(\hat{\alpha} + \Gamma_{25} + e + v\right)V_2(u+t,t) + \delta S_2(a,t) + \Gamma_{15}V_1(u+t,t)
\end{cases}
\tag{3}
$$

Theorem 1. *Let $G : \mathbb{R}^{10} \longrightarrow \mathbb{R}^{10}$ be a continuously differentiable function, given by*

$$G = \begin{pmatrix} Y_1(X) \\ Y_2(X) \\ Y_3(X) \\ . \\ . \\ . \\ Y_9(X) \\ Y_{10}(X) \end{pmatrix},$$

$$X = (S_1, E_1, I_1, R_1, V_1, S_2, E_2, I_2, R_2, V_2)^t.$$

If there exists a constant $K > 0$ such that for all indices $i \in \{1, ..., 10\}$ and every $X \in \mathbb{R}^{10}$, $\left|\frac{\partial Y_i}{\partial X}(X)\right| \leq K$, then for any initial condition $X_0 \in \mathbb{R}^{10}$, the Cauchy problem:

$$\begin{cases} X'(t) = G(X(t)), \quad t \in [0, \infty[, \\ X(0) = X_0, \end{cases}$$

admits a unique solution defined on $[0, \infty[$.

Theorem 2. *Consider system (1) with positive parameters. If the initial conditions satisfy $S_1(0) > 0, E_1(0) > 0, I_1(0) > 0, R_1(0) > 0, V_1(0) > 0, S_2(0) > 0, E_2(0) > 0, I_2(0) > 0, R_2(0) > 0, V_2(0) > 0$, then the densities $\Big(S_1(t), E_1(t), I_1(t), R_1(t), V_1(t), S_2(t), E_2(t), I_2(t), R_2(t), V_2(t)\Big)$ remaind positive for all $t > 0$.*

Proof. The proposed method takes advantage of the system's cooperativity. Consider for example the case $S_1 = 0$, the derivative then becomes
$\mu_2\hat{\theta}_2(1 - p)(a, t)E_1(a, t) + \mu_3\hat{\theta}_3 I_1 + (\mu_4\hat{\theta}_4 + \rho)R_1(a, t) + (\mu_5\theta_5 + v)V_1(a, t) + \Gamma_{21}S_2(a, t) > 0$.
According to the system (3), we have: $\frac{dS_1}{dt} > 0$, we use the same approach for others variables. Since for every variable $X = 0$ we have

$$\frac{dX}{dt} > 0$$

and the initial conditions are positive, it follows that the solution remains positive for all $t > 0$.
Hence, the positivity of the solution to problem (1) is established.

4. Stability equilibrium points

To study the stability of the equilibrium points, the delayed integral $\int_0^A \beta(a, a', t-\tau) I_1(a', t-\tau) da'$ is approximated by a linear algebraic expression $\beta N_i(t-\tau)$. Thus, we obtained a new system of ordinary differential equations governing the population dynamics given by (system 4) [5, 21, 19, 6]

$$
\begin{cases}
\frac{dS_1}{dt} = -(\hat{\alpha} + \Gamma_{11} - \mu_1\hat{\theta}_1 + \delta)S_1 - \gamma_1\beta I_1 S_1 + \mu_2\hat{\theta}_2(1-p)E_1 \\
+\mu_3\hat{\theta}_3 I_1 + (\mu_4\hat{\theta}_4 + \rho)R_1 + (\mu_5\theta_5 + v)V_1 + \Gamma_{21}S_2 \\
\frac{dE_1}{dt} = -\left(\hat{\alpha} + \Gamma_{12} + \gamma_2 - \mu_2\theta_2 p\right)E_1 + \gamma_1\beta I_1 S_1 + \Gamma_{22}E_2 \\
\frac{dI_1}{dt} = -\left(\hat{\alpha} + \gamma_3 + \alpha_{ad} + \Gamma_{13}\right)I_1 + \gamma_2 E_1 + \Gamma_{23}I_2 \\
\frac{dR_1}{dt} = -\left(\hat{\alpha} + \Gamma_{14} + \rho\right)R_1 + \gamma_3 I_1 + eV_1 + \Gamma_{24}R_2 \\
\frac{dV_1}{dt} = -\left(\hat{\alpha} + \Gamma_{15} + e + v\right)V_1 + \delta S_1 + \Gamma_{25}V_1 \\
\frac{dS_2}{dt} = -(\hat{\alpha} + \Gamma_{21} - \mu_1\hat{\theta}_1 + \delta)S_2 - \gamma_1\beta I_2 S_2 + \mu_2\hat{\theta}_2(1-p)E_2 + \\
\mu_3\hat{\theta}_3(1-q)I_2 + (\mu_4\hat{\theta}_4 + \rho)R_2 + (\mu_5\theta_5 + v)V_2 + \Gamma_{11}S_1 \\
\frac{dE_2}{dt} = -\left(\hat{\alpha} + \Gamma_{22}\gamma_2 - \mu_2\theta_2 p\right)E_1 + \gamma_1\beta I_2 S_2 + \Gamma_{12}E_1 \\
\frac{dI_2}{dt} = -\left(\hat{\alpha} + \gamma_3 + \alpha_{ad} + \Gamma_{13}\right)I_2 + \gamma_2 E_2 + \Gamma_{13}I_1 \\
\frac{dR_2}{dt} = -\left(\hat{\alpha} + \Gamma_{24} + \rho\right)R_2 + \gamma_3 I_2 + eV_2 + \Gamma_{14}R_1 \\
\frac{dV_2}{dt} = -\left(\hat{\alpha} + \Gamma_{25} + e + v\right)V_2 + \delta S_2 + \Gamma_{15}V_2
\end{cases}
\tag{4}
$$

