Journal of Contemporary Applied Mathematics
V. 17, No 2, 2027, January
ISSN 2222-5498, E-ISSN 3006-3183
https://doi.org/10.62476/jcam171.23

On the strong solvability of a nonlocal boundary value problem for the Poisson equation in weighted grand Lebesgue spaces in a rectangle

T. Gasymov*, B. Akhmadli, Ö. Bakşi, K. Demir

Abstract. This paper investigates the strong solvability of a nonlocal boundary value problem for the Poisson equation in a rectangular domain $\Pi = (0,\ 2\pi) \times (0, h)$. The study is conducted within the framework of weighted grand Sobolev spaces $W^2_{p),\nu}(\Pi)$, which are generated by the norm of the weighted grand Lebesgue space $L_{p),\nu}(\Pi)$ with a weight belonging to the Muckenhoupt class $A_p(I)$. We introduce the concept of strong solvability for this problem and establish its well-posedness by proving the existence and uniqueness of the solution. A fundamental part of our approach involves the analysis of the corresponding spectral problem. We prove the basis property (completeness, minimality, and basisness) of the system of eigenfunctions and associated functions in the separable subspaces of the weighted grand Lebesgue space. Furthermore, we obtain coercive a-priori estimates for the solution, ensuring the stability of the problem. This work generalizes the classical results obtained in semi-infinite strips by E.I. Moiseev [1], M.E. Lerner, and O.A. Repin [2] to modern functional spaces with non-standard growth and weight conditions.

Key Words and Phrases: Poisson equation, nonlocal problem, weighted grand Lebesgue space, strong solvability

2010 Mathematics Subject Classifications: 35J05, 35J25

1. Introduction

Consider the following (formal for now) nonlocal boundary value problem for Poisson's equation.

$$u_{xx} + u_{yy} = f(x, y), \quad 0 < x < 2\pi, \quad 0 < y < h, \tag{1}$$

*Corresponding author.

 © *2010 jcam* All rights reserved.

$$u(x,0)=\varphi(x),\quad u(x,h)=\psi(x),\quad 0<x<2\pi, \tag{2}$$

$$u_x(0,y)=0,\quad u(0,y)=u(2\pi,y),\quad 0<y<h. \tag{3}$$

Nonlocal boundary value problems possess distinct characteristics compared to problems with classical local conditions. This class of problems has been a subject of extensive research; notably, F.I. Frankl [5], [6, p. 453-456] investigated nonlocal boundary conditions for equations of mixed type. Another fundamental example is the Bitsadze-Samarskii problem [7] for elliptic equations, where nonlocal conditions are supported on specific parts of the boundary, independent of other boundary constraints. Furthermore, N.I. Ionkin and E.I. Moiseev [8] addressed boundary value problems for multidimensional parabolic equations with nonlocal conditions supported by characteristic and improper boundary segments. It is obvious that the solution to problem (1)-(3) can be sought as the sum of the solutions of the following two problems:

$$\left.\begin{array}{c} u_{xx}+u_{yy}=0,\\ u(x,0)=\varphi(x),\quad u(x,h)=\psi(x),\\ u_x(0,y)=0,\quad u(0,y)=u(2\pi,y). \end{array}\right\} \tag{A}$$

and

$$\left.\begin{array}{c} u_{xx}+u_{yy}=f(x,y),\\ u(x,0)=0,\quad u(x,h)=0,\\ u_x(0,y)=0,\quad u(0,y)=u(2\pi,y). \end{array}\right\} \tag{B}$$

The strong solvability of Problem (A) in the domain $\Pi=(0,2\pi)\times(0,h)$ in weighted grand Sobolev spaces was considered in [40]. In this paper, we study the question of the strong solvability of Problem (B) in a weighted grand Sobolev space with a weight from the Muckenhoupt class, which, together with the result of [40], will allow us to formulate a well-posed solvability of Problem (1)-(3) in this space. The concept of a strong solution is defined, and the well-posed solvability of Problem (B) is proved by the spectral method.

We note that the study of the solvability of elliptic equations with respect to the so-called non-standard function spaces, as well as in weighted grand Sobolev spaces, encounters certain difficulties compared to the unweighted case. Therefore, the number of studies devoted to this area has been growing in recent years (see, for example, [3, 4, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]), and the development of the corresponding theory is far from complete. These attempts are also relevant in connection with the question of the basis properties of system of eigenfunctions and associated functions of spectral problems

obtained by applying the method of separation of variables in the corresponding non-standard function spaces. The question of the basis property of the classical exponential system in weighted Lebesgue spaces was studied in [26, 27]. In this regard, we note the works [28, 29, 30, 31, 32, 33, 34], in which similar questions are studied and appropriate research methods are proposed.

2. Auxiliary concepts and facts

2.1. Notations

We employ standard mathematical notations throughout this work. Let N denote the set of positive integers, and $Z^+ = N \cup \{0\}$. For a multi-index $\alpha = (\alpha_1; \alpha_2) \in Z^+ \times Z^+$, we denote the partial derivative of order $|\alpha| = \alpha_1 + \alpha_2$ as:

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x^{\alpha_1} \partial y^{\alpha_2}}.$$

Let $|M|$ represent the Lebesgue measure of a set $M \subset R^2$ and let $\overline{M}$ denote its closure. The space $C^\infty\left(\bar{M}\right)$ consists of infinitely differentiable functions on $\bar{M}$, while $C_0^\infty(M)$ denotes the space of infinitely differentiable functions with compact support in M. Throughout this paper, p and $p^{'}$ are conjugate exponents such that $1 < p < +\infty$ and $\frac{1}{p^{'}} + \frac{1}{p} = 1$. $\mathrm{d}\sigma$ is an area element. We also accept $p_\varepsilon = p - \varepsilon$. In the context of weighted grand spaces the parameter p serves as the main exponent, while the behavior of the norm as the integration exponent approaches from below is critical for defining the grand Lebesgue space $L_{p),\nu}(\Pi)$. We assume that the weight function is a measurable, almost everywhere positive function belonging to the Muckenhoupt class, ensuring the boundedness of fundamental operators in the corresponding Sobolev framework.

