

On the Strong Solvability of a Nonlocal Boundary Value Problem for the Laplace Equation in Weighted Grand Lebesgue Spaces in Rectangle

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Abstract. The paper considers a nonlocal boundary value problem for the Laplace equation in a rectangular domain within Sobolev spaces generated by the norm of a weighted grand Lebesgue space. The concept of strong solvability for this problem is introduced, and its well-posedness (correct solvability) is established. Furthermore, the basis property of the system of eigen- and associated functions of a related spectral problem is proved in separable weighted grand Lebesgue spaces; this result is subsequently employed to justify the problem's solvability. It should be noted that similar problems in a semi-infinite strip were previously investigated in the classical setting by E.I. Moiseev [1], as well as M.E. Lerner and O.A. Repin [2].

Key Words and Phrases: Laplace equation; nonlocal problem; weighted grand Lebesgue space; strong solvability

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1. Introduction

While the theory of strong and weak solvability for linear elliptic equations in Sobolev spaces is well-established and thoroughly documented in classical monographs, many problems arising in mechanics and mathematical physics fall outside this traditional framework.

Consider the following nonlocal boundary value problem for the Laplace equation:

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 2\pi, \quad 0 < y < h, \quad (1)$$

$$u(x, 0) = \varphi(x), \quad u(x, h) = \psi(x), \quad 0 < x < 2\pi, \quad (2)$$

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$$u_x(0, y) = 0, \quad u(0, y) = u(2\pi, y), \quad 0 < y < h. \quad (3)$$

Such problems have specific peculiarities compared to the ones with local conditions. Earlier, F.I. Frankl [5]; [6, p.453-456] considered the problem with non-local boundary condition for a mixed type equation. Bitsadze-Samarski problem [7] for elliptic equations is also nonlocal with supports on a part of the boundary of domain, and these supports are free of other boundary conditions. In [8], N.I. Ionkin and E.I. Moiseev solved the boundary value problem for multi-dimensional parabolic equations with nonlocal conditions, whose supports are the characteristic and the improper parts of the boundary of domain.

We note that the study of the solvability of elliptic equations with respect to the so-called non-standard function spaces, as well as in weighted grand Sobolev spaces, encounters certain difficulties compared to the weightless case. Therefore, the number of studies devoted to this area has been growing in recent years (see, for example, [3,4, 9-25]), and the development of the corresponding theory is far from complete. These attempts are also relevant in connection with the question of the basis properties of eigenfunctions and associated functions of spectral problems obtained by applying the method of separation of variables in the corresponding non-standard function spaces. The question of the basis property of the classical exponential system in weighted Lebesgue spaces was studied in [26, 27]. In this regard, we note the works [28-34,40-41], in which similar questions are studied and appropriate research methods are proposed.

2. Auxiliary concepts and facts

We will use standard notations. N will be the set of positive integers, while $\alpha = (\alpha_1; \alpha_2) \in Z^+ \times Z^+$ will denote a multi-index, where $Z^+ = N \cup \{0\}$. Denote $\partial^a u = \frac{\partial^{|a|} u}{\partial x^{a_1} \partial y^{a_2}}$, where $|a| = a_1 + a_2$. By $|M|$ we will denote the Lebesgue measure of the set M ;

Let us define our weighted grand Sobolev space. Let, $\Pi = (0, 2\pi) \times (0, h)$. Denote by $L_{p,\nu}(\Pi)$ a Banach space of functions on Π with the mixed norm

$$\|f\|_{L_{p,\nu}(\Pi)} = \sup_{0 < \varepsilon < p-1} \int_0^h \left(\varepsilon \int_0^{2\pi} |f(x; y)|^{p-\varepsilon} \nu(x) dx \right)^{\frac{1}{p-\varepsilon}} dy, \quad 1 < p < +\infty.$$

Denote by $W_{p,\nu}^2(\Pi)$ a weighted grand Sobolev space generated by the norm

$$\|u\|_{W_{p,\nu}^2(\Pi)} = \sum_{|a| \leq 2} \|\partial^a u\|_{L_{p,\nu}(\Pi)}.$$

Now denote by $L_p) (I)$, where $I = (0, 2\pi)$, a grand Lebesgue space generated by the norm

$$\|f\|_{L_p),\nu(I)} = \sup_{0 < \varepsilon < p-1} \left(\varepsilon \int_I |f(x)|^{p-\varepsilon} \nu(x) \right)^{\frac{1}{p-\varepsilon}}.$$