4.1. Stability of disease free equilibrium point

4.1.1. Determination of the disease-free equilibrium point

At the disease-free equilibrium point we have: $E_1 = E_2 = I_1 = I_2 = R_1 = R_2 = 0$ and the derivatives are zero. Our system (4) becomes system (5):

$$\begin{cases} -(\hat{\alpha} + \Gamma_{11} - \mu_1\hat{\theta}_1 + \delta)S_1 + (\mu_5\theta_5 + v)V_1 + \Gamma_{21}S_2 = 0 \\ -(\hat{\alpha} + \Gamma_{15} + e + v)V_1 + \delta S_1 + \Gamma_{25}V_2 = 0 \\ -(\hat{\alpha} + \Gamma_{21} - \mu_1\hat{\theta}_1 + \delta)S_2 + \Gamma_{11}S_1 + (\mu_5\theta_5 + v)V_2 = 0 \\ -(\hat{\alpha} + \Gamma_{25} + e + v)V_2 + \delta S_2 + \Gamma_{15}V_1 = 0 \end{cases} \tag{5}$$

$S_1 = \dfrac{(\mu_5\theta_5)V_1 + \Gamma_{21}S_2}{A_{s1}}$; $V_1 = \dfrac{\delta S_1 + \Gamma_{25}V_2}{A_{v1}}$; $S_2 = \dfrac{\Gamma_{11}S_1 + (\mu_5\theta_5)V_2}{A_{s2}}$;
$V_2 = \dfrac{\delta S_2 + \Gamma_{15}V_1}{A_{s2}}$

By substitution, we obtain:

$$S_1 = \frac{\mu_5\theta_5\delta\Gamma_{25} + \Gamma_{21}A_{v2}A_{v1} - \Gamma_{21}\Gamma_{15}\Gamma_{25}}{A_{v2}(A_{s1}A_{v1} - \delta\mu_5\theta_5) - \Gamma_{15}\Gamma_{25}A_{s1}}S_2$$

$$V_1 = \frac{\delta\Gamma_{21}A_{v2} + \delta\Gamma_{25}A_{s1}}{A_{v2}(A_{s1}A_{v1} - \delta\mu_5\theta_5) - \Gamma_{15}\Gamma_{25}A_{s1}}S_2$$

$$V_2 = \frac{\delta(A_{s1}A_{v1} - \delta\mu_5\theta_5 + \Gamma_{15}\Gamma_{25})}{A_{v2}(A_{s1}A_{v1} - \delta\mu_5\theta_5) - \Gamma_{15}\Gamma_{25}A_{s1}}S_2$$

After normalising [4, 3] $s_1 + s_2 + s_1 + s_2 = 1$, we therefore obtain

$s_2 = \frac{A_{v2}(A_{s1}A_{v1} - \delta\mu_5\theta_5) - \Gamma_{15}\Gamma_{25}A_{s1}}{\mu_5\theta_5\delta\Gamma_{25} + \Gamma_{21}A_{v2}A_{v1} - \Gamma_{21}\Gamma_{15}\Gamma_{25} + \delta\Gamma_{21}A_{v2} + \delta\Gamma_{25}A_{s1} + \delta(A_{s1}A_{v1} - \delta\mu_5\theta_5 + \Gamma_{15}\Gamma_{25}) + A_{v2}(A_{s1}A_{v1} - \delta\mu_5\theta_5) - \Gamma_{15}\Gamma_{25}A_{s1}}$

$s_1 = \frac{\mu_5\theta_5\delta\Gamma_{25} + \Gamma_{21}A_{v2}A_{v1} - \Gamma_{21}\Gamma_{15}\Gamma_{25}}{\mu_5\theta_5\delta\Gamma_{25} + \Gamma_{21}A_{v2}A_{v1} - \Gamma_{21}\Gamma_{15}\Gamma_{25} + \delta\Gamma_{21}A_{v2} + \delta\Gamma_{25}A_{s1} + \delta(A_{s1}A_{v1} - \delta\mu_5\theta_5 + \Gamma_{15}\Gamma_{25}) + A_{v2}(A_{s1}A_{v1} - \delta\mu_5\theta_5) - \Gamma_{15}\Gamma_{25}A_{s1}}$

