

Discrete Morrey spaces over \mathbb{Z} and local estimation of the discrete Hilbert transform

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Abstract. The aim of this article is to investigate structural properties of discrete Morrey spaces $\ell_{p,\lambda}(\mathbb{Z})$ and their weak counterparts $w\ell_{p,\lambda}(\mathbb{Z})$. We establish continuous embedding theorems both within the class of discrete Morrey spaces and within weak discrete Morrey spaces, providing explicit conditions on the parameters p and λ . Furthermore, based on a local estimate for the discrete Hilbert transform derived in this work, we prove its strong and weak boundedness on discrete Morrey spaces over \mathbb{Z} .

Key Words and Phrases: Sequence space · Discrete Morrey space · Embedding · discrete Hilbert transform

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1. Introduction

Morrey spaces were first introduced by C. B. Morrey in 1938 [23] during his investigation of the local behavior of solutions to certain elliptic partial differential equations. Since their introduction, Morrey spaces have played a fundamental role in harmonic analysis, potential theory, and the theory of partial differential equations, and they have been extensively studied together with their weak and generalized variants [1, 6, 7, 8, 9, 10, 11, 15, 16, 20, 21, 22, 24, 25, 26, 29, 30].

In parallel with the development of continuous function spaces, discrete analogues have emerged as essential tools in modern analysis. Discrete function spaces naturally arise in problems where data are sampled at discrete points, such as in digital signal processing, numerical analysis, and discrete potential theory. These spaces preserve many structural features of their continuous counterparts, yet they exhibit distinct analytical phenomena inherent to the discrete setting.

The discrete Morrey spaces denoted by $\ell_{p,\lambda}(\mathbb{Z})$, play an important role in discrete harmonic analysis. In particular, weak discrete Morrey spaces $w\ell_{p,\lambda}(\mathbb{Z})$ arise naturally in the study of operators that fail to be bounded in the strong sense but satisfy weak-type inequalities, which are indispensable in both theoretical investigations and applications.

The study of embeddings between Morrey-type spaces is a one of the main topics in the theory of function spaces. Although sufficient conditions for embeddings are often available, obtaining necessary and sufficient conditions is crucial for a full understanding of their underlying structure.

Another important motivation for the study of discrete Morrey spaces comes from discrete operator theory. Classical operators such as the Hilbert transform, fractional integral operators, and maximal operators have been investigated on these spaces [2, 3, 4, 12, 13, 17, 18, 31, 32, 33, 34].

The aim of this study is to contribute to the systematic development of the theory of discrete Morrey spaces over \mathbb{Z} . We investigate the fundamental structural properties of the spaces $\ell_{p,\lambda}(\mathbb{Z})$ and $w\ell_{p,\lambda}(\mathbb{Z})$, including their completeness and quasi-Banach space structure. Furthermore, we establish continuous embedding theorems both between discrete Morrey spaces and between their weak counterparts, providing explicit conditions on the parameters p and λ . A key feature of our results is that these conditions are shown to be *necessary and sufficient*, thereby ensuring the sharpness of the obtained embeddings. These findings contribute to a deeper understanding of discrete function spaces and provide a solid foundation for further investigations of discrete operators in harmonic analysis. In particular, based on a local estimate for the discrete Hilbert transform derived in this work, we prove its strong and weak boundedness on discrete Morrey spaces over \mathbb{Z} .

2. Discrete Morrey spaces

Let us begin with some definitions and notations used in this paper. We write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $-\mathbb{N}_0 = \{\dots, -2, -1, 0\}$, $\mathbb{R}_+ = (0, \infty)$, and $\mathbb{R}_0 = [0, \infty)$.

For $j \in \mathbb{Z}$ and $R \in \mathbb{R}_0$, define the interval $J(j, R)$ as

$$J(j, R) = \{k \in \mathbb{Z} : |j - k| \leq R\}.$$

We denote $\#J(j, R)$ as the cardinality of $J(j, R)$. In fact, in some cases, it suffices to consider only $R = N \in \mathbb{N}_0$. In particular, for any $R \in \mathbb{R}_0$, there exists $N \in \mathbb{N}_0$ such that $N \leq R < N + 1$, which implies

$$\#J(j, N) = \#J(j, R) = (j + N) - (j - N) + 1 = 2N + 1.$$

Let $J \subseteq \mathbb{Z}$. We denote by $\ell_p(J)$ the discrete Lebesgue space of all sequences $a = (a_m)_{m \in J}$ with the norm

$$\|a\|_{\ell_p(J)} = \left(\sum_{m \in J} |a_m|^p \right)^{1/p}$$

for $1 \leq p < \infty$, and

$$\|a\|_{\ell_\infty(J)} = \sup_{m \in J} |a_m|$$

for $p = \infty$.