Recall that the class of Muckenhoupt weights $A_p(I)$ [26, 27] is the class of periodic functions (i.e., it is considered that the function $\nu(x)$ is periodically extended to the entire axis with period 2π) satisfying the condition

$$\sup_{J \subset I} \left(\frac{1}{|J|} \int_J \nu(t) dt \right) \left(\frac{1}{|J|} \int_J |\nu(t)|^{-\frac{1}{p-1}} \right)^{p-1} < +\infty,$$

where the supremum is taken over all intervals $J \subset I$ and $|J|$ is a length of the interval J . These spaces are nonseparable and therefore the method of biorthogonal expansion (essentially the spectral method) is not applicable for studying the solvability of differential equations with respect to these spaces. In this regard we select the subspace $N_{p),\nu}(\Pi) \subset L_{p),\nu}(\Pi)$ (separable) based on the shift operator T_δ :

$$(T_\delta u)(x; y) = \begin{cases} u(x + \delta; y), & (x + \delta; y) \in \Pi \\ 0, & (x + \delta; y) \notin \Pi \end{cases}$$

So let us assume

$$N_{p),\nu}^2(\Pi) = \left\{ W_{p),\nu}^2(\Pi) : \sum_{|\alpha| \leq 2} \|T_\delta(\partial^\alpha u) - \partial^\alpha u\|_{L_{p),\nu}(\Pi)} \rightarrow 0, \delta \rightarrow 0 \right\}.$$

2.1. Statement of the problem

For a well-posed statement of the considered boundary value problem, it is first necessary to define the trace operator corresponding to the weighted grand Sobolev space $N_{p),\nu}^2(\Pi)$ on the domain $\Pi = (0, 2\pi) \times (0, h) \subset \mathbb{R}^2$. The boundary is denoted by $\partial\Pi = I_0 \cup I_h \cup \Gamma_0 \cup \Gamma_{2\pi}$, where $I_0 = \{(x; 0) : 0 < x < 2\pi\}$, $I_h = \{(x; h) : 0 < x < 2\pi\}$, $\Gamma_0 = \{(0; y) : 0 < y < h\}$ and $\Gamma_{2\pi} = \{(2\pi; y) : 0 < y < h\}$. In the framework of grand spaces, for $1 < p < +\infty$ the embedding $L_{p),\nu}(I) \subset L_{p-\varepsilon}(I)$ holds for any small $\varepsilon > 0$. Under the assumption that the weight satisfies the Muckenhoupt condition, the continuous embedding $N_{p),\nu}^2(\Pi) \subset W_1^2(\Pi)$ is valid. Consequently, for every function $u \in N_{p),\nu}^2(\Pi)$ the traces $u|_{\partial\Pi}$ and $u_x|_{\partial\Pi}$ are well-defined as functions in $L_1(\partial\Pi; d\sigma)$ with respect to the Lebesgue measure on the boundary.

Consider the following nonlocal problem:

$$\Delta u = 0, \quad (x; y) \in \Pi, \quad (4)$$

$$u|_{I_0} = \varphi(x), \quad u|_{I_h} = \psi(x), \quad u|_{\Gamma_0} = u|_{\Gamma_{2\pi}}, \quad u_x|_{\Gamma_0} = 0, \quad (5)$$

By the solution of this problem, we mean a function $u \in N_{p;\nu}^2(\Pi)$, which satisfies the equality (4) a.e. in Π and whose traces satisfy the relations (5) on the boundary $\partial\Pi$. Taking into account the above, we arrive at the following definition of strong solvability of problem (4)-(5).

Definition 1. A function $u \in N_{p;\nu}^2(\Pi)$ is called a strong solution of the problem (4)-(5) if the equality (4) is satisfied for a.e. $(x; y) \in \Pi$ and its trace $u|_{\partial\Pi}$ satisfies the relations (5).

2.2. Some facts from basis theory

Let us present some well-known concepts and facts from basis theory. Let X be a Banach space. A system $\{u_n\}_{n \in N} \subset X$ is called a basis if any element $f \in X$ is uniquely represented as a series

$$f = \sum_{n=1}^{\infty} c_n u_n$$

convergent in the norm X . A system $\{u_n\}_{n \in N} \subset X$ is called complete in X if $\overline{\text{Sp}}\{u_n\} = X$ and minimal in X if $u_n \notin \overline{\text{Sp}}\{u_k\}_{k \neq n}$. It is known that each basis of the space X is a complete and minimal system in X , the converse is not true in general. The system $\{x_n\}_{n \in N} \subset X$ is complete in $X \iff \varphi \in X^* : \langle x_n, \varphi \rangle = \varphi(x_n) = 0, \forall n \in N \Rightarrow \varphi = 0$. The system $\{u_n\}_{n \in N}$ is minimal in X if and only if there exists a biorthogonal system i.e. there exists a system $\{\vartheta_n\}_{n \in N} \subset X^*$ such that $\langle u_n, \vartheta_k \rangle = \vartheta_k(u_n) = \delta_{nk}$, where δ_{nk} is the Kronecker symbol.

Basis criterion. The system $\{u_n\}_{n \in N} \subset X$ is a basis of the space X if and only if the following conditions are satisfied:

- 1) $\{u_n\}_{n \in N}$ is complete and minimal in X ;
- 2) The projectors

$$P_n f = \sum_{k=1}^n \langle f, \vartheta_k \rangle u_k,$$

where $\{\vartheta_k\}_{k \in N}$ is a biorthogonal system, are uniformly bounded.