$v_1 = \frac{\delta\Gamma_{21}A_{v2} + \delta\Gamma_{25}A_{s1}}{\mu_5\theta_5\delta\Gamma_{25} + \Gamma_{21}A_{v2}A_{v1} - \Gamma_{21}\Gamma_{15}\Gamma_{25} + \delta\Gamma_{21}A_{v2} + \delta\Gamma_{25}A_{s1} + \delta(A_{s1}A_{v1} - \delta\mu_5\theta_5 + \Gamma_{15}\Gamma_{25}) + A_{v2}(A_{s1}A_{v1} - \delta\mu_5\theta_5) - \Gamma_{15}\Gamma_{25}A_{s1}}$

$v_2 = \frac{\delta(A_{s1}A_{v1} - \delta\mu_5\theta_5 + \Gamma_{15}\Gamma_{25})}{\mu_5\theta_5\delta\Gamma_{25} + \Gamma_{21}A_{v2}A_{v1} - \Gamma_{21}\Gamma_{15}\Gamma_{25} + \delta\Gamma_{21}A_{v2} + \delta\Gamma_{25}A_{s1} + \delta(A_{s1}A_{v1} - \delta\mu_5\theta_5 + \Gamma_{15}\Gamma_{25}) + A_{v2}(A_{s1}A_{v1} - \delta\mu_5\theta_5) - \Gamma_{15}\Gamma_{25}A_{s1}}$.

Next, we determine the reproduction number (R_0) by using the method of the next generation matrix [15].

4.1.2. Determination of R_0 and disease free equilibrium stability

At equilibrium point, the derivatives are zero, our system (4) becomes (6):

$$\begin{cases} -(\hat{\alpha}+\Gamma_{11}-\mu_1\hat{\theta}_1+\delta)S_1-\gamma_1\beta I_1S_1+\mu_2\hat{\theta}_2(1-p)E_1 \\ +\mu_3\hat{\theta}_3I_1+(\mu_4\hat{\theta}_4+\rho)R_1+(\mu_5\theta_5+v)V_1+\Gamma_{21}S_2=0 \\ -\left(\hat{\alpha}+\Gamma_{12}+\gamma_2-\mu_2\theta_2p\right)E_1+\gamma_1\beta I_1S_1+\Gamma_{22}E_2=0 \\ -\left(\hat{\alpha}+\gamma_3+\alpha_{ad}+\Gamma_{13}\right)I_1+\gamma_2E_1+\Gamma_{23}I_2=0 \\ -\left(\hat{\alpha}+\Gamma_{14}+\rho\right)R_1+\gamma_3I_1+eV_1+\Gamma_{24}R_2=0 \\ -\left(\hat{\alpha}+\Gamma_{15}+e+v\right)V_1+\delta S_1+\Gamma_{25}V_1=0 \\ -(\hat{\alpha}+\Gamma_{21}-\mu_1\hat{\theta}_1+\delta)S_2-\gamma_1\beta I_2S_2+\mu_2\hat{\theta}_2(1-p)E_2+ \\ \mu_3\hat{\theta}_3I_2+(\mu_4\hat{\theta}_4+\rho)R_2+(\mu_5\theta_5+v)V_2+\Gamma_{11}S_1=0 \\ -\left(\hat{\alpha}+\Gamma_{22}\gamma_2-\mu_2\theta_2p\right)E_2+\gamma_1\beta I_2S_2+\Gamma_{12}E_1=0 \\ -\left(\hat{\alpha}+\gamma_3+\alpha_{ad}+\Gamma_{13}\right)I_2+\gamma_2E_2+\Gamma_{13}I_1=0 \\ -\left(\hat{\alpha}+\Gamma_{24}+\rho\right)R_2+\gamma_3I_2+eV_2+\Gamma_{14}R_1=0 \\ -\left(\hat{\alpha}+\Gamma_{25}+e+v\right)V_2+\delta S_2+\Gamma_{15}V_2=0 \end{cases} \tag{6}$$

To determine the R_0, we consider the infected states. Then, our system (6) becomes system (7):

$$\begin{cases} -\left(\hat{\alpha}+\Gamma_{12}+\gamma_2-\mu_2\theta_2p\right)E_1+\gamma_1\beta I_1S_1+\Gamma_{22}E_2=0 \\ -\left(\hat{\alpha}+\gamma_3+\alpha_{ad}+\Gamma_{13}\right)+\gamma_2E_1+\Gamma_{23}I_2=0 \\ -\left(\hat{\alpha}+\Gamma_{22}+\gamma_2-\mu_2\theta_2p\right)E_2+\gamma_1\beta I_1S_2+\Gamma_{12}E_1=0 \\ -\left(\hat{\alpha}+\gamma_3+\alpha_{ad}+\Gamma_{13}\right)+\gamma_2E_2+\Gamma_{13}I_1=0 \end{cases} \tag{7}$$

