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The Basis Property of the System of Exponents in Weighted Grand Lebesgue Space

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Abstract. The basis property of a perturbed exponential system in weighted grand Lebesgue spaces satisfying the Muckenhoupt condition is studied. Sufficient conditions for the basis property of the considered system in the weighted grand Lebesgue space are found under the assumption that the weight satisfies the Muckenhoupt condition.

Key Words and Phrases: Basis, exponential system, grand Lebesgue space, Muckenhoupt weight.

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1. Introduction

When solving problems involving mixed (or elliptic) type equations using the Fourier method, sine and cosine systems of the form

$$\{sin(nt + \beta(t))\}_{n\in N}, \quad \{cos(nt + \beta(t))\}_{n\in N}, \tag{1}$$

naturally arise (see, for example, [11, 12, 13, 18]). Here, $\beta : [0, \pi] \to \mathbb{R}$ is a real-valued function. To justify the formally constructed solution, it is essential to examine the fundamental basis properties of these systems in appropriate functional spaces-typically Lebesgue, Sobolev, or grand Lebesgue spaces (see, for example, [7, 8]). The basis properties of systems (1) in the space $L_p(0, \pi)$, $1 < p < +\infty$, have been studied for a wide class of functions $\beta(\cdot)$ (see [1, 2, 3, 4, 5, 6, 14, 15, 21, 22]). The investigation of the basis properties of system (1) closely related to those of basis properties of the following exponential system

$$\left\{e^{i(nt+\alpha(t)\mathrm{sign}n)}\right\}_{n\in\mathbb{Z}}, \tag{2}$$

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where $\alpha : [-\pi, \pi] \to \mathbb{R}$. It should be noted that the basis properties of systems (1) and (2) have been studied in Lebesgue spaces with power-type weights [9, 16, 17, 20, 23].

In this paper, we consider the basis problem for the exponential system (2) in the general weighted grand Lebesgue space $L_{p),\nu}(-\pi, \pi)$, $1 < p < +\infty$ and establish sufficient conditions for its basis property in terms of the function $\alpha(\cdot)$ and the weight function $\nu(\cdot)$.

2. Auxiliary facts

In what follows we use the standard notations:

1. $\mathbb{N}$ the set of natural numbers $\mathbb{N}_+ = \{0\} \cup \mathbb{N}$, $\mathbb{Z} = \{-\mathbb{N}\} \cup \mathbb{N}_+$,
2. $\mathbb{R}$ the set of real numbers, $\mathbb{C}$ the set of complex numbers,
3. $\omega = \{z \in \mathbb{C} : |z| < 1\}$ the unit ball in the complex plane, $\gamma = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle,
4. δ_{nm} Kronecker delta,
5. $\operatorname{sgn} n$ the sign function: $\operatorname{sgn} n = \begin{cases} 1, n \geq 0, \\ -1, n < 0. \end{cases}$

Let $L_{p),\nu}(-\pi, \pi)$, $1 < p < +\infty$, denote the weighted grand Lebesgue space of functions measurable on the interval $[-\pi, \pi]$, equipped with the norm

$$\|f\|_{p),\nu} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} |f(t)\nu(t)|^{p-\varepsilon} \, dt \right)^{\frac{1}{p-\varepsilon}} < +\infty,$$

where $\nu\colon [-\pi, \pi] \to [0, +\infty)$ is a measurable weight function, periodically extended to the whole real axis $\mathbb{R}$ by $\nu(x + 2\pi) = \nu(x)$ for all $x \in \mathbb{R}$.

We denote by $A_p(-\pi, \pi), 1 < p < +\infty$, the class of weights, referred to as the Muckenhoupt class, satisfying

$$\sup_{I \subset [-\pi,\pi]} \left(\frac{1}{|I|} \int_I \nu^p(t) dt \right) \left(\frac{1}{|I|} \int_I \nu^{-\frac{p}{p-1}}(t) dt \right)^{p-1} < +\infty,$$

where I is any finite interval, $|I|$ denotes the its length and the supremum is taken over all finite intervals $I \subset [-\pi, \pi]$.