Definition 2. A system $\{u_n\}_{n \in N} \subset X$ is called a basis with brackets in X if there exists a sequence of integers $0 = n_0 < n_1 < n_2 < \dots$, such that each element $f \in X$ is uniquely represented as a series

$$f = \sum_{k=0}^{\infty} \sum_{i=n_k+1}^{n_{k+1}} c_i u_i,$$

convergent in the norm X .

Basis criterion with brackets. *The system $\{u_n\}_{n \in N} \subset X$ is a basis with brackets of the space X if and only if the following conditions are satisfied:*

1. $\{u_n\}_{n \in N}$ is complete and minimal in X ;
2. The projectors

$$P_{n_k} f = \sum_{i=1}^{n_k} \langle f, \vartheta_i \rangle u_i,$$

where $\{\vartheta_n\}_{n \in N}$ is a biorthogonal system, are uniformly bounded.

Proposition 1. [28] *Let the system $\{u_n\}_{n \in N}$ form a basis with brackets in X and be uniformly minimal, and let the sequence $\{n_{k+1} - n_k\}_{k \in N}$ be bounded. Then the system $\{u_n\}_{n \in N}$ forms an usually basis in X .*

Applying the Fourier method to solution of the problem (4)-(5) leads to the following spectral problem

$$\left. \begin{aligned} u''(x) + \lambda u(x) &= 0, x \in I, \\ u(0) = u(2\pi), u'(0) &= 0. \end{aligned} \right\} \quad (6)$$

The eigenvalues of problem (5) are $\lambda_n = n^2$, $n \in Z^+$, and corresponding eigenfunctions are $u_{2n}(x) = \cos nx$, $n \in Z^+$. Each eigenfunction $u_{2n}(x)$, $n \in N$, has one associated function $u_{2n-1}(x) = x \sin nx$, $n \in N$. Consider the collection of root functions

$$u_0(x) = 1, u_{2n}(x) = \cos nx, u_{2n-1}(x) = x \sin nx, n \in N, \quad (7)$$

and also set

$$\vartheta_0(x) = \frac{1}{2\pi^2} (2\pi - x), \vartheta_{2n}(x) = \frac{1}{\pi^2} (2\pi - x) \cos nx,$$

$$\vartheta_{2n-1}(x) = \frac{1}{\pi^2} \sin nx, n \in N. \quad (8)$$

Note that these systems are biorthogonal conjugate, which can be verified directly. To obtain our main result, we will significantly use the following theorem.

Theorem 1. *Let $\nu \in A_p(I)$, $1 < p < +\infty$, and $\nu(x) = \nu(2\pi - x)$, $x \in I$. Then system (7) forms a basis in $N_{p;\nu}(I)$.*

Proof. To prove this, we first show the uniform boundedness of the spectral projections P_{2n} , $n \in Z_+$. Proceeding analogously to [35] and using the representation for the Green's function, we reduce the estimation of the projections P_{2n} to the estimation of integrals of the following types

$$\begin{aligned} I_1(f)(x) &= \int_0^{2\pi} \frac{|f(\xi)|}{x+\xi} d\xi, \quad I_2(f) = \\ &= \int_0^{2\pi} \frac{|f(\xi)|}{2\pi-x+\xi} d\xi, \quad I_3(f) = \int_0^{2\pi} \frac{|f(\xi)|}{2\pi-\xi+x} d\xi, \\ I_4(f) &= \int_0^{2\pi} \frac{|f(\xi)|}{4\pi-x-\xi} d\xi, \quad x \in [0, 2\pi], \quad f(x) \in N_{p;\nu}(I). \end{aligned}$$

To estimate the norm of the operator $I_1(f)$, we represent it in the form

$$I_1(f)(x) = \int_{-2\pi}^0 \frac{|f(-\xi)|}{x-\xi} d\xi.$$

Let us extend the function $f(-\xi)$ to the interval $[-2\pi, 2\pi]$, setting $f(-\xi) = 0$, $\xi \in [0, 2\pi]$. Then we obtain

$$I_1(f)(x) = \int_{-2\pi}^{2\pi} \frac{|f(-\xi)|}{x-\xi} d\xi.$$

Therefore, taking into account the boundedness of the Hilbert transform in the weighted Grand Lebesgue space $L_{p;\nu}(-2\pi, 2\pi)$ (see [39]) we have

$$\begin{aligned} \|I_1(f)\|_{N_{p;\nu}(I)} &\leq \|I_1(f)\|_{N_{p;\nu}(-2\pi, 2\pi)} \leq C \|f(-\cdot)\|_{N_{p;\nu}(-2\pi, 2\pi)} = \\ &= C \|f(-\cdot)\|_{N_{p;\nu}(-2\pi, 0)} = C \|f\|_{N_{p;\nu}(I)}. \end{aligned}$$