2.2. Weighted grand Sobolev space $W^2_{p),\nu}(\Pi)$

Define our weighted grand Sobolev space. Let $\nu : [0; 2\pi] \to [0; +\infty]$ be some weight function, $\Pi = (0,\ 2\pi) \times (0, h)$. For $1 < p < +\infty$, denote by $L_{p),\nu}(\Pi)$ a Banach space of functions on Π equipped with the following mixed norm:

$$\|f\|_{L_{p);\nu}(\Pi)} = \sup_{0<\varepsilon<p-1} \int_0^h \left(\varepsilon \int_0^{2\pi} |f(x,y)|^{p-\varepsilon} \nu(x)\, dx \right)^{\frac{1}{p-\varepsilon}} dy.$$

Denote by $W^2_{p);\nu}(\Pi)$ a grand Sobolev space generated by the norm

$$\|u\|_{W^2_{p),\nu}(\Pi)} = \sum_{|\alpha|\le 2} \|\partial^\alpha u\|_{L_{p),\nu}(\Pi)}.$$

Also denote by $L_{p),\nu}(I)$, where $I=(0,2\pi)$, a grand Lebesgue space generated by the norm

$$\|f\|_{L_{p),\nu}(I)}=\sup_{0<\varepsilon<p-1}\left(\varepsilon\int_I|f(x)|^{p-\varepsilon}\nu(x)\,dx\right)^{\frac{1}{p-\varepsilon}}.$$

These spaces are non-separable; therefore, the method of biorthogonal expansion (essentially the spectral method) is not applicable for studying the solvability of differential equations with respect to these spaces. In this regard, we select the subspace $N_{p),\nu}(\Pi)\subset L_{p),\nu}(\Pi)$ (separable) based on the shift operator T_δ:

$$(T_\delta u)(x,y)=\begin{cases}u(x+\delta,y), & (x+\delta,y)\in\Pi,\\ 0, & (x+\delta,y)\notin\Pi.\end{cases}$$

So let us assume

$$N^2_{p),\nu}(\Pi)=\left\{u\in W^2_{p),\nu}(\Pi):\sum_{|\alpha|\le 2}\|T_\delta(\partial^\alpha u)-\partial^\alpha u\|_{L_{p),\nu}(\Pi)}\to 0 \text{ as } \delta\to 0\right\}$$

and

$$N^{1,0}_{p),\nu}(\Pi)=\left\{f\in W^1_{p),\nu}(\Pi):\left\|T_\delta\left(\frac{\partial f}{\partial x}\right)-\frac{\partial f}{\partial x}\right\|_{L_{p),\nu}(\Pi)}\to 0 \text{ as } \delta\to 0\right\}.$$

2.3. Statement of the problem

For a well-posed statement of the considered boundary value problem, it is first necessary to define the trace operator corresponding to the weighted grand Sobolev space $N^2_{p),\nu}(\Pi)$ on the domain $\Pi=(0,\ 2\pi)\times(0,h)\subset R^2$. The boundary is denoted by $\partial\Pi=I_0\bigcup I_h\bigcup\Gamma_0\bigcup\Gamma_{2\pi}$, where

$$I_0=\{(x;0):0<x<2\pi\},\qquad I_h=\{(x;h):0<x<2\pi\},$$
$$\Gamma_0=\{(0;y):0<y<h\}\qquad \Gamma_{2\pi}=\{(2\pi;y):0<y<h\}.$$

In the framework of grand spaces, for $1<p<+\infty$ the embedding $L_{p),\nu}(I)\subset L_{p-\varepsilon}(I)$ holds for any small $\varepsilon>0$. Under the assumption that the weight satisfies the Muckenhoupt condition, the continuous embedding $N^2_{p),\nu}(\Pi)\subset W^2_1(\Pi)$ is valid. Consequently, for every function $u\in N^2_{p),\nu}(\Pi)$ the traces $u|_{\partial\Pi}$ and $u_x|_{\partial\Pi}$ are well-defined as functions in $L_1(\partial\Pi;d\sigma)$ with respect to the Lebesgue measure on the boundary.

Consider the following nonlocal problem:

$$\Delta u = f(x;y), \qquad (x;y) \in \Pi, \tag{4}$$

$$u|_{I_0} = 0, \quad u|_{I_h} = 0, \quad u|_{\Gamma_0} = u|_{\Gamma_{2\pi}}, \quad u_x|_{\Gamma_0} = 0. \tag{5}$$

where $f(x;y) \in L_{p),\nu}(\Pi)$. By the solution of this problem, we mean a function $u \in N^2_{p);\nu}(\Pi)$, which satisfies Eq. (4) a.e. in Π and whose traces satisfy (5) on the boundary $\partial\Pi$. Taking into account the above, we arrive at the following definition of strong solvability of problem (4)-(5).

Definition 1. *A function $u \in N^2_{p),\nu}(\Pi)$ is called a strong solution of the problem* (4)–(5) *if it satisfies Eq.* (4) *almost everywhere in* Π*, and its traces satisfy the boundary conditions* (5)*.*

2.4. Some facts from basis theory

We present some well-known concepts and facts from basis theory. Let X be a Banach space. A system $\{u_n\}_{n\in N} \subset X$ is called a basis if any element $f \in X$ is uniquely represented as a series

$$f = \sum_{n=1}^{\infty} c_n u_n$$

convergent in the norm of X. A system $\{u_n\}_{n\in N} \subset X$ is called complete in X if $\overline{Sp}\{u_n\} = X$ and minimal in X if $u_n \notin \overline{Sp}\{u_k\}_{k\neq n}$. It is known that each basis of the space X is a complete and minimal system in X, the converse is not true in general. The system $\{x_n\}_{n\in N} \subset X$ is complete in $X \Longleftrightarrow \varphi \in X^*$: $\langle x_n, \varphi\rangle = \varphi(x_n) = 0$, $\forall\ n \in N \Rightarrow\ \ \varphi = 0$. The system $\{u_n\}_{n\in N}$ is minimal in X if and only if there exists a biorthogonal system, i.e., there exists a system $\{\vartheta_n\}_{n\in N} \subset X^*$ such that $\langle u_n, \vartheta_k\rangle = \vartheta_k(u_n) = \delta_{nk}$, where δ_{nk} is the Kronecker symbol.

