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Schur Stability Analysis of Families of Polynomials: Legendre and Chebyshev Polynomials

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Abstract. In this paper, matrix families generated from Legendre and Chebyshev polynomials, the stability analysis of these matrix families, continuity theorems used to obtain interval matrices and extension of these intervals have been examined. The matrix families were introduced with the central matrices A which are generated from the Legendre and Chebyshev polynomials. The intervals which guarantee the Schur stability of the matrix families were obtained by using continuity theorems. The obtained intervals were extended with the algorithms in the literature. Afterwards, Legendre and Chebyshev polynomials with interval coefficients were constructed. Finally, examples related to the stability of the Legendre and Chebyshev polynomials were given.

Key Words and Phrases: Schur stability, sensitivity, interval matrix, Legendre polynomial, Chebyshev polynomials.

2010 Mathematics Subject Classifications: 34D20, 39A30, 93D05, 49K40

1. Introduction

The stability analysis has been one of the main research topics in applied mathematics and control theory. In particular, the determination of the stability of matrix or polynomial families remains a significant problem. In this paper, Schur stable interval polynomial families have been obtained. Specifically, Legendre, first-kind Chebyshev and second-kind Chebyshev polynomials were chosen due to their structural properties. Although there exist many studies in the literature on the Schur stability and on the orthogonal polynomials, this work differs by combining these two topics and providing a new perspective through continuity theorems [1, 3, 4, 6, 7, 9, 11, 16, 17].

Let us give the family of Schur stable matrices with S_N [21, 22]. The eigenvalues of the matrix A lie in the unit disk if and only if $A \in S_N$ [1, 12]. On the other hand, this is also known as spectral criterion in the literature. The spectral criterion can also be represented by the spectrum. $\sigma(A)$ to be spectrum of the matrix A, if it satisfies the condition $\sigma(A) \subset C_s = \{z \mid |z| < 1\}$ then the matrix A is said to be Schur stable [24]. In the literature, Schur stability problems are usually determined using eigenvalues. However, it is well-known that eigenvalue problem is an ill-conditioned problem for the non-symmetric matrices [7, 25]. Let us consider an Ostrowski-type example, when small changes are made in the entries of the given matrix, it can be observed that the eigenvalues vary significantly [1, 20, 22]. For this reason, instead of using eigenvalues, it becomes more convenient to use alternative parameters from the literature for the determination of Schur stability [1, 4, 26, 12].

The theoretical foundation of Schur stability can also be expressed via the discrete Lyapunov matrix equation $A^*HA - H + I = 0$, where $H = H^* > 0$ denotes the Lyapunov matrix associated with the system matrix A [1, 7, 12, 18, 24]. If the positive definite solution H exists, then the matrix A is said to be Schur stable [1, 7, 23, 12]. However, the Lyapunov matrix does not provide information about the quality of stability. Schur stability parameter

$$\omega(A) = \|H\| > 1; \quad H = \sum_{k=0}^{\infty} (A^*)^k A^k \tag{1}$$

is used to quantify the quality of stability [1, 7, 9]. The quality of the Schur stability improves as the parameter ω approaches 1 and deteriorates as it moves away from 1.

In another part of this study, we focus on orthogonal polynomials, which constitute a fundamental class of functions in mathematical analysis and numerical stability. Orthogonal polynomials inherently possess several advantageous properties. In particular, analyzing these polynomials via their companion matrices proves to be more practical and reliable. Here, we focus on the Legendre and Chebyshev polynomials. Due to the intervals over which they are defined, making observations regarding Schur stability becomes considerably more straightforward [2, 10, 17, 19].

In this paper, the Schur stability analysis of the polynomial families centered on the Legendre and Chebyshev orthogonal polynomials were examined. Specifically, the matrices generated from the Legendre and Chebyshev orthogonal polynomials were used as the central matrix. In Section 2, the Legendre and Chebyshev polynomials were introduced. In Section 3, matrix families centered on the Legendre and Chebyshev polynomials were constructed. In Section 4, Schur stability of the matrix families were discussed. First, the continuity theorems related to Schur stability analysis were presented. Then, a new continuity theorem was introduced. Thus, Schur stability intervals for the new matrix families were determined. In the Final Section, these intervals which preserving their Schur stability properties were extended. For this purpose, existing algorithms in the [21, 22] were modified. At the end of the paper, using the algorithm, examples were given.