Therefore, taking into account the boundedness of the Hilbert transform in the weighted Grand Lebesgue space $L_{p;\nu}(-2p, 2p)$ (see [39]) we have

$$\begin{aligned} \|I_1(f)\|_{N_{p;\nu}(I)} &\leq \|I_1(f)\|_{N_{p;\nu}(-2\pi, 2\pi)} \leq C \|f(-\cdot)\|_{N_{p;\nu}(-2\pi, 2\pi)} = \\ &= C \|f(-\cdot)\|_{N_{p;\nu}(-2\pi, 0)} = C \|f\|_{N_{p;\nu}(I)}. \end{aligned}$$

The boundedness of the operators $I_j(f)$ for $j = 2, 3, 4$ is proved similarly, by replacing the variable $\xi = 2\pi - \zeta$, and taking into account the condition $\nu(x) = \nu(2\pi - x)$. Therefore, we obtain the uniform boundedness of the system of projections $\{P_{2n}\}_{n \in Z_+}$, which, according to the basis criterion with brackets, means that system (7) is a basis with brackets in $N_{p;\nu}(I)$. Moreover, the norms of the functions from (7) and (8), are uniformly bounded in $N_{p;\nu}(I)$. and $(N_p)_{;\nu}(I)$.* respectively. Then, applying Proposition 1, we obtain that system (7) forms an usually basis for the space $N_{p;\nu}(I)$.

3. Main results

The main result of the work is the following theorem.

Theorem 2. *Let the weight function $\nu(x)$ belong to the class $A_p(I)$, $1 < p < +\infty$, $\nu(x) = \nu(2\pi - x)$ and the boundary functions $\varphi(x)$ and $\psi(x)$ belong to the space $N_{p;\nu}^2(I)$ and satisfy the conditions*

$$\varphi(0) - \varphi(2\pi) = \varphi'(0) = \psi(0) - \psi(2\pi) = \psi'(0) = 0.$$

Then problem (1)-(3) has a unique solution in $N_{p;\nu}^2(\Pi)$ and moreover it is valid the following estimate

$$\|u\|_{N_{p;\nu}^2(\Pi)} \leq c \left(\|\varphi\|_{N_{p;\nu}^2(I)} + \|\psi\|_{N_{p;\nu}^2(I)} \right),$$

where $c > 0$ is a constant independent of φ and ψ .

Proof. First of all, we note that the uniqueness of the solution follows from the results of [4]. Therefore, we proceed directly to the proof of the existence of a solution. Suppose $u(x, y) \in N_{p;\nu}^2(\Pi)$ is a solution of the problem (4)-(5). Consider $U_n(y) = \langle u(\cdot, y), \vartheta_n(\cdot) \rangle$, i.e.

$$\left. \begin{aligned} U_0(y) &= \frac{1}{2\pi} \int_0^{2\pi} u(x, y) (2\pi - x) dx, \\ U_{2n}(y) &= \frac{1}{2\pi} \int_0^{2\pi} u(x, y) (2\pi - x) \cos nx dx, \\ U_{2n-1}(y) &= \frac{1}{2\pi} \int_0^{2\pi} u(x, y) \sin nx dx, \quad n \in N. \end{aligned} \right\} \quad (9)$$

It is absolutely clear that for a.e. $x \in I$ the relation

$$u(x, y + \delta) - u(x, y) = \int_y^{y+\delta} \frac{\partial u(x, t)}{\partial t} dt, \quad \forall y \in (0, h) \quad (10)$$

holds. Since $\frac{\partial u}{\partial y} \in L_1(\Pi)$, applying Theorem 1.1.1 of [36, p.13], from (11) we obtain that the functions $U_n(y)$ are twice differentiable and can be differentiated under the integral sign. Since the function $u(x, y)$ satisfies the equation (4), multiplying it by $\sin nx$ (by $(2\pi - x) \cos nx$) and integrating over I , we obtain the following relations for $U_{2n-1}(y)$ (respectively, for $U_{2n}(y)$):

$$\left. \begin{aligned} U_0''(y) &= 0, & y \in (0, h) \\ U_{2n-1}''(y) - n^2 U_{2n-1}(y) &= 0, & y \in (0, h) \\ U_{2n}''(y) - n^2 U_{2n}(y) &= -2n U_{2n-1}(y), & y \in (0, h). \end{aligned} \right\} \quad (11)$$

Applying the Newton-Leibniz formula and taking into account the first boundary condition (5), we obtain

$$u_\xi(x) := u(x, \xi) = u(x, 0) + \int_0^\xi \frac{\partial u(x, y)}{\partial y} dy = \varphi(x) + \int_0^\xi \frac{\partial u(x, y)}{\partial \xi} d\xi, \text{ a.e. } x \in I.$$