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \begin{pmatrix} -\left(\hat{\alpha}+\Gamma_{12}+\gamma_2-\mu_2\theta_2 p\right)E_1+\Gamma_{22}E_2 \\ -\left(\hat{\alpha}+\gamma_3+\alpha_{add}+\Gamma_{13}+\gamma_2\right)E_1+\Gamma_{23}I_2 \\ -\left(\hat{\alpha}+\Gamma_{22}+\gamma_2-\mu_2\theta_2 p\right)E_2+\Gamma_{12}E_1 \\ -\left(\hat{\alpha}+\gamma_3+\alpha_{ad}+\Gamma_{23}+\gamma_2\right)E_2+\Gamma_{13}I_1 \end{pmatrix} + \begin{pmatrix}\gamma_1\beta I_1 S_1\\0\\\gamma_1\beta I_2 S_2\\0\end{pmatrix}$$

[15], [21], [10], [6] let F and T such that:

$F = \begin{pmatrix} 0 & \gamma_1\beta s_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1\beta s_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ corresponds to the matrix of transitions and

$T = \begin{pmatrix} -(A) & 0 & \Gamma_{22} & 0 \\ \gamma_2 & -(B) & 0 & \Gamma_{23} \\ \Gamma_{12} & 0 & -(C) & 0 \\ 0 & \Gamma_{13} & \gamma_2 & -(D) \end{pmatrix}$ corresponds to transmissions.

With $A = \hat{\alpha}+\Gamma_{12}+\gamma_2-\mu_2\theta_2 p$; $B = \hat{\alpha}+\gamma_3+\alpha_{ad}+\Gamma_{13}$

$C = \hat{\alpha}+\Gamma_{22}+\gamma_2-\mu_2\theta_2 p$; $D = \hat{\alpha}+\gamma_3+\alpha_{ad}+\Gamma_{23}$

$$T^{-1} = \begin{pmatrix} \frac{-C}{AC-\Gamma_{12}\Gamma_{22}} & 0 & \frac{-\Gamma_{22}}{AC-\Gamma_{12}\Gamma_{22}} & 0 \\ \frac{-\gamma_2(CD+\Gamma_{12}\Gamma_{23})}{(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}\Gamma_{13})} & \frac{-D}{BD-\Gamma_{13}\Gamma_{23}} & \frac{-\gamma_2(A\Gamma_{23}+D\Gamma_{22})}{(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}\Gamma_{13})} & \frac{-\Gamma_{23}}{BD-\Gamma_{23}\Gamma_{13}} \\ \frac{-\Gamma_{12}}{AC-\Gamma_{12}\Gamma_{22}} & 0 & \frac{-A}{AC-\Gamma_{12}\Gamma_{22}} & 0 \\ \frac{-\gamma_2(C\Gamma_{13}+B\Gamma_{12})}{(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}\Gamma_{13})} & \frac{-\Gamma_{13}}{BD-\Gamma_{23}\Gamma_{13}} & \frac{-\gamma_2(AB+\Gamma_{22}\Gamma_{13})}{(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}\Gamma_{13}} & \frac{-B}{BD-\Gamma_{23}\Gamma_{13}} \end{pmatrix}$$

-$FT^{-1}-\lambda =$

$$\begin{pmatrix} \frac{\gamma_1\beta\gamma_2 S_1(CD+\Gamma_{12}\Gamma_{23})}{(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}^2)}-\lambda & \frac{\gamma_1\beta DS_1}{BD-\Gamma_{23}^2} & \frac{\gamma_1\beta\gamma_2 S_1(A\Gamma_{23}+D\Gamma_{22}}{(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}\Gamma_{13})} & \frac{\gamma_1\beta S_1\Gamma_{23}}{BD-\Gamma_{23}\Gamma_{23}} \\ 0 & -\lambda & 0 & 0 \\ \frac{\gamma_1\beta\gamma_2 S_2(C\Gamma_{23}+B\Gamma_{12})}{(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}\Gamma_{13})} & \frac{\gamma_1\beta S_2\Gamma_{23}}{BD-\Gamma_{23}\Gamma_{13}} & \frac{\gamma_1\beta\gamma_2 S_2(AB+\Gamma_{22}\Gamma_{23})}{(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}\Gamma_{13})}-\lambda & \frac{\gamma_1\beta S_2 B}{BD-\Gamma_{23}\Gamma_{13}} \\ 0 & 0 & 0 & -\lambda \end{pmatrix}$$