Meanwhile, the discrete weak Lebesgue space $w\ell_p(J)$ consists of all sequences $a = (a_m)_{m \in J}$ equipped with the quasi-norm

$$\|a\|_{w\ell_p(J)} = \sup_{\gamma > 0} \gamma (\#\{m \in J : |a_m| > \gamma\})^{1/p},$$

for $1 \leq p < \infty$.

It can be easily shown that the inequality

$$\|a\|_{w\ell_p(J)} \leq \|a\|_{\ell_p(J)}. \quad (1)$$

is valid for any sequence $a = (a_m)_{m \in J} \in \ell_p(J)$.

The locally summable sequence space $\ell_p^{\text{loc}}(\mathbb{Z})$ is the set of all sequences $a = (a_m)_{m \in \mathbb{Z}}$ such that

$$\|a\|_{\ell_p(J)} < \infty \quad \text{for every interval } J \subseteq \mathbb{Z}.$$

Moreover, $w\ell_p^{\text{loc}}(\mathbb{Z})$ is the set of all sequences a such that

$$\|a\|_{w\ell_p(J)} < \infty \quad \text{for every interval } J \subseteq \mathbb{Z}.$$

Let $1 \leq p < \infty$ and $\lambda \in \mathbb{R}$. The discrete Morrey space $\ell_{p,\lambda}(\mathbb{Z})$ is defined as the set of all sequences $a = (a_m)_{m \in \mathbb{Z}} \in \ell_p^{\text{loc}}(\mathbb{Z})$ satisfying the condition that

$$\|a\|_{\ell_{p,\lambda}(\mathbb{Z})} = \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} (\#J(j, N))^{-\frac{\lambda}{p}} \|a\|_{\ell_p(J(j, N))} < \infty.$$

The following lemma is valid.

Lemma 1. *Let $1 \leq p < \infty$ and $\lambda \in \mathbb{R}$. The following properties are valid.*

1. *If $\lambda = 0$, then $\ell_{p,\lambda}(\mathbb{Z}) = \ell_p(\mathbb{Z})$ and $\|a\|_{\ell_{p,\lambda}(\mathbb{Z})} = \|a\|_{\ell_p(\mathbb{Z})}$.*
2. *If $\lambda < 0$, then $\ell_{p,\lambda}(\mathbb{Z}) = \{\theta\}$, where θ denotes the zero sequence.*

3. If $\lambda \geq 1$, then $\ell_{p,\lambda}(\mathbb{Z}) = \ell_\infty(\mathbb{Z})$ and $\|a\|_{\ell_{p,\lambda}(\mathbb{Z})} = \|a\|_{\ell_\infty(\mathbb{Z})}$.

4. If $0 < \lambda < 1$, then $\ell_{p,\lambda}(\mathbb{Z})$ is nontrivial.

Proof.

1. Let $\lambda = 0$. Recall that

$$\|a\|_{\ell_{p,0}(\mathbb{Z})} = \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} \|a\|_{\ell_p(J(j,N))}.$$

If $a = (a_m)_{m \in \mathbb{Z}} \in \ell_p(\mathbb{Z})$, then $\|a\|_{\ell_p(\mathbb{Z})} < \infty$. For any $J(j,N) = \{j - N, \dots, j + N\}$,

$$\|a\|_{\ell_p(J(j,N))} = \left(\sum_{m=j-N}^{j+N} |a_m|^p \right)^{1/p} \leq \|a\|_{\ell_p(\mathbb{Z})}.$$

Hence,

$$\|a\|_{\ell_{p,0}(\mathbb{Z})} = \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} \|a\|_{\ell_p(J(j,N))} \leq \|a\|_{\ell_p(\mathbb{Z})} < \infty,$$

so $a = (a_m)_{m \in \mathbb{Z}} \in \ell_{p,0}(\mathbb{Z})$.

Let $a = (a_m)_{m \in \mathbb{Z}} \in \ell_{p,0}(\mathbb{Z})$, so

$$\|a\|_{\ell_{p,0}(\mathbb{Z})} = \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} \|a\|_{\ell_p(J(j,N))} < \infty.$$

Taking $j = 0$, we have $J(0, N) = \{-N, \dots, N\}$ and

$$\sum_{m=-N}^N |a_m|^p = \|a\|_{\ell_p(J(0,N))}^p \leq \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} \|a\|_{\ell_p(J(j,N))}^p < \infty.$$

Letting $N \rightarrow \infty$ yields

$$\|a\|_{\ell_p(\mathbb{Z})} \leq \|a\|_{\ell_{p,0}(\mathbb{Z})} < \infty,$$

hence $a \in \ell_p(\mathbb{Z})$.