Basis criterion. *The system $\{u_n\}_{n\in N} \subset X$ is a basis of the space X if and only if the following conditions are satisfied:*

1) *The system $\{u_n\}_{n\in\mathbb{N}}$ is complete and minimal in X;*

2) *The projectors*

$$P_n f = \sum_{k=1}^{n} \langle f, \vartheta_k\rangle u_k,$$

where $\{\vartheta_k\}_{k\in\mathbb{N}}$ is a biorthogonal system, are uniformly bounded.
Applying the Fourier method to solve the problem (4)–(5) leads to the following spectral problem:

$$\left.\begin{aligned} u''(x)+\lambda u(x)=0, \quad x\in I,\\ u(0)=u(2\pi), \quad u'(0)=0. \end{aligned}\right\} \tag{6}$$

The eigenvalues of problem (6) are $\lambda_n = n^2$, $n = 0, 1, 2, \ldots$, and the corresponding eigenfunctions are $u_{2n}(x) = \cos nx$. For each $n \geq 1$, the eigenfunction u_{2n} has an associated function $u_{2n-1}(x) = x \sin nx$. Consider the collection of root functions

$$u_0(x)=1, \quad u_{2n}(x)=\cos nx, \quad u_{2n-1}(x)=x\sin nx, \quad n\in\mathbb{N}, \tag{7}$$

and also set

$$\begin{gathered} \vartheta_0(x)=\frac{2\pi-x}{2\pi^2}, \qquad \vartheta_{2n}(x)=\frac{2\pi-x}{\pi^2}\cos nx,\\ \vartheta_{2n-1}(x)=\frac{1}{\pi^2}\sin nx, \qquad n\in\mathbb{N}. \end{gathered} \tag{8}$$

Note that these systems are biorthogonal, as can be verified directly. To obtain our main result, we use the following theorem (see [40, Theorem 1]).

Theorem 1. *Let $\nu \in A_p(I)$, $1 < p < +\infty$, and $\nu(x) = \nu(2\pi - x)$ for $x \in I$. Then the system (7) forms a basis in $N_{p),\nu}(I)$.*

3. Main results

In this section, we will study the existence and uniqueness of a strong solution to the problem (B) in the sense of Definition 1. For $f(x, y) \in L_{p),\nu}(\Pi)$ denote $F_n(y) = \langle f(\cdot, y), \vartheta_n(\cdot)\rangle$, i.e.,

$$\left.\begin{aligned} F_0(y)&=\frac{1}{2\pi^2}\int_0^{2\pi} f(x,y)(2\pi-x)\,dx,\\ F_{2n}(y)&=\frac{1}{\pi^2}\int_0^{2\pi} f(x,y)(2\pi-x)\cos nx\,dx,\\ F_{2n-1}(y)&=\frac{1}{\pi^2}\int_0^{2\pi} f(x,y)\sin nx\,dx, \quad n\in\mathbb{N}. \end{aligned}\right\} \tag{9}$$

The main result of the work is the following theorem.

Theorem 2. *Let the weight function* $\nu(x)$ belong to the Muckenhoupt class $A_p(I)$, $1 < p < +\infty, \nu(x) = \nu(2\pi - x)$. Suppose that the function $f(x, y)$ belongs to the weighted grand Sobolev space $N^{1,0}_{p),\nu}(\Pi)$ and satisfies the periodic boundary condition $f(0, y) = f(2\pi, y)$. Then the problem (B) has a unique solution in the weighted grand Sobolev space $N^{2}_{p),\nu}(\Pi)$ and the following estimate holds:

$$\|u\|_{N^{2}_{p);\nu}(\Pi)} \leq c \|f\|_{N^{1,0}_{p),\nu}(\Pi)}$$

where $c > 0$ is a constant independent of $f(x, y)$.

Proof. First of all, we note that the uniqueness of the solution follows from the results of [4]. Therefore, we proceed directly to the proof of the existence of a solution. Suppose $u(x, y) \in N^{2}_{p);\nu}(\Pi)$ is a solution of the problem (4)-(5).
Consider $U_n(y) = \langle u(\cdot, y), \vartheta_n(\cdot)\rangle$, i.e.,

$$\left.\begin{aligned} U_0(y) &= \frac{1}{2\pi^2}\int_0^{2\pi} u(x, y)(2\pi - x)\, dx, \\ U_{2n}(y) &= \frac{1}{\pi^2}\int_0^{2\pi} u(x, y)(2\pi - x)\cos nx\, dx, \\ U_{2n-1}(y) &= \frac{1}{\pi^2}\int_0^{2\pi} u(x, y)\sin nx\, dx, \quad n \in \mathbb{N}. \end{aligned}\right\} \tag{10}$$