Consequently,

$$|u(x, \xi) - \varphi(x)| \leq \int_0^\xi \left| \frac{\partial u(x, y)}{\partial y} \right| dy, \text{ a.e. } x \in I.$$

Hence, it immediately follows that

$$\|u_\xi\|_{L_1(I)} = \int_I |u(x, \xi) - \varphi(x)| dx \leq \int_I \int_0^\xi \left| \frac{\partial u(x, y)}{\partial y} \right| dy dx. \quad (12)$$

We have $|\Pi_\xi| \rightarrow 0$ as $\xi \rightarrow +0$. Then from (12) it follows that

$$u_\xi(\cdot) \rightarrow \varphi(\cdot), \quad \xi \rightarrow +0, \quad (13)$$

in the norm of the space $L_1(I)$.

Similarly, taking into account the second boundary condition (5), we have

$$u(x, \xi) = u(x, h) - \int_\xi^h \frac{\partial u(x, y)}{\partial y} dy = \psi(x) - \int_\xi^h \frac{\partial u(x, y)}{\partial y} dy, \text{ a.e. } x \in I.$$

Hence,

$$\int_I |u(x, \xi) - \psi(x)| dx \leq \int_I \int_\xi^h \left| \frac{\partial u(x, y)}{\partial y} \right| dy dx. \quad (14)$$

As $|\Pi \setminus \Pi_\xi| \rightarrow 0$ when $\xi \rightarrow h - 0$, from (14) it follows that

$$u_\xi(\cdot) \rightarrow \psi(\cdot), \quad \xi \rightarrow h - 0, \quad (15)$$

in the norm of the space $L_1(I)$.

On the other hand, it is clear that $U_n(y) \in W_1^2(0, h)$. Hence it immediately follows that there exist the limits

$$\lim_{y \rightarrow +0} U_n(y) = U_n(0), \quad \lim_{y \rightarrow h-0} U_n(y) = U_n(h), \quad \forall n \in Z^+.$$

By (13) and (15), from the last two relations it immediately follows that

$$U_n(0) = \varphi_n, \quad U_n(h) = \psi_n, \quad \forall n \in Z^+, \quad (16)$$

where

$$\begin{cases} \varphi_0 = \frac{1}{2\pi^2} \int_0^{2\pi} \varphi(x) (2\pi - x) dx, \\ \varphi_{2n-1} = \frac{1}{\pi^2} \int_0^{2\pi} \varphi(x) \sin nx dx, \\ \varphi_{2n} = \frac{1}{\pi^2} \int_0^{2\pi} \varphi(x) (2\pi - x) \cos nx dx, n \in N; \end{cases} \quad (17)$$

$$\begin{cases} \psi_0 = \frac{1}{2\pi^2} \int_0^{2\pi} \psi(x) (2\pi - x) dx, \\ \psi_{2n-1} = \frac{1}{\pi^2} \int_0^{2\pi} \psi(x) \sin nx dx, \\ \psi_{2n} = \frac{1}{\pi^2} \int_0^{2\pi} \psi(x) (2\pi - x) \cos nx dx, n \in N. \end{cases} \quad (18)$$

Solution of the problem (9), (16) in the case of odd indices is

$$U_{2n-1}(y) = \psi_{2n-1} \frac{\sinh ny}{\sinh nh} + \varphi_{2n-1} \frac{\sinh n(h-y)}{\sinh nh}, \quad \forall n \in N, \quad (19)$$

and the solution of the problem (9), (16) in the case of even indices is

$$U_0(y) = \frac{\psi_0 - \varphi_0}{h} y + \varphi_0, \quad (20)$$

$$U_{2n}(y) = \frac{\psi_{2n} \sinh nh + h\psi_{2n-1} \cosh nh - h\varphi_{2n-1} \frac{\sinh ny}{\sinh nh} + \varphi_{2n} \frac{\sinh n(h-y)}{\sinh nh}}{\sinh nh} - y \left(\psi_{2n-1} \frac{\cosh ny}{\sinh nh} - \varphi_{2n-1} \frac{\cosh n(h-y)}{\sinh nh} \right), \quad \forall n \in N. \quad (21)$$

Consider the function

$$u(x, y) = U_0(y) + \sum_{n=1}^{\infty} U_n(y) \quad u_n(x) = U_0(y) + \sum_{k=1}^{\infty} (U_{2k}(y) \cos kx + U_{2k-1}(y) x \sin kx), \quad (x, y) \in \Pi, \quad (22)$$

where the coefficients $U_0(y)$, $U_{2k}(\cdot)$, $U_{2k-1}(\cdot)$, $k \in N$, are defined by (19)-(21). Let's show that the function $u(x, y)$ belongs to $N_{p;\nu}^2(\Pi)$. Denote by $u_{\alpha,\beta}(x, y)$ the sum of the series obtained by the formal differentiation of the series (22), i.e.