The space $L_{p),\nu}(-\pi,\pi)$, endowed with the given norm, is not separable. Therefore, the basis problem for the system will be considered in the subspace

$$N_{p),\nu}\left(-\pi,\pi\right)=\left\{f\in L_{p)}(-\pi,\pi):\|f(\cdot+\eta)-f(\cdot)\|_{L_{p),\nu}(-\pi,\pi)}\to 0,\eta\to 0\right\},$$

which is induced by the shift operator.

The space associated with the weighted grand Lebesgue space denote by $\left(L_{p),\nu}(-\pi,\pi)\right)'$, $p>1$. It is defined by the norm

$$\|g\|_{{L'}_{p),\nu}}=\sup_{f\in\mathbb{S}^{p)}}\|fg\|_{L_{1,\nu}}<+\infty,$$

where $S^{p)}=\left\{f\in L_{p),\nu}(-\pi,\pi):\|f\|_{p),\nu}\le 1\right\}$ is unit sphere. According to [24] $\left(N_{p),\nu}(-\pi,\pi)\right)^*=\left(N_{p),\nu}(-\pi,\pi)\right)'$, and thus each functional $\eta\in\left(N_{p),\nu}(-\pi,\pi)\right)^*$ is determined by a unique function $g\in\left(N_{p),\nu}(-\pi,\pi)\right)'$ via

$$\eta\left(f\right)=\int_{-\pi}^{\pi}f\left(t\right)\overline{g\left(t\right)}dt,\forall f\in N_{p),\nu}\left(-\pi,\pi\right).$$

We now introduce the weighted grand Lebesgue spaces, Hardy classes, and their grand analogs-grand-Hardy classes- along with several foundational theorems essential for our analysis. In the unit disk, the weighted Hardy classes of analytic functions f satisfying

$$\sup_{0<r<1}\int_{-\pi}^{\pi}\left|f\left(re^{it}\right)\nu(t)\right|^{p}dt<+\infty,$$

are denoted by $H^{+}_{p,\nu},p>1$(for more details, see [28]) with norm

$$\|f\|_{H^{+}_{p,\nu}}=\sup_{0<r<1}\left(\int_{-\pi}^{\pi}\left|f\left(re^{it}\right)\nu(t)\right|^{p}dt\right)^{1/p}.$$

Suppose $f(z)$ is analytic outside the unit disk and has finite order at infinity. Its Laurent expansion about infinity is

$$f(z)=\sum_{k=-\infty}^{m}a_kz^k=f_0(z)+f_1(z),z\to+\infty,$$

where $f_0(z) = \sum\limits_{k=-\infty}^{0} a_k z^k$ is the regular part and $f_1(z) = \sum\limits_{k=1}^{m} a_k z^k$ is the principal part. If $m < 1$ then $f_1(z) \equiv 0$. If $\overline{f_0\left(\frac{1}{\bar{z}}\right)} \in H^+_{p,\nu}, p > 1$, we say that $f(z)$ belongs to the class ${}_mH^-_{p,\nu}, p > 1$. For more detailed information about Hardy classes, refer to works [25-27].

Let $H^+_{p),\nu)}, p > 1$, denote the weighted grand Hardy class of analytic functions in the disc ω, defined by the following norm

$$\|f\|_{H^+_{p),\nu}} = \sup_{0<r<1} \|f(re^{it})\|_{p),\nu} < +\infty.$$

It is clear that if $f\nu \in L_1(-\pi,\pi)$, then any function, $f \in H^+_{p,\nu}, p > 1$, has non-tangential limit values $f^+\left(e^{it}\right)$ almost everywhere on the boundary γ as $r \to 1$. We will denote by ${}_mH^-_{p),\nu}, p > 1$, the weighted grand Hardy class that satisfies the condition $f \in_m H^-_{p-\varepsilon,\nu}, 0 < \varepsilon < p-1 : \overline{f_0\left(\frac{1}{\bar{z}}\right)} \in H^+_{p),\nu}$. For these classes, the following continuous inclusions hold:

$$H^+_{p,\nu} \subset H^+_{p),\nu} \subset H^+_{p-\varepsilon,\nu}, \forall\varepsilon \in (0, p-1). \tag{3}$$