Since both inclusions hold, we have

$$\ell_{p,0}(\mathbb{Z}) = \ell_p(\mathbb{Z}) \quad \text{and} \quad \|a\|_{\ell_{p,0}(\mathbb{Z})} = \|a\|_{\ell_p(\mathbb{Z})}.$$

2. Let $\lambda < 0$. Recall that

$$\|a\|_{\ell_{p,\lambda}(\mathbb{Z})} = \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} (\#J(j, N))^{-\lambda/p} \|a\|_{\ell_p(J(j, N))}.$$

Then $-\lambda/p > 0$, so $(\#J(j, N))^{-\lambda/p} = (2N + 1)^{-\lambda/p} \rightarrow \infty$ as $N \rightarrow \infty$. Suppose $a = (a_m)_{m \in \mathbb{Z}} \in \ell_{p,\lambda}(\mathbb{Z})$ is nonzero. Then there exists $m_0 \in \mathbb{Z}$ such that $|a_{m_0}| > 0$. For $N \in \mathbb{N}_0$,

$$\|a\|_{\ell_p(J(m_0, N))} = \left(\sum_{m=m_0-N}^{m_0+N} |a_m|^p \right)^{1/p} \geq (|a_{m_0}|^p)^{1/p} = |a_{m_0}|.$$

So,

$$\begin{aligned} \|a\|_{\ell_{p,\lambda}(\mathbb{Z})} &\geq (\#J(m_0, N))^{-\lambda/p} \|a\|_{\ell_p(J(m_0, N))} \\ &\geq |a_{m_0}| (\#J(m_0, N))^{-\lambda/p} = |a_{m_0}| (2N + 1)^{-\lambda/p} \rightarrow \infty, \quad N \rightarrow \infty, \end{aligned}$$

which contradicts the finiteness of $\|a\|_{\ell_{p,\lambda}(\mathbb{Z})}$. Hence no nonzero sequence belongs to $\ell_{p,\lambda}(\mathbb{Z})$, that is $\ell_{p,\lambda}(\mathbb{Z}) = \{\theta\}$, where θ denotes the zero sequence.

3. Let $\lambda \geq 1$. Let $a = (a_m)_{m \in \mathbb{Z}} \in \ell_\infty(\mathbb{Z})$, i.e., $\|a\|_{\ell_\infty(\mathbb{Z})} = \sup_{m \in \mathbb{Z}} |a_m| < \infty$. Then for any $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$,

$$\begin{aligned} \|a\|_{\ell_p(J(j, N))} &= \left(\sum_{m \in J(j, N)} |a_m|^p \right)^{1/p} \\ &\leq (\#J(j, N) \|a\|_{\ell_\infty(\mathbb{Z})}^p)^{1/p} = (\#J(j, N))^{1/p} \|a\|_{\ell_\infty(\mathbb{Z})}, \end{aligned}$$

so

$$(\#J(j, N))^{-\lambda/p} \|a\|_{\ell_p(J(j, N))} \leq (\#J(j, N))^{(1-\lambda)/p} \|a\|_{\ell_\infty(\mathbb{Z})}.$$

Since $\lambda \geq 1$, $(1-\lambda)/p \leq 0$, and thus $(\#J(j, N))^{(1-\lambda)/p} = (2N+1)^{(1-\lambda)/p} \leq 1$ for all $N \geq 0$. Hence

$$\|a\|_{\ell_{p,\lambda}(\mathbb{Z})} \leq \|a\|_{\ell_\infty(\mathbb{Z})} < \infty,$$

Therefore, $a \in \ell_{p,\lambda}(\mathbb{Z})$.