2. Legendre Polynomial and Chebyshev Polynomials

Orthogonal polynomials play a central role in approximation theory, spectral methods and stability analysis of dynamical systems. Among these, the Legendre and Chebyshev polynomials constitute three of these orthogonal polynomials due to their elegant analytical properties and wide range of applications in numerical analysis and control theory. In this section, the main characteristics of the Legendre polynomial, Chebyshev polynomials of the first and second kinds are briefly summarized and the Schur stability of these polynomials is discussed [2, 10, 17, 19].

2.1. Legendre Polynomials

The Legendre polynomials $P_n(x)$ form a well-known family of orthogonal polynomials defined on $[-1, 1]$. They arise naturally as solutions to the Legendre differential equation,

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0. \tag{2}$$

An explicit representation of the n-th Legendre polynomial is given by Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \big[(x^2 - 1)^n\big]. \tag{3}$$

The Legendre polynomials satisfy the Cauchy difference equation

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x), \quad P_0(x) = 1, \quad P_1(x) = x. \tag{4}$$

Several members of Legendre polynomials are

$$P_2(x) = \tfrac{1}{2}(3x^2 - 1), \quad P_3(x) = \tfrac{1}{2}(5x^3 - 3x), \quad P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3). \tag{5}$$

2.2. Chebyshev Polynomials of the First Kind

The Chebyshev polynomials of the first kind, denoted by $T_n(x)$, are defined over the interval $[-1, 1]$ by the trigonometric relation

$$T_n(x) = \cos(n \arccos x), \quad x \in [-1, 1]. \tag{6}$$

The Chebyshev polynomials of the first kind satisfy the well-known Cauchy difference equation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad T_0(x) = 1, \quad T_1(x) = x. \tag{7}$$

Several members of Chebyshev polynomials of the first kind are

$$T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1. \tag{8}$$

Their roots are distributed as

$$x_k = \cos\left(\frac{(2k-1)\pi}{2n}\right), \quad k = 1, 2, \dots, n. \tag{9}$$

2.3. Chebyshev Polynomials of the Second Kind

The Chebyshev polynomials of the second kind, denoted by $U_n(x)$, share many structural similarities with $T_n(x)$ but differ in their orthogonality properties. They are defined by

$$U_n(x) = \frac{\sin\big((n+1)\arccos x\big)}{\sqrt{1-x^2}}, \quad x \in [-1, 1]. \tag{10}$$

Their Cauchy difference equation takes a form analogous to that of the first kind:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad U_0(x) = 1, \quad U_1(x) = 2x. \tag{11}$$

Several members of Chebyshev polynomials of the second kind are

$$U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x, \quad U_4(x) = 16x^4 - 12x^2 + 1. \tag{12}$$

The zeros of $U_n(x)$ are given by

$$x_k = \cos\left(\frac{k\pi}{n+1}\right), \quad k = 1, 2, \dots, n. \tag{13}$$

2.4. Schur Stabilty of Polynomials

Let us examine equations (9) and (13), which are given for the Chebyshev polynomials of the first and second kinds, respectively. These equations define the roots of the corresponding orthogonal polynomials.

For the Chebyshev polynomials of the first kind, an examination of their roots yields

$$\frac{\pi}{2} \leq \frac{(2k-1)\pi}{2n} < \pi, \quad k = 1, 2, \dots, n. \tag{14}$$

Consequently, it follows that $|x_k| < 1, k = 1, 2, \ldots, n$. This result guarantees that all roots of Chebyshev polynomials of the first kind lie strictly within the interval $(-1, 1)$.

Similarly, for the Chebyshev polynomials of the second kind, an analysis of their roots leads to the same conclusion. Hence, the roots of the Chebyshev polynomials of the second kind are also contained in the interval $(-1, 1)$.

In contrast to the Chebyshev polynomials of the first and second kinds, there is no simple closed-form trigonometric expression for the roots of the Legendre polynomials. These roots are typically computed approximately using numerical methods such as the Newton–Raphson method and exact expressions are generally unavailable. Nevertheless, it is well known that the roots of the Legendre polynomials are also located within the interval $(-1, 1)$. Indeed, the locations of these roots are explicitly characterized in [17], Theorem 62.