We will denote the restriction of functions from the weighted grand Hardy class $H^+_{p),\nu}$ to the γ as $L^+_{p),\nu}$. The operator Φ_+ defined by the formula $\Phi_+ f(\tau) = f^+(\tau)$, $\tau \in \gamma$, establishes an isomorphism between the spaces $L^+_{p),\nu}$ and $H^+_{p),\nu}$. Denote by $N^+_{p),\nu}$ the intersection of spaces $N_{p),\nu}$ and $L^+_{p),\nu}$. It is clear that the space $N^+_{p),\nu}$ is a subspace of $L^+_{p),\nu}$. Let us denote this fact as $NH^+_{p),\nu} = \Phi_+^{-1}\left(N^+_{p),\nu}\right)$. Accordingly, define the operator $\Phi_- f(\tau) = f^-(\tau)$ and the space $\Phi_-\left({}_mH^-_{p),\nu}\right) =_m L^-_{p),\nu}$. Let us define the space ${}_mN^-_{p),\nu}$ as $N_{p),\nu} \bigcap L^-_{p),\nu}$ and accept ${}_mNH^-_{p),\nu} = \Phi_-^{-1}\left({}_mN^-_{p),\nu}\right)$. Φ_-: ${}_mH^-_{p),\nu} \to {}_mL^-_{p),\nu}$ is an isomorphism between these spaces.

We will need the following theorem about singular operator S.

Theorem 1. *([30]) Suppose that the operator S is bounded on the weighted spaces $L_{p,\nu}(-\pi,\pi)$ and $L_{p-\varepsilon_0,\nu}(-\pi,\pi)$ for some $\varepsilon_0 \in (0, p-1)$ and $\nu \in A_p \cap A_{p-\varepsilon_0}$. Then S is bounded on $L_{p),\nu}(-\pi,\pi)$, where operator S is Cauchy type singular operator and defined by*

$$S(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(\tau)}{\tau - t} d\tau.$$

We will also need the following theorem concerning these spaces.

Theorem 2. *([29]) Suppose there exists a number $\varepsilon \in (0, p-1)$ such that $\nu^{-1} \in L_{\frac{1}{p-\varepsilon-1}}(-\pi,\pi)$. Then the space $L_{p,\nu}(-\pi,\pi)$ is dense in $N_{p),\nu}(-\pi,\pi)$.*

Additionally, the following theorem on the basis property of the exponential system in grand Lebesgue spaces holds.

Theorem 3. *([29]) Suppose $\nu(\cdot)$ is a 2π periodic function and $\nu \in A_p(-\pi,\pi)$. Then the system $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$ forms a basis in the space $N_{p),\nu}(-\pi,\pi)$, $1<p<+\infty$.*

We will also need the following theorem about the basis property of exponential systems $\left\{e^{\mathrm{int}}\right\}_{n\in Z_+}$ and $\left\{e^{-\mathrm{int}}\right\}_{n\in N}$.

Theorem 4. *Suppose $\nu \in A_p(-\pi,\pi)$, $1<p<\infty$ then:*
1) The system $\left\{e^{int}\right\}_{n\in Z_+}$ forms a basis in the space $N^+_{p),\nu}(-\pi,\pi)$;
2) The system $\left\{e^{-\mathrm{int}}\right\}_{n\in N}$ forms a basis in the space $N^-_{p),\nu}(-\pi,\pi)$.

Proof. The following continuous inclusions are valid

$$L_{p,\nu} \subset N_{p),\nu} \subset L_{p),\nu} \subset L_{p-\varepsilon,\nu} \quad , \quad \varepsilon \in (0,p-1). \tag{4}$$

It is known that for $\forall p \in (1,+\infty)$ if $\nu \in A_p$ then the direct sum $L_{p,\nu} = L^+_{p,\nu} \bigoplus L^-_{p,\nu}$ is valid, where $L^+_{p,\nu}\left(L^-_{p,\nu}\right) \subset L_{p,\nu}$ is a subspace obtained by restricting the Hardy class $H^+_{p,\nu}\left({}_{-1}H^-_{p,\nu}\right)$ onto γ. Clearly, the inclusion $N_{p),\nu} \subset L_{p-\varepsilon,\nu}$,$\forall\varepsilon \in (0,p-1)$ holds. Let us take any function f from the space $N^+_{p),\nu}(-\pi,\pi)$. Since the system $\left\{e^{\mathrm{int}}\right\}_{n\in\mathbb{Z}}$ forms a basis in the space $N_{p),\nu}(-\pi,\pi)$ (see Theorem 3) we can write the expansion of f with respect to this system