It is absolutely clear that for a.e. $x \in I$ the relation

$$u(x, y + \delta) - u(x, y) = \int_y^{y+\delta} \frac{\partial u(x,t)}{\partial t}\, dt, \quad \forall\, y \in (0, h). \tag{11}$$

holds. Since $\frac{\partial u}{\partial y} \in L_1(\Pi)$, applying Theorem 1.1.1 of [36, p.13], from (11) we obtain that the functions $U_n(y)$ are twice differentiable and can be differentiated under the integral sign. Since the function $u(x, y)$ satisfies the equation (4), multiplying it by $\sin nx$ (by $(2\pi - x)\cos nx$) and integrating over I, we obtain the following relations for $U_{2n-1}(y)$ (respectively, for $U_{2n}(y)$):

$$\left.\begin{aligned} U_0''(y) &= F_0(y), && y \in (0, h), \\ U_{2n-1}''(y) - n^2 U_{2n-1}(y) &= F_{2n-1}(y), && y \in (0, h), \\ U_{2n}''(y) - n^2 U_{2n}(y) &= -2nU_{2n-1}(y) + F_{2n}(y), && y \in (0, h). \end{aligned}\right\} \tag{12}$$

where the functions $F_k(y)$, k$\in Z_+$ are defined by formulas (9).
Applying the Newton-Leibniz formula and taking into account the first boundary condition in (5), we obtain

$u_\xi(x) := u(x,\xi) = u(x,0) + \int_0^\xi \frac{\partial u(x,y)}{\partial y}\, dy = \int_0^\xi \frac{\partial u(x,y)}{\partial y}\, dy, \quad a.e. \quad x \in I.$
Consequently,

$$|u(x,\xi)| \le \int_0^\xi \left|\frac{\partial u(x,y)}{\partial y}\right| dy, \qquad a.e. \quad x \in I.$$

Hence, it immediately follows that

$$\|u_\xi\|_{L_1(I)} = \int_I |u(x,\xi)|\, dx \le \int_I \int_0^\xi \left|\frac{\partial u(x,y)}{\partial y}\right| dy\, dx. \tag{13}$$

Set $\Pi_\xi := I \times (0,\xi)$ for $0 < \xi < h$. Then $|\Pi_\xi| \to 0$ as $\xi \to +0$. By (13), it follows that

$$u_\xi(\cdot) \to 0, \qquad \xi \to +0. \tag{14}$$

in the norm of the space $L_1(I)$.
Similarly, taking into account the second boundary condition in (5), we have

$$u(x,\xi) = u(x,h) - \int_\xi^h \frac{\partial u(x,y)}{\partial y}\, dy = -\int_\xi^h \frac{\partial u(x,y)}{\partial y}\, dy, \quad a.e. \quad x \in I.$$

Hence,

$$\int_I |u(x,\xi)|\, dx \le \int_I \int_\xi^h \left|\frac{\partial u(x,y)}{\partial y}\right| dy\, dx. \tag{15}$$

As $|\Pi \backslash \Pi_\xi| \to 0$ when $\xi \to h - 0$, it follows from (15) that

$$u_\xi(\cdot) \to 0, \qquad \xi \to h - 0. \tag{16}$$

in the norm of the space $L_1(I)$. On the other hand, it is clear that $U_n(y) \in W_1^2(0,h)$. Hence, it immediately follows that there exist limits:

$$\lim_{y \to +0} U_n(y) = U_n(0), \qquad \lim_{y \to h-0} U_n(y) = U_n(h), \qquad \forall\, n \in \mathbb{Z}_+.$$

By (14) and (16), it follows immediately from the last two relations that

$$U_n(0) = 0, \quad U_n(h) = 0, \quad \forall\, n \in \mathbb{Z}_+, \tag{17}$$

The solution of system (12) for odd indices $2n-1$, $n \in \mathbb{N}$, subject to the boundary conditions (17), is given by

$$\begin{aligned} U_{2n-1}(y) = & -\frac{\sinh n(h-y)}{n \sinh nh} \int_0^y \sinh nt\, F_{2n-1}(t)\, dt \\ & -\frac{\sinh ny}{n \sinh nh} \int_y^h \sinh n(h-t)\, F_{2n-1}(t)\, dt, \end{aligned} \tag{18}$$

and the solution of system (12) for even indices $2n$, $n \in \mathbb{N}$, subject to the boundary conditions (17), is given by

$$U_0(y) = -\frac{y}{h}\int_0^h (h-t)F_0(t)\,dt + \int_0^y (y-t)F_0(t)\,dt. \tag{19}$$

$$\begin{aligned}
U_{2n}(y) = &-\frac{\sinh n(h-y)}{n\sinh nh}\int_0^y F_{2n}(t)\sinh nt dt - \frac{\sinh ny}{n\sinh nh}\int_y^h F_{2n}(t)\sinh n(h-t)dt \\
&- \frac{2\sinh n(h-y)}{n(\sinh nh)^2}\int_0^y \sinh nt\,\sinh n(h-t)\left(\int_0^t F_{2n-1}(\tau)\sinh n\tau\,d\tau\right)dt \\
&- \frac{2\sinh n(h-y)}{n(\sinh nh)^2}\int_0^y (\sinh nt)^2\left(\int_t^h F_{2n-1}(\tau)\sinh n(h-\tau)\,d\tau\right)dt \\
&- \frac{2\sinh ny}{n(\sinh nh)^2}\int_y^h \big(\sinh n(h-t)\big)^2\left(\int_0^t F_{2n-1}(\tau)\sinh n\tau\,d\tau\right)dt \\
&- \frac{2\sinh ny}{n(\sinh nh)^2}\int_y^h \sinh nt\,\sinh n(h-t) \\
&\quad\times\left(\int_t^h F_{2n-1}(\tau)\sinh n(h-\tau)\,d\tau\right)dt.
\end{aligned} \tag{20}$$

Consider the function

$$\begin{aligned}
u(x,y) &= U_0(y) + \sum_{n=1}^{\infty} U_n(y)u_n(x) \\
&= U_0(y) + \sum_{k=1}^{\infty}\left(U_{2k}(y)\cos kx + U_{2k-1}(y)x\sin kx\right), \quad (x,y)\in\Pi.
\end{aligned} \tag{21}$$

where the coefficients $U_0(y)$, $U_{2k}(y)$, and $U_{2k-1}(y)$, $k \in \mathbb{N}$, are defined by (18)–(20). We now show that u belongs to $N^2_{p),\nu}(\Pi)$. Denote by $u_{\alpha_1,\alpha_2}(x,y)$ the sum of the series obtained by the formal differentiation of the series (21), i.e.