Therefore, it is clear that the roots of the polynomials $P_n(x)$, $T_n(x)$, and $U_n(x)$ all lie in the interval $(-1, 1)$. As a result, these polynomials are inherently Schur stable orthogonal polynomials due to their intrinsic structure.

3. Matrix Families Centered on the Legendre and Chebyshev Polynomials

While Legendre and Chebyshev polynomials are primarily studied for their orthogonality properties and their role in approximation theory, it is often insightful to analyze them from an algebraic and matrix-theoretic perspective. In particular, the concept of the companion matrix provides a direct connection between polynomial equations and linear algebra: the roots of a polynomial correspond to the eigenvalues of its associated companion matrix. This relationship allows one to translate properties of orthogonal polynomials, such as the boundedness of their roots within the interval $(-1, 1)$, into spectral characteristics of matrices. In this section, after introducing the companion matrix, the matrix families obtained from these companion matrices have been presented. Thus, new matrix families centered on the Legendre and Chebyshev polynomials have been constructed.

3.1. From Polynomials to Companion Matrices

Given a monic polynomial of degree n,

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \tag{15}$$

its companion matrix C_p is defined in [12, 11, 8] as follows:

$$C_p = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}. \tag{16}$$

The eigenvalues of the matrix C_p coincide exactly with the roots of $p(x)$. In this study, the polynomials $p(x)$ are denoted by

- $P_n(x)$ Legendre polynomials,
- $T_n(x)$ Chebyshev polynomials of the first kind,
- $U_n(x)$ Chebyshev polynomials of the second kind.

Consequently, when $p(x)$ is chosen as these polynomials, the companion matrix inherits the spectral properties of the corresponding polynomial. Since all roots of these polynomials lie strictly within the interval $(-1, 1)$, the eigenvalues of the companion matrix are contained within the unit disk, implying that such matrices are Schur stable.

3.2. Matrix Families Centered on the Legendre and Chebyshev Polynomials

Building upon this framework, matrix families centered on the Legendre and Chebyshev polynomials were introduced. Let us consider the matrix family $\mathcal{L}$,

$$\mathcal{L} = \mathcal{L}(A, B) = \{A(r) = A + rB \mid A, B \in M_n(\mathbb{C})\}.$$

In this construction, the matrix A is taken as the companion matrix of the Legendre and Chebyshev polynomials and the perturbation matrices B are defined as combinations of the elementary matrices E_{ni}, $i \leq n$, then the following matrix families are defined:

$$\mathcal{L}|_{P_n} = L(A, B)|_{A=C_{P_n}} = \left\{ A(r) = C_{P_n} + rB \mid B = \sum_{i=1}^{n} \delta_i E_{ni} \right\},$$

$$\mathcal{L}|_{T_n} = L(A, B)|_{A=C_{T_n}} = \left\{ A(r) = C_{T_n} + rB \mid B = \sum_{i=1}^{n} \delta_i E_{ni} \right\},$$

$$\mathcal{L}|_{U_n} = L(A, B)|_{A=C_{U_n}} = \left\{ A(r) = C_{U_n} + rB \mid B = \sum_{i=1}^{n} \delta_i E_{ni} \right\}.$$

Here, the values δ_i defined as 0 or 1 and $\sum_{i=1}^{n} \delta_i^2 \neq 0$ simultaneously. On the other hand, E_{ni} is a real matrix which the elements at position (n, i) are equal to 1 and the other elements are equal to 0. For the sake of brevity, the remainder of the paper will use the notations $\mathcal{L}|_{P_n}$, $\mathcal{L}|_{T_n}$ and $\mathcal{L}|_{U_n}$.

Remark 1. *The reason for choosing the matrix B in the form of E_{ni} is to avoid disrupting the general structure of the companion matrix obtained from the polynomials. Thus, with the perturbation matrix B, the new matrix is ensured to conform to the companion matrix form.*

4. Schur Stability of the Matrix Families

In this section, before giving the stability of the matrix families, present the continuity theorems for Schur stability. These theorems exist in the literature are determines the sensitivity of the Schur stability. Let us remember the family of Schur stable matrices as $S_N = \{A \in M_N(\mathbb{C}) \mid \omega(A) < \infty\}$.