$$f = \sum_{n=-\infty}^{+\infty} c_n e^{int}. \tag{5}$$

Since $f \in L^+_{p-\varepsilon,\nu}(-\pi,\pi)$ the expansion (5) is also valid for $\forall\varepsilon \in (0,p-1)$ in the space $L_{p-\varepsilon,\nu}(-\pi,\pi)$. From this, it follows that for $n<0 \Rightarrow c_n = 0$. Hence, the expansion (5) takes the following form

$$f = \sum_{n=0}^{+\infty} c_n e^{int}.$$

The uniqueness of the expansion is obtained by the minimality of the system $\left\{e^{\mathrm{int}}\right\}_{n\in\mathbb{Z}}$ in the space $N^+_{p),\nu}(-\pi,\pi)$. Thus, any function f from the space $N^+_{p),\nu}(-\pi,\pi)$ is uniquely expanded into a series with respect to the system $\left\{e^{\mathrm{int}}\right\}_{n\in Z_+}$ and therefore, the system $\left\{e^{\mathrm{int}}\right\}_{n\in Z_+}$ forms a basis in the space $N^+_{p),\nu}(-\pi,\pi)$.

Similarly, it is established the basis property of the system $\left\{e^{-\mathrm{int}}\right\}_{n\in N}$ in the space $N^-_{p),\nu}(-\pi,\pi)$, $1<p<+\infty$. The theorem is proved.

3. The Basis Property of Perturbed Exponential System

Let us consider the following exponential system

$$\left\{e^{i(nt+\alpha(t)\mathrm{sign}n)}\right\}_{n\in\mathbb{Z}},$$

where $\alpha(t)$ is a piecewise Hölder function defined on the interval $[-\pi,\pi]$. Denote the discontinuity points of the function $\alpha(t)$ by $\{t_k\}_{k=0}^{n} : -\pi = t_0 < t_1 < t_2 < \cdots < t_n < \pi$ and define

$$h_k = \alpha(t_k+0) - \alpha(t_k-0), k = 1, \ldots, n,$$

$$h_0 = \alpha(+\pi) - \alpha(-\pi),$$

where h_k represents the jumps of the function $\alpha(\cdot)$ at the points $\{t_k\}_{k=0}^{n}$.

The following main theorem is valid.

Theorem 5. *Suppose $\nu \in A_p(-\pi,\pi)$ and let the weight function $\omega(t)$ be defined by*

$$\omega(t) = \prod_{k=0}^{n}\left|\sin\frac{t-t_k}{2}\right|^{-\frac{h_k}{2\pi}}, \ t_0 = -\pi,$$

where $h_0 = \alpha(+\pi) - \alpha(-\pi), h_k = \alpha(t_k+0) - a(t_k-0), k = 1, \ldots, n,$ and $\alpha(\cdot)$ is a piecewise Hölder function with jumps at the points $-\pi < t_1 < t_2 < \cdots < t_n < \pi$.

If $\omega(t)\nu(t) \in A_p$, $\frac{h_k}{2\pi} < 1$ for k=1,...,n then the system

$$\left\{e^{i(nt+\alpha(t)\mathrm{sign}n)}\right\}_{n\in\mathbb{Z}},$$

forms a basis in the space $N_{p),\nu}(-\pi,\pi)$, $1 < p < +\infty$.

Proof. For the arbitrary function $f \in N_{p),\nu}(-\pi,\pi)$, let us consider the following non-homogeneous Riemann problem in the grand-Hardy class $NH^{+}_{p),\nu} \times_{-1} NH^{-}_{p),\nu}$:

$$e^{-i\alpha(t)}F^{+}\left(e^{it}\right) - e^{i\alpha(t)}F^{-}\left(e^{it}\right) = f(t), \text{ a.e. } t \in (-\pi,\pi). \tag{6}$$