$$u_{\alpha,\beta}(x, y) = U_0^{(\beta)}(y) + \sum_{n=1}^{\infty} u_n^{(\alpha)}(x) U_n^{(\beta)}(y), \quad (23)$$

where $a, \beta \in Z^+$, $a + \beta = 0, 1, 2$, and

$$u_{0,0}(x, y) = u(x, y), \quad u_n^{(\alpha)}(x) = \frac{d^\alpha u_n(x)}{dx^\alpha}, \quad U_n^{(\beta)}(y) = \frac{d^\beta U_n(y)}{dy^\beta}.$$

Let's first consider the series

$$u_1(x, y) = \sum_{k=1}^{\infty} U_{2k-1}(y) x \sin kx.$$

Differentiating this series formally term-by-term, we have

$$\frac{\partial^2 u_1}{\partial y^2} = \sum_{k=1}^{\infty} U''_{2k-1}(y) x \sin kx = \sum_{k=1}^{\infty} k^2 U_{2k-1}(y) x \sin kx, \quad (24)$$

$$\frac{\partial u_1}{\partial x} = \sum_{k=1}^{\infty} U_{2k-1}(y) \sin kx + \sum_{k=1}^{\infty} k U_{2k-1}(y) x \cos kx, \quad (25)$$

$$\frac{\partial^2 u_1}{\partial x^2} = 2 \sum_{k=1}^{\infty} k U_{2k-1}(y) \cos kx - \sum_{k=1}^{\infty} k^2 U_{2k-1}(y) x \sin kx. \quad (26)$$

Denote

$$w(x, y) = \sum_{k=1}^{\infty} k^2 U_{2k-1}(y) x \sin kx.$$

Let's show that the function $w(x, y)$ belongs to $L_{p); \nu}(\Pi)$.

$$\varphi''_{2k-1} = \frac{1}{p^2} \int_0^{2\pi} \varphi''(x) \sin kx dx, \quad \psi''_{2k-1} = \frac{1}{\pi^2} \int_0^{2\pi} \psi''(x) \sin kx dx.$$

From (17), integrating by parts, we obtain

$$\begin{aligned} \varphi_{2k-1} &= -\frac{1}{\pi^2 k} \int_0^{2\pi} \varphi(x) d\cos kx = \\ &= -\frac{1}{\pi^2 k} \left(\varphi(2\pi) - \varphi(0) - \int_0^{2\pi} \varphi'(x) \cos kx dx \right) = \\ &= \frac{1}{\pi^2 k} \int_0^{2\pi} \varphi'(x) \cos kx dx = \frac{1}{\pi^2 k^2} \int_0^{2\pi} \varphi''(x) \sin kx dx = \frac{1}{k^2} \varphi''_{2k-1}. \end{aligned}$$

Similarly, from (18) we have

$$\psi_{2k-1} = -\frac{1}{k^2} \psi''_{2k-1}.$$

Thus,

$$w(x, y) = \sum_{k=1}^{\infty} \left(\psi''_{2k-1} \frac{\sinh ky}{\sinh kh} + \psi''_{2k-1} \frac{\sinh k(h-y)}{\sinh kh} \right) x \sin kx.$$

It is known that if $\nu \in A_p(I)$, $1 < p < +\infty$, then $\exists \alpha > 1 : \nu \in L_\alpha(I)$ (see, e.g., [37, p. 395]). Let $\varepsilon \in (0, p-1)$ be an arbitrary number. We have the following continuous embeddings: $L_p(I) \subset L_{p-\varepsilon}(I) \subset L_1(I)$. Let us suppose that $\beta = \frac{p}{p-\varepsilon} \implies \frac{1}{\beta'} = 1 - \frac{p-\varepsilon}{p} = \frac{\varepsilon}{p}$. Then from Holder's inequality we obtain

$$\begin{aligned} \int_0^{2\pi} |f|^{p-\varepsilon} \nu dx &= \int_0^{2\pi} |f|^{p-\varepsilon} \nu^{\frac{1}{\beta}} \nu^{\frac{1}{\beta'}} dx \leq \left(\int_0^{2\pi} |f|^p \nu dx \right)^{\frac{1}{\beta}} \left(\int_0^{2\pi} \nu dx \right)^{\frac{1}{\beta'}} \implies \\ \implies \left(\varepsilon \int_0^{2\pi} |f|^{p-\varepsilon} \nu dx \right)^{\frac{1}{p-\varepsilon}} &\leq \left(\int_0^{2\pi} |f|^{p-\varepsilon} \nu dx \right)^{\frac{1}{p}} \left(\int_0^{2\pi} \nu dx \right)^{\frac{\varepsilon}{p-\varepsilon} \cdot \frac{1}{p}} \varepsilon^{\frac{1}{p-\varepsilon}} \leq \\ &\leq c \left(\int_0^{2\pi} |f|^p \nu dx \right)^{\frac{1}{p}}, \end{aligned}$$

where $c > 0$ is a constant which is independent of f and ε . This immediately gives

$$\|f\|_{L_{p,\nu}(I)} \leq c \|f\|_{L_{p,\nu}(I)}, \forall f \in L_{p,\nu}(I).$$