Let $a = (a_m)_{m \in \mathbb{Z}} \in \ell_{p,\lambda}(\mathbb{Z})$. For any $m \in \mathbb{Z}$, $N = 0$ set $J(m, 0) = \{m\}$ and $\#J(m, 0) = 1$. Then

$$\|a\|_{\ell_p(J(m, 0))} = |a_m|.$$

By definition of the discrete Morrey norm,

$$\begin{aligned} \|a\|_{\ell_{p,\lambda}(\mathbb{Z})} &= \sup_{m \in \mathbb{Z}, N \in \mathbb{N}_0} (\#J(m, N))^{-\frac{\lambda}{p}} \|a\|_{\ell_p(J(m, N))} \\ &\geq (\#J(m, 0))^{-\frac{\lambda}{p}} \|a\|_{\ell_p(J(m, 0))} = \|a\|_{\ell_p(J(m, 0))} = |a_m|. \end{aligned}$$

So $|a_m| \leq \|a\|_{\ell_{p,\lambda}(\mathbb{Z})}$ for all $m \in \mathbb{Z}$. Hence, $\|a\|_{\ell_\infty(\mathbb{Z})} \leq \|a\|_{\ell_{p,\lambda}(\mathbb{Z})} < \infty$ and $a \in \ell_\infty(\mathbb{Z})$.

Since both inclusions hold, we have

$$\ell_{p,\lambda}(\mathbb{Z}) = \ell_\infty(\mathbb{Z}) \quad \text{and} \quad \|a\|_{\ell_{p,\lambda}(\mathbb{Z})} = \|a\|_{\ell_\infty(\mathbb{Z})}.$$

4. Let $0 < \lambda < 1$. Show that $\ell_{p,\lambda}(\mathbb{Z})$ is not trivial.

Consider the sequence $a = (a_m)_{m \in \mathbb{Z}}$ defined by

$$a_m = \begin{cases} 1, & m = 0, \\ 0, & m \neq 0. \end{cases}$$

The ℓ_p norm of $a = (a_m)_{m \in \mathbb{Z}}$ over the interval $J(j, N)$ is equal to 0 or 1, depending on whether $0 \in J(j, N)$ or not:

– If $0 \in J(j, N)$ (i.e., $|j| \leq N$), then

$$\|a\|_{\ell_p(J(j, N))} = |a_0| = 1.$$

– If $0 \notin J(j, N)$ (i.e., $|j| > N$), then

$$\|a\|_{\ell_p(J(j, N))} = 0.$$

Then for this sequence,

$$(\#J(j, N))^{-\lambda/p} \|a\|_{\ell_p(J(j, N))} = \begin{cases} (2N + 1)^{-\lambda/p}, & |j| \leq N, \\ 0, & |j| > N. \end{cases}$$

Hence we obtain that

$$\|a\|_{\ell_{p,\lambda}(\mathbb{Z})} = \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} (\#J(j, N))^{-\lambda/p} \|a\|_{\ell_p(J(j, N))} = \sup_{N \in \mathbb{N}_0} (2N + 1)^{-\lambda/p} = 1.$$

Therefore, discrete Morrey spaces are nontrivial for $0 < \lambda < 1$:

$$\ell_{p,\lambda}(\mathbb{Z}) \neq \{\theta\}.$$

Lemma 2. *Let $1 \leq p < \infty$ and $0 \leq \lambda < 1$. Then $(\ell_{p,\lambda}(\mathbb{Z}), \|\cdot\|_{\ell_{p,\lambda}(\mathbb{Z})})$ is a Banach space.*

Proof. Clearly $\|a\|_{\ell_{p,\lambda}(\mathbb{Z})} \geq 0$ for every $a = (a_m)_{m \in \mathbb{Z}} \in \ell_{p,\lambda}(\mathbb{Z})$. If $\|a\|_{\ell_{p,\lambda}(\mathbb{Z})} = 0$, then for every $j \in \mathbb{Z}$, and $N \in \mathbb{N}_0$ we have $\|a\|_{\ell_p(J(j,N))} = 0$, so every coordinate of a is zero; hence $a = \theta$, and conversely, if $a = (a_m)_{m \in \mathbb{Z}} = \theta$, then $\|a\|_{\ell_{p,\lambda}(\mathbb{Z})} = 0$.

For scalar $\alpha \in \mathbb{R}$,

$$\begin{aligned} \|\alpha a\|_{\ell_{p,\lambda}(\mathbb{Z})} &= \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} (\#J(j, N))^{-\lambda/p} \|\alpha a\|_{\ell_p(J(j,N))} \\ &= |\alpha| \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} (\#J(j, N))^{-\lambda/p} \|a\|_{\ell_p(J(j,N))} = \alpha \|a\|_{\ell_{p,\lambda}(\mathbb{Z})}. \end{aligned}$$