A solution of the boundary value problem (6) is understood as a pair $(F^{+},F^{-}) \in NH^{+}_{p),\nu} \times_{-1} NH^{-}_{p),\nu}$ whose non-tangential boundary values $F^{+}\left(e^{it}\right)$ (inside the unit disk) and $F^{-}\left(e^{it}\right)$ (outside the unit disk) satisfy (6) on γ. The solution of the problem (6) is known to be unique (see [6, 7]) and is given by

$$F_1(z) = \frac{\mathrm{Z}(z)}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)}{\mathrm{Z}^{+}\left(e^{it}\right)}\frac{1}{1-e^{-it}z}dt, z \notin \gamma,$$

where $Z(z)$ is the canonical solution of the following homogeneous boundary value problem

$$e^{-i\alpha(t)}Z^{+}\left(e^{it}\right)-e^{i\alpha(t)}Z^{-}\left(e^{it}\right)=0,\ t\in(-\pi,\pi).$$

The canonical solution (see [7]) is given by:

$$Z(z)=\begin{cases} X(z), & |z|<1,\\ X^{-1}(z), & |z|\geq 1,\end{cases}$$

where

$$X(z)\equiv\exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi}\alpha(t)\frac{e^{it}+z}{e^{it}-z}dt\right\}.$$

Applying the Sokhotski-Plemelj formulas to the Cauchy-type integral for $F_1(z)$ we obtain

$$F_1^{+}\left(e^{it}\right)=\frac{1}{2}f(t)-\frac{Z^{+}\left(e^{it}\right)}{2\pi}\int_{-\pi}^{\pi}\frac{f(s)}{Z^{+}\left(e^{is}\right)}\frac{e^{is}}{e^{is}-e^{it}}ds\equiv\frac{1}{2}f(t)-Z^{+}\left(e^{it}\right)S\left(\frac{f}{Z^{+}}\right),$$

$$F_1^{-}\left(e^{it}\right)=-\frac{1}{2}\frac{Z^{-}\left(e^{it}\right)}{Z^{+}\left(e^{it}\right)}f(t)-\frac{Z^{-}\left(e^{it}\right)}{2\pi}\int_{-\pi}^{\pi}\frac{f(s)}{Z^{+}\left(e^{is}\right)}\frac{e^{is}}{e^{is}-e^{it}}ds\equiv$$

$$\equiv-\frac{1}{2}\frac{Z^{-}\left(e^{it}\right)}{Z^{+}\left(e^{it}\right)}f(t)-Z^{-}\left(e^{it}\right)S\left(\frac{f}{Z^{-}}\right),t\in(-\pi,\pi),$$

where singular operator S is given by

$$S(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(s)\frac{e^{is}}{e^{is}-e^{it}}ds.$$

It is known (see [25]) that

$$\left|\mathrm{Z}^{+}\left(e^{it}\right)\right|\sim\omega(t)=\prod_{\hat{k}=0}^{n}\left|\sin\frac{t-t_{\mathrm{k}}}{2}\right|^{-\frac{h_{\mathrm{k}}}{2\pi}}.$$

Denote the integral term in the above formula by

$$K(f)=\omega S\left(f\omega^{-1}\right).$$

We now show that the Cauchy-type operator is bounded in the space $N_{p),\nu}(-\pi,\pi)$, $1<p<+\infty$.

Suppose that $\{\omega\nu\}\in A_p$. Then, by Theorem 1, there exists $M>0$ such that

$$\|K(f)\|_{p),\nu}=\sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi}\left(|K(f)\nu|\right)^{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}}=$$

$$= \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} \left(\left| \omega S \left(f\omega^{-1} \right) \nu \right| \right)^{p-\varepsilon} \right)^{\frac{1}{p-\varepsilon}} =$$

$$= \left\| \omega S \left(f\omega^{-1} \right) \right\|_{p),\nu} \leq M \left\| f\omega^{-1} \right\|_{p),\omega\nu} = M \left\| f \right\|_{p),\nu}, \forall f \in L_{p),\nu}\left(-\pi,\pi\right).$$

Hence, K is bounded in $N_{p),\nu}(-\pi,\pi)$.

Now, let us show that the inclusion $F_1^+ \in N^+_{p),\nu}(-\pi,\pi) \left(F_1^- \in N^+_{p),\nu}(-\pi,\pi) \right)$ is valid. By Theorem 2 we have $\overline{L_{p,\nu}(-\pi,\pi)} = N_{p),\nu}(-\pi,\pi)$. Then for every $f \in N_{p),\nu}(-\pi,\pi)$,there exists a sequence $\{f_n\} \in L_{p,\nu}(-\pi,\pi)$, such that $f_n \to f$ as $n \to \infty$ in $L_{p),\nu}(-\pi,\pi)$. The boundedness of operator $K(\cdot)$, implies $K(f_n) \to K(f)$ as $n \to \infty$ in $L_{p),\nu}(-\pi,\pi)$. From this, we deduce that $K(f) \in N_{p),\nu}(-\pi,\pi)$.