$$det(-FT^{-1}-\lambda I)=\lambda^2\Big[\Big(\frac{\gamma_1\beta\gamma_2 S_1(CD+\Gamma_{12}\Gamma_{23})}{(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}\Gamma_{13})}-\lambda\Big)\Big(\frac{\gamma_1\beta\gamma_2 S_2(AB+\Gamma_{22}\Gamma_{23})}{(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}\Gamma_{13})}-\lambda\Big)-$$
$$\frac{\gamma_1^2\beta^2\gamma_2^2 S_1S_2(C\Gamma_{23}+B\Gamma_{12})}{(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}\Gamma_{13})}\times\frac{(A\Gamma_{23}+D\Gamma_{22})}{(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}\Gamma_{13})}\Big]$$

$$=\lambda^2\Big(\lambda^2-\frac{\gamma_1\beta\gamma_2(S_1(CD+\Gamma_{12}\Gamma_{23})+S_2(AB+\Gamma_{22}\Gamma_{23}))}{(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}\Gamma_{13})}\lambda+$$
$$\frac{\gamma_1^2\beta^2\gamma_2^2 S_1S_2(CD+\Gamma_{12}\Gamma_{22})(AB+\Gamma_{22}\Gamma_{23})}{(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}\Gamma_{13})}-\frac{\gamma_1^2\beta^2\gamma_2^2 S_1S_2(C\Gamma_{23}+B\Gamma_{12})}{(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}\Gamma_{13})}\times$$
$$\frac{(A\Gamma_{23}+D\Gamma_{22})}{2(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}\Gamma_{13})}\Big)$$

$$R_0=max(\lambda_i)=$$
$$=\frac{\gamma_1\beta\gamma_2}{2(AC-\Gamma_{12}\Gamma_{22})(BD-\Gamma_{23}\Gamma_{13})}\times\Big(s_1(CD+\Gamma_{12}\Gamma_{23})+s_2(AB+\Gamma_{22}\Gamma_{23})+$$
$$\sqrt{(s_1(CD+\Gamma_{12}\Gamma_{23}))^2-2s_1(CD+\Gamma_{12}\Gamma_{23})s_2(\Gamma_{23}\Gamma_{22}+AB)+(s_2(AB+\Gamma_{22}\Gamma_{23}))^2+4s_1s_2(C\Gamma_{23}+B\Gamma_{12})(A\Gamma_{23}+D\Gamma_{22})}\Big)$$

Theorem 3. *The disease-free equilibrium point DFE is asymptotically stable if* $\hat{\alpha}+\gamma_2-\mu_2\theta_2 p>0$ *and* $R_0<1$.

Proof. Consider the Lyapunov function $L=E_1(t)+I_1(t)+E_2(t)+I_2(t)$
$L>0$, $L(0,0,0,0)=0$ and

$$\frac{dL}{dt}=\frac{dE_1}{dt}+\frac{dI_1}{dt}+\frac{dE_2}{dt}+\frac{dI_2}{dt}=$$

$$-AE_1+\Gamma_{22}E_2-CE_2+\Gamma_{21}E_1-BI_1+\gamma_2E_1+\Gamma_{23}I_2-DI_2+\gamma_2E_2+\Gamma_{13}I_1$$

$$=-(\hat{\alpha}+\gamma_2-\mu_2\theta_2 p)(E_1+E_2)-(\hat{\alpha}+\gamma_3+\alpha_{ad})(I_1+I_2)$$

if $\hat{\alpha}+\gamma_2-\mu_2\theta_2 p>0$, then $\frac{dL}{dt}<0$.
The conditions of Lyapunov's theorem are verified, hence the disease-free equilibrium point is asymptotically stable under the conditions of the theorem (3)

Also by numerical simulation using the Python program, we find that all the eigenvalues have negative real parts, we conclude that the disease-free equilibrium point is asymptotically stable for all $\tau\geq 0$. With Python program, we obtain

the results in the following diagram (3):
We give the parameter values as follows [21]: $\hat{\alpha} = 0.13$, $\alpha_{add} = 0.05$, $\mu_1 = 0.6$, $\mu_2 = 0.5$, $\mu_3 = 0.5$, $\mu_4 = 0.3$, $\mu_5 = 0.5$, $\theta_1 = 0.4$, $\theta_2 = 0.1$, $\theta_3 = 0.1$, $\theta_4 = 0.2$, $\theta_5 = 0.4$, $\gamma_1 = 0.1$, $\gamma_2 = 0.16$, $\gamma_3 = 0.1$, $v = 0.01$, $\beta = 0.0023$, $p = 0.1$, $\rho = 0.02$, $\Gamma_{11} = 0.12$, $\Gamma_{21} = 0.12$, $\Gamma_{12} = 0.11$, $\Gamma_{22} = 0.11$, $\Gamma_{13} = 0.04$, $\Gamma_{23} = 0.04$, $\Gamma_{14} = 0.07$, $\Gamma_{24} = 0.07$, $\Gamma_{15} = 0.1$, $\Gamma_{25} = 0.1$

Real parts of the eigenvalues

-1.44329026	-1.24329026	-0.72934704	-0.02441337	-0.02441337
-0.59153596	-0.52934704	-0.39153596	-0.22441337	-0.22441337

We note that all the eigenvalues are in negative real parts, hence according to Hartman Grobman's theorem the disease-free equilibrium point is stable for all $\tau \geqslant 0$.