Let $a, b \in \ell_{p,\lambda}(\mathbb{Z})$. By Minkowski's inequality,

$$\begin{aligned} \|a + b\|_{\ell_{p,\lambda}(\mathbb{Z})} &= (\#J(j, N))^{-\frac{\lambda}{p}} \|a + b\|_{\ell_p(J(j,N))} \\ &= \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} (\#J(j, N))^{-\frac{\lambda}{p}} \left(\sum_{m \in J(j,N)} |a_m + b_m|^p \right)^{\frac{1}{p}} \\ &\leq \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} \left[\left((\#J(j, N))^{-\frac{\lambda}{p}} \sum_{m \in J(j,N)} |a_m|^p \right)^{\frac{1}{p}} + \left((\#J(j, N))^{-\frac{\lambda}{p}} \sum_{m \in J(j,N)} |b_m|^p \right)^{\frac{1}{p}} \right] \\ &= \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} \left(\|a\|_{\ell_p(J(j,N))} + \|b\|_{\ell_p(J(j,N))} \right) \\ &\leq \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} \|a\|_{\ell_p(J(j,N))} + \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} \|b\|_{\ell_p(J(j,N))} \\ &= \|a\|_{\ell_{p,\lambda}(\mathbb{Z})} + \|b\|_{\ell_{p,\lambda}(\mathbb{Z})}. \end{aligned}$$

Thus $\|\cdot\|_{\ell_{p,\lambda}(\mathbb{Z})}$ is a norm.

Let us show that $\ell_{p,\lambda}(\mathbb{Z})$ is a Banach space. Let $(a^{(n)})_{n \in \mathbb{N}}$ be a Cauchy sequence in $\ell_{p,\lambda}(\mathbb{Z})$. Then for every $\varepsilon > 0$ there exists N_ε such that for all $n, m \geq N_\varepsilon$,

$$\sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} (\#J(j, N))^{-\lambda/p} \|a^{(n)} - a^{(m)}\|_{\ell_p(J(j,N))} < \varepsilon.$$

For any interval $J(j, N)$, we obtain

$$\|a^{(n)} - a^{(m)}\|_{\ell_p(J(j,N))} \leq (\#J(j, N))^{\lambda/p} \varepsilon \quad \text{for all } n, m \geq N_\varepsilon. \quad (2)$$

In particular, taking $N = 0$, $k \in \mathbb{Z}$ (so $J(k, 0) = \{k\}$) we obtain

$$\varepsilon > \|a^{(n)} - a^{(m)}\|_{\ell_p(J(k,0))} = \left(\sum_{l \in J(k,0)} |a_l^{(n)} - a_l^{(m)}|^p \right)^{\frac{1}{p}} = |a_k^{(n)} - a_k^{(m)}|.$$

Hence we conclude that for each fixed $k \in \mathbb{Z}$ the scalar sequence $(a_k^{(n)})_n$ is Cauchy in \mathbb{R} and hence converges in \mathbb{R} . Define $a = (a_k)_{k \in \mathbb{Z}}$ by

$$a_k = \lim_{n \rightarrow \infty} a_k^{(n)}.$$

We now show $a \in \ell_{p,\lambda}(\mathbb{Z})$ and $a^{(n)} \rightarrow a$ in the Morrey norm. By (2) we obtain that

$$\begin{aligned} \|a^{(n)} - a\|_{\ell_p(J(j,N))} &= \left(\sum_{k \in J(j,N)} |a_k^{(n)} - a_k|^p \right)^{\frac{1}{p}} = \lim_{m \rightarrow \infty} \left(\sum_{k \in J(j,N)} |a_k^{(n)} - a_k^{(m)}|^p \right)^{\frac{1}{p}} \\ &= \lim_{m \rightarrow \infty} \|a^{(n)} - a^{(m)}\|_{\ell_p(J(j,N))} \leq (\#J(j,N))^{\lambda/p} \varepsilon. \end{aligned}$$

Multiplying by $(\#J(j,N))^{-\lambda/p}$ and taking the supremum over $j \in \mathbb{Z}$ and $N \in \mathbb{N}$ yields

$$\sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} (\#J(j,N))^{-\lambda/p} \|a^{(n)} - a\|_{\ell_p(J(j,N))} \leq \varepsilon.$$

Hence $\|a^{(n)} - a\|_{\ell_{p,\lambda}(\mathbb{Z})} \leq \varepsilon$ for all $n \geq N_\varepsilon$, which proves $a^{(n)} \rightarrow a$ in $\ell_{p,\lambda}(\mathbb{Z})$.

Finally, since $\|a\|_{\ell_{p,\lambda}(\mathbb{Z})} \leq \|a - a^{(n)}\|_{\ell_{p,\lambda}(\mathbb{Z})} + \|a^{(n)}\|_{\ell_{p,\lambda}(\mathbb{Z})}$ and the right-hand side is finite for sufficiently large n , we conclude $a \in \ell_{p,\lambda}(\mathbb{Z})$.