Thus, $f \in N_{p),\nu}(-\pi,\pi)$ and $K(f) \in N_{p),\nu}(-\pi,\pi)$ imply $F_1^+ \in N_{p),\nu}(-\pi,\pi)$. Since the space $N^+_{p),\nu}(-\pi,\pi)$ is the intersection of $N_{p),\nu}(-\pi,\pi)$ and $L^+_{p),\nu}(-\pi,\pi)$, it follows that $F_1^+ \in N^+_{p),\nu}(-\pi,\pi)$.

Similarly, one shows $F_1^+ \in N^-_{p),\nu}(-\pi,\pi)$.

Since $\nu \in A_p(-\pi,\pi)$, Theorem 4 ensures that the functions $F^+\left(e^{it}\right)$ and $F^-\left(e^{it}\right)$ admit the expansions

$$F^+\left(e^{it}\right) = \sum_{n=0}^{\infty} W_n^+\left(F^+\right) e^{int}, \qquad F^-\left(e^{it}\right) = \sum_{n=1}^{\infty} W_n^-\left(F^-\right) e^{-int},$$

where

$$W_n^+\left(F^+\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F^+(t)\, e^{-int}dt, W_n^-\left(F^-\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F^-(t)\, e^{int}dt.$$

We now show that $W_n^+ \in \left(NH^+_{p,\nu}\right)^*$ for all $\forall n \in Z_+$, and $W_n^- \in \left(NH^-_{p,\nu}\right)^*$ for all $\forall n \in N$

Indeed

$$\left| W_n^+(f) \right| = \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}dt \right| \leq$$

$$\leq \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} |f(t)\nu(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} \left(\int_{-\pi}^{\pi} \left| \nu^{-1}(t) \right|^{\frac{p-\varepsilon}{p-\varepsilon-1}} dt \right)^{\frac{p-\varepsilon-1}{p-\varepsilon}} < +\infty.$$

Due to $\nu \in A_p(-\pi,\pi)$ the last parenthesis is bounded and and hence $\{W_n^+\} \subset \left(N^+_{p),\nu}(-\pi,\pi) \right)^*$.

Consider now the minimality of the system $\left\{ e^{i(nt+\alpha(t)\mathrm{sign} n)} \right\}_{n\in \mathrm{Z}}$ in the space $N_{p),\nu}(-\pi,\pi)$. We have

$$e^{-i\alpha(t)} \sum_{n=0}^{\infty} W_n^+\left(F^+\right) e^{int} - e^{i\alpha(t)} \sum_{n=1}^{\infty} W_n^-\left(F^-\right) e^{-\mathrm{int}} = f(t). \tag{7}$$

By $T_+ (T_-)$ we denote the linear operators that map the function $f(\cdot)$ to $F^+(\cdot)\,(F^-(\cdot)) : F^{\pm} = T_{\pm} f$ where $F^{\pm}$ are the solutions of the problem (6). Then we have

$$W_n^+ \left(F^+ \left(e^{it}\right)\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F^+ \left(e^{it}\right) e^{-\text{int}} dt =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (T_+ f)\, e^{-\text{int}} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f \left(T_+^* e^{-\text{int}}\right) dt, n \in Z_+,$$

$$W_n^- \left(F^- \left(e^{it}\right)\right) = \frac{1}{2\pi} \int^{\pi} F^- \left(e^{it}\right) e^{\text{i}kt} dt =$$

$$= \frac{1}{2\pi} \int^{\pi} (T_- f)\, e^{\text{i}kt} dt = \frac{1}{2\pi} \int^{\pi} f \left(T_-^* e^{\text{i}kt}\right) dt, k \in N.$$

It follows that

$$T_+ : N_{p),\nu}(-\pi,\pi) \to N^+_{p),\nu}(-\pi,\pi); T_- : N_{p),\nu}(-\pi,\pi) \to {}_m N^-_{p),\nu}(-\pi,\pi).$$