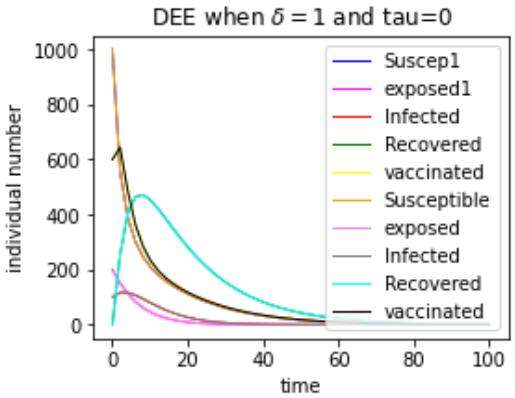

Figure 2: **Asymptotic stability of the Disease-free Equilibrium point**

The curves become horizontal as time passes, which further proves the stablity of the disease-free equilibrium.

4.2. Determination of the endemic equilibrium point and stability

We consider the following system:

$$\begin{cases} -(\hat{\alpha}+\Gamma_{11}-\mu_1\hat{\theta}_1+\delta)S_1^*-\gamma_1\beta I_1^*S_1^*+\mu_2\hat{\theta}_2(1-p)E_1^* \\ +\mu_3\hat{\theta}_3I_1^*+(\mu_4\hat{\theta}_4+\rho)R_1^*+(\mu_5\theta_5+v)V_1^*+\Gamma_{21}S_2^*=0 \\ -(\hat{\alpha}+\Gamma_{12}+\gamma_2-\mu_2\theta_2p)\,E_1^*+\gamma_1\beta I_1^*S_1^*+\Gamma_{22}E_2^*=0 \\ -(\hat{\alpha}+\gamma_3+\alpha_{ad}+\Gamma_{13})\,I_1^*+\gamma_2E_1^*+\Gamma_{23}I_2^*=0 \\ -(\hat{\alpha}+\Gamma_{14}+\rho)\,R_1^*+\gamma_3I_1^*+eV_1^*+\Gamma_{24}R_2^*=0 \\ -(\hat{\alpha}+\Gamma_{15}+v+e)\,V_1^*+\delta S_1^*+\Gamma_{25}V_2^*=0 \\ -(\hat{\alpha}+\Gamma_{21}-\mu_1\hat{\theta}_1+\delta)S_2^*-\gamma_1\beta I_2^*S_2^*+\mu_2\hat{\theta}_2(1-p)E_2^*+ \\ \mu_3\hat{\theta}_3(1-q)I_2^*+(\mu_4\hat{\theta}_4+\rho)R_2^*+(\mu_5\theta_5+v)V_2^*+\Gamma_{11}S_1^*=0 \\ -(\hat{\alpha}+\Gamma_{22}+\gamma_2-\mu_2\theta_2p)\,E_2^*+\gamma_1\beta I_2^*S_2^*+\Gamma_{21}E_1^*=0 \\ -(\hat{\alpha}+\gamma_3+\alpha_{ad}+\Gamma_{13})\,I_2^*+\gamma_2E_2^*+\Gamma_{23}I_1^*=0 \\ -(\hat{\alpha}+\Gamma_{24}+\rho)\,R_2^*+\gamma_3I_2^*+eV_2^*+\Gamma_{14}R_1^*=0 \\ -(\hat{\alpha}+\Gamma_{25}+v+e)\,V_2^*+\delta S_2^*+\Gamma_{15}V_1^*=0 \end{cases} \tag{8}$$

By substitution, we obtain: $V_1^* = \dfrac{-(\delta A_{v2}S_1^*+\delta\Gamma_{25}S_2^*)}{\Gamma_{15}\Gamma_{25}-A_{v2}A_{v1}}$; $V_2^* = \dfrac{-(\delta A_{v1}S_2^*+\delta\Gamma_{15}S_1^*)}{\Gamma_{15}\Gamma_{25}-A_{v2}A_{v1}}$

$R_1^* = \dfrac{-A_{r2}\gamma_3I_1^*-A_{r2}(1-v)V_1^*-\Gamma_{24}\gamma_3I_2^*-\Gamma_{24}(1-v)V_2^*}{\Gamma_{14}\Gamma_{24}-A_{r2}A_{r1}}$; $R_2^* = \dfrac{\gamma_3I_2^*+(1-v)V_2^*+\Gamma_{14}R_1^*}{A_{v2}}$