$$u_{\alpha_1,\alpha_2}(x,y) = \sum_{n=0}^{\infty} U_n^{(\alpha_2)}(y)\,u_n^{(\alpha_1)}(x), \tag{22}$$

where $\alpha_1, \alpha_2 \in Z^+, \alpha_1+\alpha_2 = 0,1,2$, and

$$u_{0,0}(x,y) = u(x,y), \quad u_n^{(\alpha_1)}(x) = \frac{d^{\alpha_1}u_n(x)}{dx^{\alpha_1}}, \quad U_n^{(\alpha_2)}(y) = \frac{d^{\alpha_2}U_n(y)}{dy^{\alpha_2}}.$$

First, consider the series

$$u_1(x,y)=\sum_{k=1}^{\infty}U_{2k-1}(y)x\sin kx.$$

Differentiating this series formally, term by term, we obtain

$$\begin{aligned}\frac{\partial^2 u_1}{\partial y^2}&=\sum_{k=1}^{\infty}U''_{2k-1}(y)x\sin kx\\&=\sum_{k=1}^{\infty}k^2U_{2k-1}(y)x\sin kx+\sum_{k=1}^{\infty}F_{2k-1}(y)x\sin kx,\end{aligned}\tag{23}$$

$$\frac{\partial u_1}{\partial x}=\sum_{k=1}^{\infty}U_{2k-1}(y)\sin kx+\sum_{k=1}^{\infty}kU_{2k-1}(y)x\cos kx,\tag{24}$$

$$\frac{\partial^2 u_1}{\partial x^2}=2\sum_{k=1}^{\infty}kU_{2k-1}(y)\cos kx-\sum_{k=1}^{\infty}k^2U_{2k-1}(y)x\sin kx.\tag{25}$$

Set

$$w(x,y)=\sum_{k=1}^{\infty}k^2U_{2k-1}(y)x\sin kx.$$

We now show that w belongs to $N_{p),\nu}(\Pi)$. Using (9), integrating by parts, and taking into account that $f(2\pi,y)=f(0,y)$, we obtain

$$\begin{aligned}F_{2k-1}(y)&=\frac{1}{\pi^2}\int_0^{2\pi}f(x,y)\sin kx\,dx\\&=-\frac{f(2\pi,y)-f(0,y)}{\pi^2k}+\frac{1}{\pi^2k}\int_0^{2\pi}f_x(x,y)\cos kx\,dx\\&=\frac{1}{\pi^2k}\int_0^{2\pi}f_x(x,y)\cos kx\,dx=\frac{1}{k}F^1_{2k}(y),\end{aligned}$$

where

$$F^1_{2k}(y)=\frac{1}{\pi^2}\int_0^{2\pi}f_x(x,y)\cos kx\,dx,\qquad k\in\mathbb{N}.$$

It is known that if $\nu\in A_p(I)$, $1<p<+\infty$, then $\exists\ \alpha>1:\ \nu\in L_\alpha(I)$ (see, e.g., [37, p. 395]). Let $\varepsilon\in(0,p-1)$ be an arbitrary number. We have the following continuous embeddings: $L_p(I)\ \subset L_{p-\varepsilon}(I)\subset L_1(I)$. Let us suppose that $\beta_\varepsilon=\frac{p}{p-\varepsilon}$ and $g\in L_{p,\nu}(I)$. Then

$$\frac{1}{\beta'_\varepsilon}=1-\frac{p-\varepsilon}{p}=\frac{\varepsilon}{p}$$

and from Hölder's inequality, we obtain

$$\int_0^{2\pi} |g|^{p-\varepsilon}\nu dx = \int_0^{2\pi} |g|^{p-\varepsilon}\nu^{\frac{1}{\beta_\varepsilon}}\nu^{\frac{1}{\beta'_\varepsilon}} dx \le \left(\int_0^{2\pi} |g|^p \nu dx\right)^{\frac{1}{\beta_\varepsilon}} \left(\int_0^{2\pi} \nu dx\right)^{\frac{1}{\beta'_\varepsilon}}$$

$$\Longrightarrow \left(\varepsilon \int_0^{2\pi} |g|^{p-\varepsilon}\nu dx\right)^{\frac{1}{p-\varepsilon}} \le \left(\int_0^{2\pi} |g|^p \nu dx\right)^{\frac{1}{p}} \left(\int_0^{2\pi} \nu dx\right)^{\frac{\varepsilon}{p-\varepsilon}\bullet\frac{1}{p}} \varepsilon^{\frac{1}{p-\varepsilon}}$$

$$\le c \left(\int_0^{2\pi} |g|^p \nu dx\right)^{\frac{1}{p}},$$

where

$$c = \sup_{0<\varepsilon<p-1} \left(\int_0^{2\pi} \nu(x)\, dx\right)^{\frac{\varepsilon}{p(p-\varepsilon)}} \varepsilon^{\frac{1}{p-\varepsilon}}.$$

Thus, $c \in (0, +\infty)$ is a constant independent of g and ε. This immediately yields

$$\|g\|_{L_{p),\nu}(I)} \le c\|g\|_{L_{p,\nu}(I)}, \qquad \forall\, g \in L_{p,\nu}(I).$$

Applying Hölder's inequality again, we obtain

$$\left(\int_0^{2\pi} |w(x,y)|^p \nu(x)\, dx\right)^{1/p} \le \left(\int_0^{2\pi} \nu(x)^\alpha\, dx\right)^{1/(\alpha p)} \left(\int_0^{2\pi} |w(x,y)|^{p\alpha'}\, dx\right)^{1/(p\alpha')}$$
$$= c_1 \left(\int_0^{2\pi} |w(x,y)|^{p_1}\, dx\right)^{1/p_1},$$

where $c_1 = \|\nu\|^{1/p}_{L_\alpha(I)}$ is independent of w, $p_1 = p\alpha'$, and $\alpha' = \alpha/(\alpha-1)$.