Therefore every Cauchy sequence converges in $\ell_{p,\lambda}(\mathbb{Z})$, and the space is complete (Banach).

Remark 1. Lemma 2 was proved in [12]. For the sake of completeness, we include the proof in our setting.

Theorem 1. Let $1 \leq p_1 \leq p_2 < \infty$ and $0 \leq \lambda_1, \lambda_2 < 1$. Then

$$\frac{1 - \lambda_1}{p_1} \leq \frac{1 - \lambda_2}{p_2} \quad (3)$$

is a necessary and sufficient condition for

$$\ell_{p_1, \lambda_1}(\mathbb{Z}) \hookrightarrow \ell_{p_2, \lambda_2}(\mathbb{Z}) \text{ with } \|a\|_{\ell_{p_1, \lambda_1}(\mathbb{Z})} \leq \|a\|_{\ell_{p_2, \lambda_2}(\mathbb{Z})} \quad (4)$$

for any $a \in \ell_{p_2, \lambda_2}(\mathbb{Z})$ to hold.

Proof. Sufficiency. Assume (3) holds and set

$$\alpha := \frac{1 - \lambda_1}{p_1} - \frac{1 - \lambda_2}{p_2} \leq 0.$$

Let $a = (a_m)_{m \in \mathbb{Z}} \in \ell_{p_2, \lambda_2}(\mathbb{Z})$. Define $f(t) = t^{p_1/p_2}$ for $t > 0$. Since $0 < p_1/p_2 \leq 1$, the function f is concave. Fix $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$. Applying Jensen's inequality to the finite average of the numbers $\{|a_m|^{p_2} : m \in J(j, N)\}$ with equal weights $1/\#J(j, N)$ yields

$$\frac{1}{\#J(j, N)} \sum_{m \in J(j, N)} f(|a_m|^{p_2}) \leq f\left(\frac{1}{\#J(j, N)} \sum_{m \in J(j, N)} |a_m|^{p_2}\right).$$

That is,

$$\frac{1}{\#J(j, N)} \sum_{m \in J(j, N)} |a_m|^{p_1} \leq \left(\frac{1}{\#J(j, N)} \sum_{m \in J(j, N)} |a_m|^{p_2}\right)^{p_1/p_2}.$$

Taking the $1/p_1$ power of both sides gives

$$(\#J(j, N))^{-1/p_1} \|a\|_{\ell_{p_1}(J(j, N))} \leq (\#J(j, N))^{-1/p_2} \|a\|_{\ell_{p_2}(J(j, N))}.$$

Multiply both sides by $(\#J(j, N))^{(1-\lambda_1)/p_1}$ to obtain

$$(\#J(j, N))^{-\lambda_1/p_1} \|a\|_{\ell_{p_1}(J(j, N))} \leq (\#J(j, N))^\alpha (\#J(j, N))^{-\lambda_2/p_2} \|a\|_{\ell_{p_2}(J(j, N))}.$$

Since $\alpha \leq 0$ we have $(\#J(j, N))^\alpha = (2N + 1)^\alpha \leq 1$, hence

$$(\#J(j, N))^{-\lambda_1/p_1} \|a\|_{\ell_{p_1}(J(j, N))} \leq (\#J(j, N))^{-\lambda_2/p_2} \|a\|_{\ell_{p_2}(J(j, N))}.$$

Taking the supremum over all $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$ yields

$$\begin{aligned} \|a\|_{\ell_{p_1, \lambda_1}(\mathbb{Z})} &= \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} (\#J(j, N))^{-\lambda_1/p_1} \|a\|_{\ell_{p_1}(J(j, N))} \\ &\leq \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} (\#J(j, N))^{-\lambda_2/p_2} \|a\|_{\ell_{p_2}(J(j, N))} = \|a\|_{\ell_{p_2, \lambda_2}(\mathbb{Z})}. \end{aligned}$$

This proves the sufficiency of (3).

Necessity. We use proof by contradiction. Suppose (3) fails, i.e.

$$\alpha := \frac{1 - \lambda_1}{p_1} - \frac{1 - \lambda_2}{p_2} > 0.$$

We construct a sequence $a \in \ell_{p_2, \lambda_2}(\mathbb{Z})$ for which $a \notin \ell_{p_1, \lambda_1}(\mathbb{Z})$.