Applying Holder's inequality again, we obtain

$$\begin{aligned} \left(\int_0^{2\pi} |w(x,y)|^p \nu dx \right)^{\frac{1}{p}} &\leq \left(\int_0^{2\pi} \nu(x) dx \right)^{\frac{1}{\alpha}} \left(\int_0^{2\pi} |w(x,y)|^{p\alpha'} dx \right)^{\frac{1}{p\alpha'}} = \\ &= c \left(\int_0^{2\pi} |w(x,y)|^{p_1} dx \right)^{\frac{1}{p_1}}, \end{aligned}$$

where $c = (2\pi)^{\frac{1}{\alpha}}$ (consequently, c does not depend on $w(x,y)$) and $p_1 = p\alpha'$. Consider the following separate cases regarding p .

I. First, let us assume $p \geq 2$. Then we have $p_1 = p\alpha' > 2$. Applying the classical Hausdorff-Young theorem (see, e.g., [38, p.154]) to the function $w(x,y)$, from inequality we have

$$\begin{aligned} \|w(x,y)\|_{L_{p,\nu}} &\leq \left(\int_0^{2\pi} |w(x,y)|^p \nu(x) dx \right)^{\frac{1}{p}} \leq \\ &\leq c \sum_{k=1}^{\infty} |U_{2k-1}(y)| \left(\int_0^{2\pi} |u_{2k-1}(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \leq \\ &\leq c_1 \sum_{k=1}^{\infty} k^2 |U_{2k-1}(y)| \leq c_1 \sum_{k=1}^{\infty} \left| \psi_{2k-1}'' \frac{\sinh k y}{\sinh k h} + \varphi_{2k-1}'' \frac{\sinh k(h-y)}{\sinh k h} \right| \leq \end{aligned}$$

$$\leq c_1 \sum_{k=1}^{\infty} \left(\left| \psi''_{2k-1} \right| \frac{\sinh k y}{\sinh k h} + \left| \varphi''_{2k-1} \right| \frac{\sinh k (h-y)}{\sinh k h} \right).$$

Hence, first integrating with respect to $y \in (0, h)$ and then applying Holder's inequality for any $\beta \in (1, \infty)$, we obtain

$$\begin{aligned} \|w\|_{L_{p,\nu}(\Pi)} &\leq c_1 \sum_{k=1}^{\infty} \left(\frac{\psi''_{2k-1}}{\sinh k h} \int_0^h \sinh k y \, dy + \frac{|\varphi''_{2k-1}|}{\sinh k h} \int_0^h \sinh k (h-y) \, dy \right) \leq \\ &\leq c_1 \sum_{k=1}^{\infty} \frac{|\psi''_{2k-1}| + |\varphi''_{2k-1}|}{\sinh k h} \int_0^h \sinh k y \, dy \leq \\ &\leq c_1 \sum_{k=1}^{\infty} \frac{\cosh k h - 1}{k \sinh k h} \left(|\psi''_{2k-1}| + |\varphi''_{2k-1}| \right) \leq \\ &\leq c_2 \sum_{k=1}^{\infty} \frac{1}{k} \left(|\psi''_{2k-1}| + |\varphi''_{2k-1}| \right) \leq c_2 \left(\sum_{k=1}^{\infty} \frac{1}{k^{\beta'}} \right)^{\frac{1}{\beta'}} \left(\left(\sum_{n=1}^{\infty} |\varphi''_n|^{\beta} \right)^{\frac{1}{\beta}} + \left(\sum_{n=1}^{\infty} |\psi''_n|^{\beta} \right)^{\frac{1}{\beta}} \right). \end{aligned}$$

Now, assuming $\beta \geq 2$ and applying classical Hausdorff-Young inequality (see, e.g. [38, p.154]) we have

$$\|w\|_{L_{p,\nu}(\Pi)} \leq c_3 \left(\|\psi''\|_{L_{\beta'}(I)} + \|\varphi''\|_{L_{\beta'}(I)} \right). \quad (27)$$

where $\beta \in [2; +\infty]$ is some number with $\frac{1}{\beta} + \frac{1}{\beta'} = 1$.