We consider the following two cases.

I. Suppose first that $p \ge 2$. Then $p_1 = p\alpha' > 2$. Applying Hölder's inequality to the function $w(x,y)$, we obtain

$$\|w(\cdot,y)\|_{L_{p),\nu}(I)} \le c \left(\int_0^{2\pi} |w(x,y)|^p \nu(x)\, dx\right)^{1/p}$$
$$\le cc_1 \sum_{k=1}^{\infty} k^2 |U_{2k-1}(y)| \left(\int_0^{2\pi} |u_{2k-1}(x)|^{p_1}\, dx\right)^{1/p_1}$$

$$
\begin{aligned}
&\le c_2 \sum_{k=1}^{\infty} k^2 |U_{2k-1}(y)| \\
&= c_2 \sum_{k=1}^{\infty} \frac{k}{\sinh kh} \bigg| \sinh k(h-y) \int_0^y \sinh kt \, F_{2k-1}(t)\, dt \\
&\qquad\qquad + \sinh ky \int_y^h \sinh k(h-t)\, F_{2k-1}(t)\, dt \bigg| \\
&\le c_2 \sum_{k=1}^{\infty} \frac{k}{\sinh kh} \bigg(\sinh k(h-y) \int_0^y \sinh kt \, |F_{2k-1}(t)|\, dt \\
&\qquad\qquad + \sinh ky \int_y^h \sinh k(h-t)\, |F_{2k-1}(t)|\, dt \bigg).
\end{aligned}
$$

Integrating with respect to $y \in (0,h)$, interchanging the order of integration, and then applying Hölder's inequality with $\beta \in (1,\infty)$, we obtain

$$
\begin{aligned}
\|w\|_{L_{p),\nu}(\Pi)} &\le c_2 \sum_{k=1}^{\infty} \frac{k}{\sinh kh} \bigg[\int_0^h \sinh k(h-y) \bigg(\int_0^y \sinh kt \, |F_{2k-1}(t)|\, dt \bigg) dy \\
&\qquad + \int_0^h \sinh ky \bigg(\int_y^h \sinh k(h-t)\, |F_{2k-1}(t)|\, dt \bigg) dy \bigg] \\
&\le c_3 \sum_{k=1}^{\infty} \int_0^h |F_{2k-1}(t)|\, dt \\
&= c_3 \int_0^h \sum_{k=1}^{\infty} |F_{2k-1}(t)|\, dt \\
&= c_3 \int_0^h \sum_{k=1}^{\infty} \frac{1}{k} |F_{2k}^1(t)|\, dt \\
&\le c_3 \bigg(\sum_{k=1}^{\infty} \frac{1}{k^{\beta'}} \bigg)^{1/\beta'} \int_0^h \bigg(\sum_{k=1}^{\infty} |F_{2k}^1(t)|^{\beta} \bigg)^{1/\beta} dt,
\end{aligned}
$$

where $\beta' = \beta/(\beta-1)$. Now, taking $\beta \ge 2$ and applying the classical Hausdorff–Young inequality (see, e.g., [38, p. 154]), we obtain

$$
\|w\|_{L_{p),\nu}(\Pi)} \le c_4 \int_0^h \bigg\| \frac{\partial f}{\partial x}(\cdot,t) \bigg\|_{L_{\beta'}(I)} dt, \tag{26}
$$

where $\beta \in [2, +\infty)$ and $\frac{1}{\beta} + \frac{1}{\beta'} = 1$.
It is known (see, e.g., [26]) that if $\nu \in A_p(I)$, $1 < p < +\infty$, then there exists $q \in (1, p)$ such that $\nu \in A_q(I)$. Choose $p_0 \in (q, p)$ and set $r = p_0/q$. Let $g \in L_{p),\nu}(I)$. Then $1 < r < p_0$, and Hölder's inequality gives

$$\begin{aligned}\left(\int_I |g|^r\,dx\right)^{1/r} &= \left(\int_I |g|^{p_0/q}\nu^{1/q}\nu^{-1/q}\,dx\right)^{1/r}\\ &\le \left(\int_I |g|^{p_0}\nu\,dx\right)^{1/(qr)}\left(\int_I \nu^{-q'/q}\,dx\right)^{1/(q'r)}\\ &= \left(\int_I |g|^{p_0}\nu\,dx\right)^{1/p_0}\left(\int_I \nu^{-1/(q-1)}\,dx\right)^{(q-1)/p_0}\\ &\le c_5\|g\|_{L_{p),\nu}(I)}\left(\int_I \nu^{-1/(q-1)}\,dx\right)^{(q-1)/p_0}.\end{aligned}$$

Taking into account that $\frac{q'}{q} = \frac{1}{q-1}$, the relation $\nu^{-\frac{1}{q-1}} \in L_1(I)$ follows from $\nu \in A_q(I)$. Then from the previous inequality, we directly obtain

$$\|g\|_{L_r(I)} \le c_6\|g\|_{L_{p),\nu}(I)}, \tag{27}$$

where

$$c_6 = c_5\left(\int_I \nu(x)^{-1/(q-1)}\,dx\right)^{(q-1)/p_0}.$$

Thus, $c_6 > 0$ is a constant independent of g. Moreover, the continuous embedding $L_{p),\nu}(I) \hookrightarrow L_\gamma(I)$ holds for every $\gamma \in (1, r)$. Choose $\beta \ge 2$ sufficiently large that $1 < \beta' < r$. Then

$$\|g\|_{L_{\beta'}(I)} \le c_7\|g\|_{L_{p),\nu}(I)}.$$

Combining (26) and (27), we obtain

$$\|w\|_{N_{p),\nu}(\Pi)} \le c_8\left\|\frac{\partial f}{\partial x}\right\|_{N_{p),\nu}(\Pi)}.$$

II. Now let $p \in (1, 2)$. Therefore, choosing $\alpha > 1$ close enough to 1, we can provide that $p_1 = p\alpha' > 2$ (this is possible, because $\alpha' \to +\infty$ as $\alpha \to 1 + 0$). With this, further considerations are carried out similar to the previous case.
Other series from (23)-(25), and, consequently, all series from (21) are estimated in a similar way. So, as a result, we obtain

$$\|u\|_{N^2_{p),\nu}(\Pi)} \le c\|f\|_{N^{1,0}_{p),\nu}(\Pi)},$$

where $c > 0$ is a constant independent of f. It can be verified directly that u satisfies Eq. (4). The boundary conditions (5) follow from the trace relations established in [40].
Thus, combining Theorem 2 with the results of [40], we obtain the following theorem on the well-posedness of problem (1)–(3).