Define a sequence $a = (a_m)_{m \in \mathbb{Z}}$ by

$$a_m := \begin{cases} 0, & m = 0, \\ |m|^{-\beta}, & m \neq 0 \end{cases} \quad \text{with} \quad \beta := \frac{1 - \lambda_2}{p_2}.$$

Step 1. Show that $a \in \ell_{p_2, \lambda_2}(\mathbb{Z})$.

Fix $j \in \mathbb{Z}, N \in \mathbb{N}_0$ and consider two cases.

- (i) Case 1: If $|j| \leq 2N + 1$, then for $m \in J(j, N)$, $|m| \leq |j| + N \leq 3N + 1$. That is, $J(j, N) \subseteq J(0, 3N + 1)$. Hence

$$\begin{aligned}
\sum_{m \in J(j, N)} |a_m|^{p_2} &= \sum_{m \in J(j, N), m \neq 0} |m|^{-(1-\lambda_2)} \leq \sum_{m \in J(0, 3N+1), m \neq 0} |m|^{-(1-\lambda_2)} \\
&= \sum_{m=-(3N+1), m \neq 0}^{3N+1} |m|^{-(1-\lambda_2)} = 2 \sum_{m=1}^{3N+1} m^{-(1-\lambda_2)} \\
&= 2 \left(1 + \sum_{m=2}^{3N+1} m^{-(1-\lambda_2)} \right) \leq 2 + 2 \int_1^{3N+1} x^{-(1-\lambda_2)} dx \\
&= 2 + \frac{2}{\lambda_2} x^{\lambda_2} \Big|_1^{3N+1} = 2 + \frac{2}{\lambda_2} \left((3N+1)^{\lambda_2} - 1 \right).
\end{aligned}$$

Then

$$\begin{aligned}
&(\#J(j, N))^{-\lambda_2/p_2} \left(\sum_{m \in J(j, N)} |a_m|^{p_2} \right)^{1/p_2} \\
&\leq (2N+1)^{-\lambda_2/p_2} \left(2 + \frac{2}{\lambda_2} \left((3N+1)^{\lambda_2} - 1 \right) \right)^{1/p_2} \\
&= \frac{\left(2 + \frac{2}{\lambda_2} \left((3N+1)^{\lambda_2} - 1 \right) \right)^{1/p_2}}{(2N+1)^{\lambda_2/p_2}} \leq \frac{\left(2 + \frac{2}{\lambda_2} \left((3N+1)^{\lambda_2} \right) \right)^{1/p_2}}{(2N+1)^{\lambda_2/p_2}} \\
&\leq \frac{\left((3N+1)^{\lambda_2} + \frac{2}{\lambda_2} \left((3N+1)^{\lambda_2} \right) \right)^{1/p_2}}{(2N+1)^{\lambda_2/p_2}} \\
&= \frac{\left(\left(1 + \frac{2}{\lambda_2} \right) \left((3N+1)^{\lambda_2} \right) \right)^{1/p_2}}{(2N+1)^{\lambda_2/p_2}} = \left(1 + \frac{2}{\lambda_2} \right)^{1/p_2} \left(\frac{3N+1}{2N+1} \right)^{\lambda_2/p_2} \\
&\leq 2^{\lambda_2/p_2} \left(1 + \frac{2}{\lambda_2} \right)^{1/p_2}.
\end{aligned}$$

- (ii) Case 2: If $|j| > 2N + 1$, then for $m \in J(j, N)$, $|m| = |j - (j - m)| \geq |j| - |j - m| \geq |j| - N \geq |j| + \frac{1-|j|}{2} = \frac{|j|+1}{2} > \frac{|j|}{2}$. Therefore

$$\sum_{m \in J(j, N)} |a_m|^{p_2} = \sum_{m \in J(j, N), m \neq 0} |m|^{-(1-\lambda_2)} \leq (2N+1) \left(\frac{|j|}{2} \right)^{-(1-\lambda_2)}.$$

Hence

$$(\#J(j, N))^{-\lambda_2/p_2} \left(\sum_{m \in J(j, N)} |a_m|^{p_2} \right)^{1/p_2} \leq$$

$$(2N+1)^{-\lambda_2/p_2} (2N+1)^{1/p_2} \left(\frac{|j|}{2}\right)^{-(1-\lambda_2)/p_2} = \left(\frac{2N+1}{|j|/2}\right)^{(1-\lambda_2)/p_2}.$$

Since $|j| > 2N+1$, we have $\frac{2N+1}{|j|/2} < 2$. Therefore,

$$(\#J(j, N))^{-\lambda_2/p_2} \|a\|_{\ell_{p_2}(J(j, N))} \leq 2^{(1-\lambda_2)/p_2}.$$

Combining the two cases and taking the supremum over all $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$ shows $\|a\|_{\ell_{p_2, \lambda_2}(\mathbb{Z})} < \infty$, so $a \in \ell_{p_2, \lambda_2}(\mathbb{Z})$.