It is known (see, e.g., [26]) that if $\nu \in A_p(I)$, $1 < p < +\infty$, then $\exists q : 1 < q < p - \varepsilon < p \Rightarrow \nu \in A_q(I)$. Let us suppose that, $r = \frac{p\varepsilon}{q}$ and $g \in L_{p,\nu}(I)$. Then $1 < r < p_\varepsilon$ and we have

$$\begin{aligned} \left(\int_I |g|^r dx \right)^{\frac{1}{r}} &= \left(\int_I |g|^{\frac{p\varepsilon}{q}} \nu^{\frac{1}{q}} \nu^{-\frac{1}{q}} dx \right)^{\frac{1}{r}} \leq \left(\int_I |g|^{p\varepsilon} \nu dx \right)^{\frac{1}{q'r}} \left(\int_I \nu^{-\frac{q'}{q}} dx \right)^{\frac{1}{q'r}} = \\ &= \left(\int_I |g|^p \nu dx \right)^{\frac{1}{p}} \left(\int_I \nu^{-\frac{1}{q-1}} dx \right)^{\frac{q-1}{p}} \end{aligned}$$

Taking into account that $\frac{q'}{q} = \frac{1}{q-1}$, the relation $\nu^{-\frac{1}{q-1}} \in L_1(I)$ follows from $\nu \in A_q(I)$. Then from previous inequality we directly obtain

$$\|g\|_{L_r(I)} \leq c \|g\|_{L_{p_\varepsilon}(I)} \quad (28)$$

Then, the last inequality means and where $c > 0$ is a constant independent of g . Also note that the continuous embedding $L_p(I) \subset L_\alpha(I)$ is true for every $\alpha \in (1, r)$. Let us choose β big enough to satisfy the condition

$$1 < \beta' < r \Rightarrow \|g\|_{L_{\beta'}(I)} \leq c\|g\|_{L_r(I)}$$

is satisfied. Then from (27), (28) we obtain

$$\|w\|_{L_p, \nu(\Pi)} \leq c \left(\|\psi''\|_{L_p, \nu(I)} + \|\varphi''\|_{L_p, \nu(I)} \right)$$

II. Now let $p \in (1, 2)$. Therefore, choosing $\alpha > 1$ close enough to 1, we can provide that $p_1 = p\alpha' > 2$ (this is possible, because $\alpha' \rightarrow +\infty$ as $\alpha \rightarrow 1+0$). With this, further+ considerations are carried out similar to the previous case. Other series from (24)-(26), and, consequently, all series from (22) are estimated in a similar way. So, as a result, we obtain

$$\|u\|_{W_p^2, \nu(\Pi)} \leq c \left(\|\varphi\|_{W_p^2, \nu(I)} + \|\psi\|_{W_p^2, \nu(I)} \right),$$

where $c > 0$ is a constant independent of φ and ψ . The fulfillment of equation (4) by $u(\cdot, \cdot)$ can be verified directly. Let's verify the fulfillment of boundary conditions. Denote the trace operators on $\Gamma_0, \Gamma_{2\pi}, I_0$ and I_h by $\theta_0, \theta_{2\pi}, T_0$ and T_h , respectively. Let's show that $T_0 u = \varphi$. It is clear that, $T_0 u \in L_1(I)$ and $\varphi \in L_1(I)$. From the boundedness of the operator $T_0 \in [W_p^2, \nu(\Pi); L_p, \nu(I)]$, $\forall p \geq 1$, it follows that if $u_m \rightarrow u$ in $W_p^2, \nu(\Pi)$, then $u_m/I \rightarrow u/I$ in $L_p, \Pi(I)$.

Now, let's consider the following functions:

$$u_m(x, y) = U_0(y) + \sum_{n=1}^m (U_{2n}(y) \cos nx + U_{2n-1}(y) x \sin nx), \quad (x; y) \in \Pi, \quad m \in N.$$

We have

$$\begin{aligned} T_0 u_m &= u_m(x, 0) = U_0(0) + \sum_{n=1}^m (U_{2n}(0) \cos nx + U_{2n-1}(0) x \sin nx) = \\ &= \frac{1}{2p^2} \int_0^{2\pi} \varphi(\tau) (2\pi - \tau) d\tau + \sum_{n=1}^m \frac{1}{\pi^2} \int_0^{2\pi} \varphi(t) (2\pi - \tau) \cos n\tau d\tau \cos nx + \\ &\quad + \frac{1}{\pi^2} \int_0^{2\pi} \varphi(\tau) \sin n\tau d\tau \sin nx. \end{aligned}$$

It is clear that $T_0 u_m \rightarrow T_0 u$. On the other hand the basicity of the system (7) for $N_{p,\nu}(I)$ implies $T_0 u_m \rightarrow \varphi, m \rightarrow \infty$, in $L_p(I)$. Consequently, $T_0 u = \varphi$, a.e. on I .

Absolutely similar we can show that $T_h u_m \rightarrow \psi, m \rightarrow \infty$, in $L_{p,\nu}(I)$. Consequently, $T_h u = \psi$, a.e. on I .

Consider the operators θ and θ . It is clear that $\theta_0 u_m = \theta u_m, \forall m \in N$. Obviously, $\theta_0 u_m \rightarrow \theta_0 u$ and $\theta_{2\pi} u_m = \theta_{2p} u \Rightarrow \theta_0 u = \theta_{2\pi} u$. Thus, the boundary conditions (5) are fulfilled.

The theorem is proved.

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