Thus

$$T_+^* : \left(N^+_{p),\nu}(-\pi,\pi)\right)^* \to \left(N_{p),\nu}(-\pi,\pi)\right)^*; T_-^* : \left({}_m N^-_{p),\nu}(-\pi,\pi)\right)^* \to \left(N_{p),\nu}(-\pi,\pi)\right)^*.$$

Here, T_+^* and T_-^* denote the adjoint to the T_+ and T_- operators, correspondingly. It is clear that the following relations are satisfied

$$T_+^* e^{-\text{int}} \in \left(N_{p),\nu}(-\pi,\pi)\right)^*, \forall n \in Z_+, T_-^* e^{\text{ikt}} \in \left(N_{p),\nu}(-\pi,\pi)\right)^*, \forall k \in N.$$

Let us denote $V_n^{\pm} = T_{\pm}^* W_n^{\pm}$. Thus, we have

$$W_n^{\pm} \left(F^{\pm}\right) = W_n^{\pm} \left(T_{\pm} f\right) = T_{\pm}^* W_n^{\pm}(f) = V_n^{\pm}(f).$$

For the expression (6), taking the expansion of $f(\cdot)$ into account, we obtain

$$f(t) = e^{-i\alpha(t)} \sum_{n=0}^{\infty} V_n^+ (f)\, e^{int} + e^{i\alpha(t)} \sum_{n=1}^{\infty} V_n^- (f)\, e^{-int}.$$

Clearly, here

$$F^+ \left(e^{it}\right) = \sum_{n=0}^{\infty} V_n^+ (f)\, e^{int},$$

$$F^- \left(e^{it}\right) = \sum_{n=1}^{\infty} V_n^-(f) e^{-int}.$$

And now in the Riemann problem (6) instead of the function $f(t)$ take $f(t) = e^{-ia(t)}e^{ikt}$, where $k \in Z_+$ is a fixed integer. Then the solution of (6) takes the following form

$$\begin{cases} F^+(z) = \sum\limits_{n=0}^{\infty} V_n^+\left[e^{-i\alpha(t)}e^{i\alpha t}\right] z^n, \\ F^-(z) = \sum\limits_{n=1}^{\infty} V_n^-\left[e^{-i\alpha(t)}e^{i\alpha t}\right] z^{-n}. \end{cases} \tag{8}$$

On the other hand, the following pair of functions are also solutions of (6):

$$\begin{cases} F^+(z) = z^n, \ \ |z| < 1, \\ F^-(z) = 0, \ \ |z| > 1. \end{cases} \tag{9}$$

By comparing the solutions (8) and (9) and using the uniqueness of the solution, we conclude

$$V_n^+\left[e^{-i\alpha(t)}e^{\mathrm{ikt}}\right] = \delta_{n\mathrm{k}}$$
$$V_n^-\left[e^{-i\alpha(t)}e^{-\mathrm{ikt}}\right] = 0, \forall n \in N, \forall k \in Z.$$

Similarly, if we take $f(t) = e^{i\alpha(t)}e^{-\mathrm{ikt}}$ we obtain

$$V_n^+\left[e^{i\alpha(t)}e^{-\mathrm{ikt}}\right] = 0, \forall k \in N, \forall n \in Z_+,$$
$$V_n^-\left[e^{i\alpha(t)}e^{-\mathrm{ikt}}\right] = \delta_{n\mathrm{k}}, \forall k, n \in N.$$

From these relations, it follows that the system $\left\{V_n^+, V_{n+1}^-\right\}_{n\in Z_+}$ is biorthogonal to the system (2) and consequently, the system (2) is a minimal in $N_{p),\nu}(-\pi,\pi)$.

The theorem proved.

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References

[1] Bilalov, B.T. (1988). On uniform convergence of series with regard to some system of sines. Differentsial'nye Uravneniya, 24(1), 175–177.

[2] Bilalov, B.T. (1989). Basicity of some systems of functions. Differentsial'nye Uravneniya, 25, 163–164.

[3] Bilalov, B.T. (1990). Basicity of some systems of exponents, cosines and sines. Differentsial'nye Uravneniya, 26(1), 10–16.

[4] Bilalov, B.T. (1999). On basicity of systems of exponents, cosines and sines in L_p. Dokl. Math. 365(1) , 7–8.