$E_2^* = \dfrac{\gamma_1\beta(I_1^*S_1^*+I_2^*S_2^*)}{CA-\Gamma_{21}\Gamma_{22}}$; $E_1^* = \dfrac{\gamma_1\beta I_1^*S_1^*+\Gamma_{22}E_2^*}{A}$; $I_2^* = \dfrac{\gamma_2(\Gamma_{23}E_1^*+BE_2^*)}{BD-\Gamma_{23}^2}$;

$I_1^* = \dfrac{\gamma_2E_1^*+\Gamma_{23}I_2^*}{B}$; $S_1^* = \dfrac{A_{v1}V_1^*-\Gamma_{25}V_2^*}{\delta}$; $S_2^* = \dfrac{A_{v2}V_2^*-\Gamma_{15}V_1^*}{\delta}$

Assumption

Suppose that the number of the susceptible population moving from patch 1

to patch 2, and patch 2 to patch 1 are equals, this means: $\Gamma_{11}S_1^* = \Gamma_{21}S_2^* \Rightarrow S_2^* = \frac{\Gamma_{11}S_1^*}{\Gamma_{21}}$. Otherwise there exists a real $k > 0$ such that $S_2^* = k\frac{\Gamma_{11}S_1^*}{\gamma_{21}}$.
At the endemic equilibrium point, we have two cases:
1st case $\tau = 0$:
For τ= 0 and $\delta = 0$, we have the eigenvalues and the following figure (3): **Real parts of the eigenvalues when $\tau = 0$ and $\delta = 0$**

-0.66163869	0.01264207	0.01264207	-0.44512919	-0.33834645
-0.18260655	-0.18260655	-0.1419567	-1.33	-1.13

We note that all the eigenvalues do not have negative real parts, hence the Endemic Equilibrium point is not stable when $\tau = 0$ and δ=0.0.

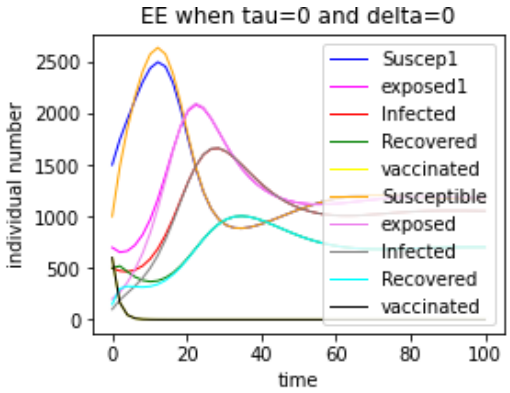

Figure 3: **late stability of Endemic Equilibrium point when $\tau = 0$ and δ=0.0**

The graph indicates that the curves oscillate and don't converge to a stable equilibrium point. This oscillatory behavior shows that the system does not settle into a stable fixe point. We can conclude that the Endemic equilibrium point is not stable when $\tau = 0$ and $\delta = 0$ (see figure (3)).
For $\tau = 0$ and $\delta = 0.2$, we have figure (4). **Real parts of the eigenvalues when $\tau = 0$ and $\delta = 0.2$**

-1.37241293	-1.17245135	-0.67397177	-0.44512919	-0.32171797
-0.26792452	-0.06572346	-0.07608494	-0.32171797	-0.11543897

we notice that the endemic equilibrium point EE is asymptotically stable for $\tau = 0$ and $\delta = 0.2$ because all the eigenvalues have negative real parts. According to figure (4), we notice that the graph also shows a stationary equilibrium (the curves become horizontal). There is a gradual disappearance of the disease, hence the stability of the Endemic Equilibrium point (EE) when $\tau = 0$ and δ =0.2.
when $\tau = 0$ and $\delta = 0.6$, we have figure (5).

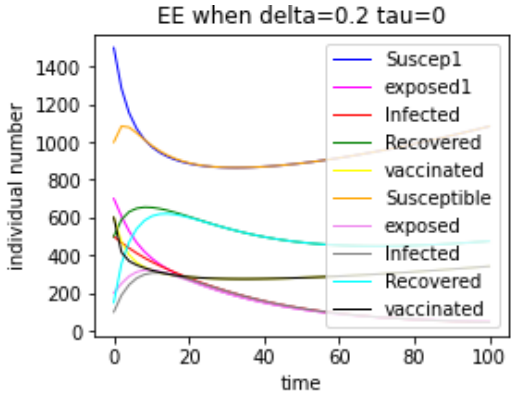

Figure 4: **Asymptotic stability of the Endemic Equilibrium point $\tau = 0$ and δ =0.2**

Real parts of the eigenvalues when $\tau = 0$ and $\delta = 0.6$

-1.50499731	-1.30389694	-0.73726301	-0.00620616	-0.5119286
-0.51192862	-0.14285271	-0.20750353	-0.32123097	-0.33919212

We note that all the eigenvalues have negative real parts, hence the Endemic Equilibrium point is stable when $\tau = 0$ and $\delta = 0.6$.