Theorem 1. *Let $A \in S_N$. If $\|B\| < \sqrt{\|A\|^2 + \frac{1}{\omega(A)}} - \|A\|$ then the matrix $A + B \in S_N$ and*

$$\omega(A+B) \leq \frac{\omega(A)}{1-(2\|A\|+\|B\|)\|B\|\omega(A)}$$

holds [5, 11].

Now, considering Theorem 1, let us give the following theorem obtained by Topcu and Aydın.

Theorem 2. *If $A \in S_N$, $B \in M_N(\mathbb{C})$ and $r \in \mathcal{I}_{\mathcal{L}} = [\underline{r}, \overline{r}]$ then $\mathcal{L}(A, B) \subset S_N$, where $-l = u = -\frac{\|A\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A\|^2 + \frac{1}{\omega(A)}}$, $l < \underline{r} < \overline{r} < u$ [20, 22].*

Thanks to these continuity theorems, intervals preserving the Schur stability of the given matrix families are obtained. It should be noted that the matrix A is Schur stable and an extension preserving Schur stability has been constructed by means of the perturbation matrix B.

On the other hand, when the matrix A is obtained from the Legendre and Chebyshev polynomials for the matrix families, there is an important point to emphasize. Due to the intrinsic structure of these polynomials, all of their roots lie strictly within the interval $(-1, 1)$. As a consequence, the eigenvalues of the companion matrices associated with these polynomials are contained within the unit disk, implying that the Legendre and Chebyshev polynomials are inherently Schur stable. With the Theorem 2, we can conclude that the matrix families $\mathcal{L}|_{P_n}$, $\mathcal{L}|_{T_n}$ and $\mathcal{L}|_{U_n}$ are Schur stable on the intervals $\mathcal{I}_{P_n}, \mathcal{I}_{T_n}$ and $\mathcal{I}_{U_n}$, respectively.

Let us present the continuity theorem for the matrix families as follows:

Theorem 3. *If the matrix A is equal to one of these companion matrices C_{P_n}, C_{T_n} and C_{U_n} and the matrix B is equal to $\sum_{i=1}^{n} \delta_i E_{ni}$ for the r values selected from the intervals $\mathcal{I}_{P_n}, \mathcal{I}_{T_n}$ and $\mathcal{I}_{U_n}$ then the matrix families $\mathcal{L}|_{P_n}$, $\mathcal{L}|_{T_n}$ and $\mathcal{L}|_{U_n}$ are the Schur stable, respectively.*

Proof. Theorem 3 is obtained from Theorem 2. Therefore, the proof of Theorem 3 is omitted. For further details, see [20, 22].

Example 1. *Let us take the third-degree Legendre and Chebyshev polynomials and the perturbation matrix $B_4 = E_{31} + E_{32}$, respectively. There exist seven different alternatives for the matrix B. They are denoted as $B_1 = E_{31}, B_2 = E_{32}, B_3 = E_{33}, B_5 = E_{31} + E_{33}, B_6 = E_{32} + E_{33}$ and $B_7 = E_{31} + E_{32} + E_{33}$. However, in this analysis, only the matrix B_4 has been considered. Similar stability intervals can be obtained for the other matrices B as well.*

With the application of the continuity theorem which is Theorem 3 the following Schur stability intervals are obtained for the matrix families $\mathcal{L}|_{P_n}$, $\mathcal{L}|_{T_n}$ and $\mathcal{L}|_{U_n}$, respectively:

$$\mathcal{I}_{P_n} = (-0.0559, 0.0559)\,,$$
$$\mathcal{I}_{T_n} = (-0.0341, 0.0341)\,,$$
$$\mathcal{I}_{U_n} = (-0.0699, 0.0699)\,.$$

5. Extending the Stability Intervals and Obtaining the Interval Polynomials Which Legendre/Chebyshev Centered

In the last example, the obtained intervals preserve the Schur stability of the given matrix families. However, upon further analysis, it has been observed that there also exist points outside these intervals that ensure the Schur stability.

To include these points into the obtained intervals, algorithms have been given by Topcu and Aydın [20, 22]. By means of these algorithms, extended intervals are obtained for the given matrix families while preserving the Schur stability. In this study, these algorithms are modified. Before presenting the algorithm, let us specify the step size, which is determined by Theorem 2 as follows:

$$r_k = -\frac{\|A_k\|}{\|B\|} + \frac{1}{\|B\|}\sqrt{\|A_k\|^2 + \frac{1}{\omega(A_k)}}. \tag{17}$$

Algorithm for the Schur Stability

1. Input; $P_n/T_n/U_n$ (type of polynomial),

   $B = \sum_{i=1}^{n} \delta_i E_{ni}$, r^* (stopping parameter).