Theorem 3. Let the weight function $\nu(x)$ belong to the class $A_p(I)$, $1 < p < +\infty$, $\nu(x) = \nu(2\pi - x)$ and the boundary functions $\varphi(x)$ and $\psi(x)$ belong to $N^2_{p);\nu}(I)$ and satisfy the conditions

$\varphi(0) - \varphi(2\pi) = \varphi^{'}(0) = 0$, $\psi(0) - \psi(2\pi) = \psi^{'}(0) = 0$, and the function $f(x, y) \in N^{1,0}_{p),\nu}(\Pi)$ and satisfies the condition $f(0, y) = f(2\pi, y)$. Then problem (1)-(3) has a unique solution in $N^2_{p),\nu}(\Pi)$ and moreover, the following estimate is valid

$$\|u\|_{N^2_{p);\nu}(\Pi)} \leq c \left(\|\varphi\|_{N^2_{p);\nu}(I)} + \|\psi\|_{N^2_{p);\nu}(I)} + \|f\|_{N^{1,0}_{p),\nu}(\Pi)} \right),$$

where $c > 0$ is a constant independent of φ, ψ and f.

Acknowledgements

The third and fourth authors acknowledge support from the Scientific and Technological Research Council of Türkiye (TÜBİTAK) under the 1002 Short-Term Support Program, Project No. 126F314.

References

[1] Moiseev E.I. (1999). Solving one nonlocal boundary value problem by spectral method. Differential Equations, 35(8), 1094–1100. (in Russian)

[2] Lerner M.E., Repin O.A. (1999). On Frankl-type problems for some elliptic equations with different kinds of degeneration. Differential Equations, 35(8), 1087–1093. (in Russian)

[3] Mammadov T.J. (2023). Strong solvability of a nonlocal problem for the Laplace equation in weighted grand Sobolev spaces. Azerbaijan Journal of Mathematics, 13(1), 188–204.

[4] Gasymov T., Akhmadli B., İldız Ü. (2024). On strong solvability of one nonlocal boundary value problem equation in rectangle. Turkish Journal of Mathematics, 48(1), 21–33.

[5] Frankl F.I. (1956). Unterschallströmungen um Profile mit Überschallzonen, die mit geradem Verdichtungsstoß enden. Prikl. Matematika i Mekhanika, 20(2), 196–202. (in Russian)

[6] Frankl F.I. (1973). Selected Works on Gas Dynamics. Moscow, 711 p. (in Russian)

[7] Bitsadze A.V., Samarski A.A. (1969). On some simplest generalizations of linear elliptic boundary value problems. Dokl. AN SSSR, 185(4), 739–740. (in Russian)

[8] Ionkin N.I., Moiseev E.I. (1979). On the problem for heat equation with two-point boundary conditions. Differential Equations, 15(7), 1284–1295. (in Russian)

[9] Bilalov B.T., Ahmadov T.M., Zeren Y., Sadigova S.R. (2022). Solution in the small and interior Schauder-type estimate for the m-th order elliptic operator in Morrey-Sobolev spaces. Azerbaijan Journal of Mathematics, 12(2), 190–219.

[10] Bilalov B.T., Ahmedzadeh N.R., Garayev T.Z. (2022). Some remarks on solvability of Dirichlet problem for Laplace equation in non-standard function spaces. Mediterranean Journal of Mathematics, 19, Article 133, 25 p.

[11] Bilalov B.T., Sadigova S.R. (2021). On solvability in the small of higher order elliptic equations in grand-Sobolev spaces. Complex Variables and Elliptic Equations, 63(3), 2117–2130. doi: 10.1080/17476933.2020.1807965

[12] Bilalov B.T., Sadigova S.R. (2021). Interior Schauder-type estimates for higher-order elliptic operators in grand-Sobolev spaces. Sahand Communications in Mathematical Analysis, 1(2), 129–148. doi: 10.22130/SCMA.2021.521544.893

[13] Bilalov B.T., Sadigova S.R. (2022). On local solvability of higher order elliptic equations in rearrangement invariant spaces. Sibirskii Matematicheskii Zhurnal, 63(3), 516–530. doi: 10.33048/smzh.2022.63.304

[14] Bilalov B.T., Sadigova S.R. (2024). On the Fredholmness of the Dirichlet problem for a second-order elliptic equation in grand-Sobolev spaces.

Ricerche di Matematica, 73, 283–322. https://doi.org/10.1007/s11587-021-00599-9

[15] Bilalov B.T., Zeren Y., Sadigova S.R., Seyma C. (2022). Solvability in the small of m-th order elliptic equations in weighted grand-Sobolev spaces. Turkish Journal of Mathematics, 46, 2078–2095. doi: 10.55730/1300-0098.3255

[16] Bui T.Q., Bui T.A., Duong X.T. (2020). Global regularity estimates for non-divergence elliptic equations on weighted variable Lebesgue spaces. Communications in Contemporary Mathematics, Article 2050014. doi: 10.1142/S0219199720500145

[17] Byun S.S., Lee M. (2015). On weighted estimates for elliptic equations with BMO coefficients in non-divergence form. International Journal of Mathematics, 26(1), Article 1550001. doi: 10.1142/S0129167X15500019

[18] Caso L., D'Ambrosio R., Softova L. (2020). Generalized Morrey spaces over unbounded domains. Azerbaijan Journal of Mathematics, 10(1), 193–208.