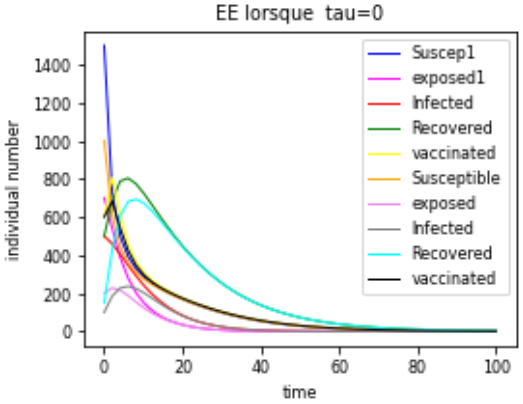

Figure 5: **Asymptotic stability of the Endemic Equilibrium point when $\tau = 0$ and δ=0.6**

According to figure (5), when $\tau = 0$ and $\delta = 0.6$, we notice a stability of the endemic equilibrium point. Thus the three diagrams (figures (3), (4) and (5)) show that for diseases whose latency time is neglected, the stability of the equilibrium point strongly depends on the vaccination rate.
2nd case $\tau > 0$, figure (6), figure (7) and figure (8). At this level the simulations show that stability depends on the vaccination rate.
Figure (6) shows a Hopf bifurcation which disrupts the stability.

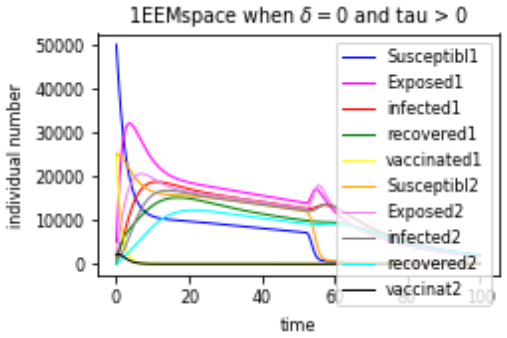

Figure 6: **A Hopf bifurcation when $\tau > 0$ and δ=0.**

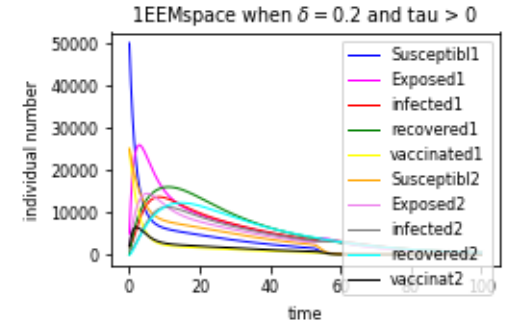

Figure 7: **Asymptotic stability of the Endemic Equilibrium point when $\tau > 0$ and δ=0.2**

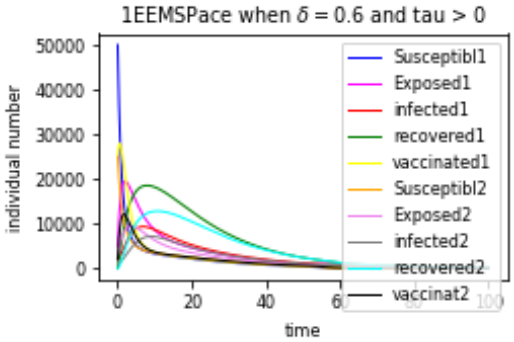

Figure 8: **Asymptotic stability of the Endemic Equilibrium point when $\tau > 0$ and δ=0.6**

According to the three graphs (figures (6), (7) and (8)) we see that vaccination accelerates Asymptotic Stability.

5. Conclusion

In this article, we have studied an age-structured model with delay applied to tuberculosis, with the aim of evaluating the impact of vaccination in relation to the latency period of the disease and population mobility between two patches [18], [8]. We first established existence, uniqueness and positivity of solution. Then basic reproduction number R_0 was computed, and the stability of the equilibrium points was analyzed. For the disease-free equilibrium point, we constructed a Lyapunov function and proved, both analytically and numerically, its asymptotic stability. The stability of the endemic equilibrium point was investigated numerically. Our results show that population mobility between two patches significantly influences the dynamics of tuberculosis transmission. In particular, while the endemic equilibrium is stable in a single patch when $\tau = 0$ (see [5]), in the two-patch setting, stability is achieved only in the presence of vaccination. This study highlights that uncontrolled migration can substantially increase the risk of tuberculosis spread. Therefore, effective control strategies should include not only the restriction of movement for infected individuals but also measures to limit the mobility of susceptible populations in order to reduce transmission.

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Received 08 January 2026
Accepted 04 April 2026