[5] Bilalov, B.T. (2001). On basicity of some systems of exponents, cosines and sines in L_p. Dokl. Math., 379(2), 7–9.

[6] Bilalov, B.T. (2004). The basis properties of some systems of exponential functions, cosines, and sines. Sibirskii Matematicheskii Zhurnal, 45(2), 264–273.

[7] Danilyuk, I.I. (1975). Non-Regular Boundary Value Problems in the Plane. Nauka, Moscow.

[8] Garnett, G.J. (1984). Bounded Analytic Functions. Mir, Moscow.

[9] Kasumov, Z.A. (2010). Basicity of a system of exponents in weighted Lebesgue space. Estestv. i Tekhnicheskiye Nauki, Moscow, 6(50), 35–41.

[10] Kazaryan, K.S. & Lizorkin, P.I. (1989). Multipliers, bases and unconditional bases in the weighted spaces B and SB. Proc. Steklov Inst. Math., 187, 111–130.

[11] Moiseev, E.I. (1992). Some boundary value problems for mixed-type equations. Differ. Equ., 28(1), 105–115.

[12] Moiseev, E.I. (1992). Solution of the Frankl problem in a special domain. Differentsial'nye Uravneniya, 28(4), 721–723.

[13] Moiseev, E.I. (1994). On existence and uniqueness of solution a classical problem. Dokl. Math., 336(4), 448–450.

[14] Moiseev, E.I. (1984). On basicity of systems of sines and cosines. DAN SSSR, 275(4), 794–798.

[15] Moiseev, E.I. (1987). On basicity of a system of sines. Differentsial'nye Uravneniya, 23(1), 177–179.

[16] Moiseev, E.I. (1998). On basicity of systems of cosines and sines in weight space. Differentsial'nye Uravneniya, 34(1), 40–44.

[17] Moiseev, E.I. (1999). The basicity in the weight space of a system of eigen functions of a differential operator. Differentsial'nye Uravneniya, 35(2), 200–205.

[18] Ponomarev, S.M. (1979). On an eigenvalue problem. DAN SSSR, 249(5), 1068–1070.

[19] Privalov, I.I. (1950). Boundary Properties of Analytic Functions. M-L, Gostekhizdat.

[20] Pukhov, S.S. & Sedletskii, A.M. (2009). Bases of exponents, sines and cosines in weight spaces on finite interval. Dokl. Math., 425(4), 452–455.

[21] Sadigova, S.R. (2014). The general solution of the homogeneous Riemann problem in the weighted Smirnov classes. Proceedings of the Institute of Mathematics and Mechanics, 40(2), 115–124.

[22] Sedletskii, A.M. (1982). Biorthogonal expansions of functions in series of exponents on intervals of the real axis. Uspekhi Matematicheskikh Nauk, 37(5), 51–95.

[23] Sadigova, S.R. & Guliyeva, A.E. (2022). Bases of the perturbed system of exponents in weighted Lebesgue space with a general weight. Kragujevac Journal of Mathematics, 46(3), 477–486.

[24] Hagverdi, T. (2021). On stability of bases consisting of perturbed exponential systems in grand Lebesgue spaces, Journal of Contemporary Applied Mathematics, 11(2), 81-92.

[25] Danilyuk, I.I. (1975). *Nonregular Boundary Value Problems in the Plane.* Nauka, Moscow (in Russian).

[26] Gakhov, F.D. (1966). Boundary value problems. Science Direct, Moscow.

[27] Koosis, P. (2008). Introduction to Hp Spaces. Cambridge Univ. Press, New York.

[28] Bilalov, B.T., Huseynli, A.A. & Seyidova, F. (2017). Riemann boundary value problems in generalized weighted Hardy. Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, 43(2), 240–251.

[29] Ismailov, M.I. & Aliyarova, I.F. (2024). On the Basis Property of the System of Exponents and the Trigonometric Systems of Sines and Cosines in Weighted Grand Spaces of Lebesgue. Vestnik Moskov. Univ. Ser. 1. Mat. Mekh., 2, 15–25.

[30] Kokilashvili, V.M., Meskhi, A., Rafeiro, H. & Samko, S. (2016). Integral operators in nonstandard function spaces, Birkhauser.

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