[19] Castillo R.E., Rafeiro H., Rojas E.M. (2021). Unique continuation of the quasilinear elliptic equation on Lebesgue spaces L_p. Azerbaijan Journal of Mathematics, 11(1), 136–153.

[20] Coifman R.R., Fefferman C. (1974). Weighted norm inequalities for maximal functions and singular integrals. Studia Mathematica, 51, 241–250.

[21] Dong H., Kim D. (2018). On L_p-estimates for elliptic and parabolic equations with A_p weights. Transactions of the American Mathematical Society, 370(7), 5081–5130. doi: 10.1090/tran/7161

[22] Dynkin E.M., Osilenker V.R. (1983). Weighted estimates of singular integrals and their applications. Matematicheskiy Analiz, VINITI, Moscow, 21, 42–129. (in Russian)

[23] Palagachev D.K., Softova L.G. (2021). Elliptic systems in generalized Morrey spaces. Azerbaijan Journal of Mathematics, 11(2), 153–162.

[24] Gasymov T.B., Akhmadli B.Q. (2024). On strong solvability of one nonlocal boundary value problem for Laplace equation in grand Sobolev space in rectangle. Caspian Journal of Applied Mathematics, Ecology and Economics, 12(1), 3–15.

[25] Gasymov T.B., Akhmadli B.Q., Behbudova J.H. (2025). On strong solvability of one nonlocal boundary value problem for Poisson's equation in grand Sobolev space in rectangle. Transactions of the National Academy of Sciences of Azerbaijan, Series of Physical-Technical and Mathematical Sciences: Mathematics, 45(4), 1–8.

[26] Muckenhoupt B. (1972). Weighted norm inequalities for the Hardy maximal function. Transactions of the American Mathematical Society, 165, 207–226.

[27] Hunt R.A., Muckenhoupt B., Wheeden R. (1973). Weighted norm inequalities for the conjugate function and Hilbert transform. Transactions of the American Mathematical Society, 176, 227–252.

[28] Bilalov B.T., Gasymov T.B., Maharramova G.V. (2019). Basis property of eigenfunctions in Lebesgue spaces for a spectral problem with a point of discontinuity. Differential Equations, 55(12), 1544–1553. doi: 10.1134/S0012266119120024

[29] Bilalov B.T., Gasymov T.B., Maharramova G.V. (2016). On basicity of eigenfunctions of one discontinuous spectral problem in Morrey type spaces. The Aligarh Bulletin of Mathematics, 35(1–2), 119–129.

[30] Gasymov T.B., Maharramova G.V., Jabrailova A.N. (2018). Spectral properties of the problem of vibration of a loaded string in Morrey type spaces. Proceedings of the Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan, 44(1), 116–122.

[31] Gasymov T.B., Akhtyamov A.M., Ahmedzade N.R. (2020). On the basicity of eigenfunctions of a second-order differential operator with a discontinuity point in weighted Lebesgue spaces. Proceedings of the Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan, 46(1), 32–44. doi: 10.29228/proc.15

[32] Gasymov T.B., Guliyeva A.A. (2018). On basicity of the system of exponents with linear phase in Morrey-Lebesgue space. Advances in Analysis, 3(2), 113–120. doi: 10.22606/aan.2018.32005

[33] Gasymov T.B., Hashimov Ch.M. (2018). On atomic decomposition in Banach spaces. Sahand Communications in Mathematical Analysis, 9(1), 15–32. doi: 10.22130/scma.2018.22984

[34] Sirin F., Zeren Y., Gasymov T. (2022). On basicity of eigenfunctions of one discontinuous differential operator in Banach function spaces. Authorea, November 2, 1–30. doi: 10.22541/au.166736702.23017975/v1

[35] Benzinger H.E. (1972). L^p behavior of eigenfunction expansions. Transactions of the American Mathematical Society, 174, 333–344.

[36] Mikhlin S.G. (1977). Linear Partial Equations. Moscow, 432 p. (in Russian)

[37] Garcia-Cuerva J., Rubio de Francia J.L. (1985). Weighted Norm Inequalities and Related Topics. North-Holland, 615 p.

[38] Zygmund A. (1965). Trigonometric Series. Moscow, 538 p. (in Russian)

[39] Kokilashvili V.M., Meskhi A. (2009). A note on the boundedness of the Hilbert transform in weighted grand Lebesgue spaces. Georgian Mathematical Journal, 16(3), 547–551.

[40] Gasymov T., Akhmadli B., Feyzullayev I. (2027). On the strong solvability of a nonlocal boundary value problem for the Laplace equation in weighted grand Lebesgue spaces in rectangle. Journal of Contemporary Applied Mathematics, 17(1), 13-29 https://doi.org/10.62476/jcam171.2

Telman B. Gasymov
Baku State University, Baku, Azerbaijan
Institute of Mathematics, The Ministry of Science and Education of the Republic of Azerbaijan, Baku, Azerbaijan
*E-mail:*telmankasumov@rambler.ru

Baharchin Akhmadli
Institute of Mathematics, Ministry of Science and Education of the Republic of Azerbaijan, Baku, Azerbaijan
Azerbaijan National Aviation Academy, Baku, Azerbaijan
E-mail: baharchin.ahmadli@naa.edu.az

Özlem Bakşi
Department of Mathematics, Yıldız Technical University, Istanbul, Türkiye
E-mail: baksi@yildiz.edu.tr

Kemal Demir
Department of Mathematics, Yıldız Technical University, Istanbul, Türkiye
E-mail: kemal.demir@std.yildiz.edu.tr

Received 11 March 2026
Accepted 20 October 2026