2. Create the companion matrix; $A := C_{P_n}/C_{T_n}/C_{U_n}$.

3. Calculate; $\|A\|, \|B\|$,

   $\omega(A)$ from equation (1),

   u from Theorem 2.

4. If $u < r^*$ then write "The interval cannot be extended based on the available data." and finish the algorithm.

5. Data renewal;

   $A_0 := A,\ r_0 := u,\ u_0 := r_0,\ l_0 := -r_0,\ k =: 0.$

For Lower Bound

6. Calculate;

   $A_{k+1} = A_k - r_k B$,

   $\|A_{k+1}\|$,

   $\omega(A_{k+1})$,

   r_{k+1} from equation (17).

7. If $r_{k+1} \geq r^*$ then

   calculate $l_{k+1} = l_k - r_{k+1}$,

   take $k := k + 1$,

   go to the (6). step.

8. Take $n := k$ and write the lower bound of the interval as $l^e = l_n$.

For Upper Bound

6. Calculate;

   $A_{k+1} = A_k + r_k B$,

   $\|A_{k+1}\|$,

   $\omega(A_{k+1})$,

   r_{k+1} from equation (17).

7. If $r_{k+1} \geq r^*$ then

   calculate $u_{k+1} = u_k + r_{k+1}$,

   take $k := k + 1$,

   go to the (6). step.

8. Take $m := k$ and write the upper bound of interval as $u^e = u_m$.

9. Construct;

   $\mathcal{I}^e_{P_n}/\mathcal{I}^e_{T_n}/\mathcal{I}^e_{U_n} = [l^e, u^e]$,

   $\mathcal{C}^e_{P_n}/\mathcal{C}^e_{T_n}/\mathcal{C}^e_{U_n} = A + [l^e, u^e]\, B$.

10. Generate the Legendre/Chebyshev-centered interval polynomials;

    $P^e_n(x)/T^e_n(x)/U^e_n(x)$

Let us examine the given algorithm. By taking the Legendre/Chebyshev polynomials as the central polynomials, the Schur stable interval polynomial families are obtained. Here, the matrix A corresponds to the Legendre/Chebyshev polynomials of the degree specified by the user, while the matrix B is provided by the user as well. However, a certain restriction is imposed on the matrix B in order not to disrupt the structure of the companion matrix. On the other hand, the parameter r^* is defined by the user as the practical parameter for the step size [13, 14, 15]. With this stopping criterion, the algorithm yields more efficient and reliable results.

Once executed, the algorithm performs the necessary computations in accordance with the user-defined inputs and yields the extended interval $\mathcal{I}^e_{P_n}/\mathcal{I}^e_{T_n}/\mathcal{I}^e_{U_n} = [l^e, u^e]$ that preserves the Schur stability. At this stage, an interval companion matrix is constructed using the interval $\mathcal{I}^e_{P_n}/\mathcal{I}^e_{T_n}/\mathcal{I}^e_{U_n}$. Finally, by means of the companion matrix, a Legendre/Chebyshev-centered Schur stable interval polynomial family is generated.

The descriptions of the symbols used were given below:

P_n :	Legendre polynomials of degree n.
T_n :	Chebyshev polynomials of first kind of degree n.
U_n :	Chebyshev polynomials of second kind of degree n.
${}_k\mathcal{I}_{P_n}/{}_k\mathcal{I}_{T_n}/{}_k\mathcal{I}_{U_n}$:	Schur stability intervals of the polynomials $P_n/T_n/U_n$ for the perturbation matrix B_k.
${}_k\mathcal{I}^e_{P_n}/{}_k\mathcal{I}^e_{T_n}/{}_k\mathcal{I}^e_{U_n}$:	Extended Schur stability intervals.
${}_k\mathcal{C}^e_{P_n}/{}_k\mathcal{C}^e_{T_n}/{}_k\mathcal{C}^e_{U_n}$:	Interval companion matrices constructed from the interval ${}_k\mathcal{I}^e_{P_n}/{}_k\mathcal{I}^e_{T_n}/{}_k\mathcal{I}^e_{U_n}$.
${}_kP^e_n(x)/{}_kT^e_n(x)/{}_kU^e_n(x)$:	Legendre / Chebyshev (first kind) / Chebyshev (second kind)-centered extended Schur stable interval polynomials.

With the aid of the algorithm for the Schur stability, the following theorem is obtained.

Theorem 4. *If the matrix A is equal to one of these companion matrices C_{P_n}, C_{T_n} and C_{U_n} and the matrix B is equal to $\sum_{i=1}^{n} \delta_i E_{ni}$ for the r values selected from the intervals $\mathcal{I}^e_{P_n}, \mathcal{I}^e_{T_n}$ and $\mathcal{I}^e_{U_n}$ then the matrix families $\mathcal{L}|_{P_n}$, $\mathcal{L}|_{T_n}$ and $\mathcal{L}|_{U_n}$ are the Schur stable, respectively.*

Proof. This theorem is presented as a result of the algorithm. The algorithm is constructed based on the continuity theorems, which are Theorem 2 and Theorem 3. Therefore, the obtained intervals guarantee Schur stability.

Example 2. *Let us take the 3rd degree Legendre and Chebyshev polynomials*

$$\begin{array}{rcl} P_3(x) & = & \frac{1}{2}(5x^3-3x) \quad \longrightarrow \quad C_{P_3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{3}{5} & 0 \end{pmatrix} \\ T_3(x) & = & 4x^3-3x \quad \longrightarrow \quad C_{T_3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{3}{4} & 0 \end{pmatrix} \\ U_3(x) & = & 8x^3-4x \quad \longrightarrow \quad C_{U_3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

and the perturbation matrices B *as follows,*

$$B_1 = E_{31},\ B_2 = E_{32},\ B_3 = E_{33},\ B_4 = E_{31}+E_{32},$$
$$B_5 = E_{31}+E_{33},\ B_6 = E_{32}+E_{33},\ B_7 = E_{31}+E_{32}+E_{33}$$

Let us examine the Table 1. The matrices A*,* B *and the parameter* r^* *are the input elements selected by the users.* l *and* u *are the lower and upper bounds which are calculated with the continuity theorems, respectively.* l^e *and* u^e *are the extended lower and upper bounds obtained by the Algorithm for the Schur Stability.* N^l *and* N^u *indicate step number of the algorithm stopped for lower and upper bounds, respectively. It should be noted here that the matrix* B *has been examined under seven different cases. Beyond these options, no other choice of the matrix* B *is admissible. The computations were also carried out for all seven perturbation matrices* B*. However, due to the similarity of the results, only the outcomes corresponding to* B_1, B_4 *and* B_7 *are presented in the Table 1.*

To illustrate, a comparison with Example 1 shows that the extended intervals for $r^* = 0.001$ *were obtained as follows:*

- *For the matrices* C_{P_3} *and* B_4*;*
  - *Lower bound is* $l^e = -0.8573$ *with* 20 *steps and upper bound is* $u^e = 0.1979$ *with* 10 *steps.*
  - *Extended interval is* ${}_4\mathcal{I}^e_{P_3} = [-0.8573, 0.1979]$.
  - *Interval companion matrix for the matrix family* ${}_4\mathcal{L}^e|_{P_3}$ *is*
    $${}_4\mathcal{C}^e_{P_3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ [-0.8573, 0.1979] & [-0.2573, 0.7979] & 0 \end{pmatrix}.$$
  - *Legendre-centered Schur stable interval polynomial is*
    ${}_4P^e_3(x) = x^3 + [-0.7979, 0.2573]\,x + [-0.1979, 0.8573]$.

A	B	r^*	l	l^e	N^l	u	u^e	N^u
C_{P_3}	B_1	0.01 0.001	-0.079	-0.3684 -0.3977	8 15	0.079	0.3684 0.3977	8 15
	B_4	0.01 0.001	-0.0559	-0.8398 -0.8573	15 20	0.0559	0.175 0.1979	5 10
	B_7	0.01 0.001	-0.0456	-0.3562 -0.3966	11 21	0.0456	0.1195 0.1317	4 7
C_{T_3}	B_1	0.01 0.001	-0.0482	-0.2167 -0.2468	7 14	0.0482	0.2167 0.2468	7 14
	B_4	0.01 0.001	-0.0341	-0.8898 -0.9121	20 26	0.0341	0.0977 0.1233	4 10
	B_7	0.01 0.001	-0.0278	-0.2075 -0.2455	10 20	0.0278	0.0646 0.0818	3 7
C_{U_3}	B_1	0.01 0.001	-0.0988	-0.4755 -0.4976	9 15	0.0988	0.4755 0.4976	9 15
	B_4	0.01 0.001	-0.0699	-0.7986 -0.8204	13 19	0.0699	0.2323 0.2478	6 10
	B_7	0.01 0.001	-0.0571	-0.459 -0.4958	12 21	0.0571	0.1509 0.1652	4 7

Table 1: The values l, l^e, u, u^e for the data A, B, r^*

- *For the matrices C_{T_3} and B_4;*
  - *Lower bound is $l^e = -0.9121$ with 26 steps and upper bound is $u^e = 0.1233$ with 10 steps.*
  - *Extended interval is ${}_4\mathcal{I}^e_{T_3} = [-0.9121, 0.1233]$.*
  - *Interval companion matrix for the matrix family ${}_4\mathcal{L}^e|_{T_3}$ is*
  $${}_4\mathcal{C}^e_{T_3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ [-0.9121, 0.1233] & [-0.1621, 0.8733] & 0 \end{pmatrix}.$$
  - *Chebyshev of first kind-centered Schur stable interval polynomial is ${}_4T^e_3(x) = x^3 + [-0.8733, 0.1621]\,x + [-0.1233, 0.9121]$.*
- *For the matrices C_{U_3} and B_4;*
  - *Lower bound is $l^e = -0.8204$ with 19 steps and upper bound is $u^e = 0.2478$ with 10 steps.*
  - *Extended interval is ${}_4\mathcal{I}^e_{U_3} = [-0.8204, 0.2478]$.*
  - *Interval companion matrix for the matrix family ${}_4\mathcal{L}^e|_{U_3}$ is*

$$
{}_4\mathcal{C}^e_{U_3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ [-0.8204, 0.2478] & [-0.3204, 0.7478] & 0 \end{pmatrix}.
$$

– *Chebyshev of second kind-centered Schur stable interval polynomial is* ${}_4U^e_3(x) = x^3 + [-0.7478, 0.3204]\, x + [-0.2478, 0.8204]$.

Note. *Python has been used for the computations. The development of the Python procedure was supported through the use of artificial intelligence.*

6. Conclusion

The main objective of this study is to obtain the Legendre/Chebyshev-centered Schur stable interval polynomial families. Accordingly, the following results have been obtained:

- Using the companion matrix, Legendre/Chebyshev-centered matrix families were constructed.
- With the aid of the continuity theorems, the intervals $\mathcal{I}_{P_n}, \mathcal{I}_{T_n}$ and $\mathcal{I}_{U_n}$ that ensure the Schur stability of the matrix families $\mathcal{L}|_{P_n}$, $\mathcal{L}|_{T_n}$ and $\mathcal{L}|_{U_n}$ were determined, respectively.
- The intervals $\mathcal{I}_{P_n}, \mathcal{I}_{T_n}$ and $\mathcal{I}_{U_n}$ were extended through the algorithm while preserving the Schur stability property.
- The companion matrices $\mathcal{C}^e_{P_n}, \mathcal{C}^e_{T_n}$ and $\mathcal{C}^e_{U_n}$ corresponding to the extended intervals $\mathcal{I}^e_{P_n}, \mathcal{I}^e_{T_n}$ and $\mathcal{I}^e_{U_n}$ were constructed, respectively.
- A Legendre/Chebyshev-centered Schur stable interval polynomial families $P^e_n(x), T^e_n(x)$ and $U^e_n(x)$ were obtained.

For illustrative purposes, examples based on third-degree Legendre and Chebyshev polynomials are presented. Moreover, the algorithms can also be applied to higher-degree Legendre and Chebyshev polynomials.

On the other hand, this study is based on the continuity theorems. Thus, the Legendre and Chebyshev polynomials have been taken as the central polynomials and the stability intervals have been extended with the continuity theorems while preserving its Schur stability property. Therefore, the study contributes a novel perspective to the literature.

This study provides a foundation for extending the proposed approach to higher-degree polynomials. In addition, it opens the way for constructing new polynomial-based matrix families with guaranteed Schur stability.

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