Step 2. Let us now show that $a \notin \ell_{p_1, \lambda_1}(\mathbb{Z})$. Consider the intervals $J(0, N) = \{-N, \dots, N\}$. For $N \geq 1$,

$$\|a\|_{\ell_{p_1}(J(0, N))}^{p_1} = \sum_{m=-N}^N |a_m|^{p_1} \geq 2 \sum_{m=1}^N m^{-p_1 \beta} = 2 \sum_{m=1}^N m^{-s},$$

where $s = p_1 \beta = \frac{p_1(1-\lambda_2)}{p_2}$.

Because $\alpha > 0$,

$$s = \frac{p_1(1-\lambda_2)}{p_2} = p_1 \left(\frac{1-\lambda_1}{p_1} - \alpha \right) = 1 - \lambda - \alpha p_1 < 1.$$

Moreover,

$$1 - s - \lambda_1 = 1 - \lambda_1 - \frac{p_1(1-\lambda_2)}{p_2} = p_1 \left(\frac{1-\lambda_1}{p_1} - \frac{1-\lambda_2}{p_2} \right) = p_1 \alpha > 0.$$

Since $s < 1$, we estimate using an integral:

$$\sum_{m=1}^N m^{-s} \geq \int_1^{N+1} x^{-s} dx = \frac{(N+1)^{1-s} - 1}{1-s}.$$

Therefore

$$\begin{aligned} (\#J(0, N))^{-\lambda_1/p_1} \|a\|_{\ell_{p_1}(J(0, N))} &\geq (2N+1)^{-\lambda_1/p_1} 2^{1/p_1} \left(\frac{(N+1)^{1-s} - 1}{1-s} \right)^{1/p_1} \\ &= 2^{1/p_1} \left(\frac{(N+1)^{1-s} - 1}{(1-s)(2N+1)^{\lambda_1}} \right)^{1/p_1}. \end{aligned}$$

Since $1-s > \lambda_1$, then the right-hand side tends to ∞ as $N \rightarrow \infty$. Hence

$$\sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} (2N+1)^{-\lambda_1/p_1} \|a\|_{\ell_{p_1}(J(j, N))} = \infty,$$

i.e. $\|a\|_{\ell_{p_1, \lambda_1}} = \infty$. This contradicts the assumed inequality, so the inequality cannot hold when $\alpha > 0$.

Combining sufficiency and necessity, we conclude that (3) is the necessary and sufficient condition for (4) to hold.

Corollary 1. [12, Proposition 2.4] *Let $1 \leq p < \infty$. Then $0 \leq \lambda_2 \leq \lambda_1 < 1$ is a necessary and sufficient condition for*

$$\ell_{p, \lambda_1}(\mathbb{Z}) \hookrightarrow \ell_{p, \lambda_2}(\mathbb{Z}) \text{ with } \|a\|_{\ell_{p, \lambda_1}(\mathbb{Z})} \leq \|a\|_{\ell_{p, \lambda_2}(\mathbb{Z})}$$

for any $a \in \ell_{p, \lambda_2}(\mathbb{Z})$ to hold.

3. Discrete weak Morrey spaces

Let $1 \leq p < \infty$ and $\lambda \in \mathbb{R}$. The discrete weak Morrey space $w\ell_{p, \lambda}(\mathbb{Z})$ is defined as the set of all sequences $a = (a_m)_{m \in \mathbb{Z}} \in w\ell_p^{loc}(\mathbb{Z})$ such that

$$\|a\|_{w\ell_{p, \lambda}(\mathbb{Z})} = \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} (\#J(j, N))^{\frac{-\lambda}{p}} \|a\|_{w\ell_p(J(j, N))} < \infty.$$

The following proposition will be used to prove the second part of the next lemma.

Proposition 1. *Let $1 \leq p < \infty$, $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$. Then for any sequence $a = \{a_m\}_{m \in \mathbb{Z}} \in w\ell_p(J(j, N))$, the following inequality*

$$|a_j| \leq \|a\|_{w\ell_p(J(j, N))} \quad (5)$$

holds.

Proof. Since $J(j, N) = \{j - N, \dots, j + N\}$, then $j \in J(j, N)$. We consider

$$\beta = |a_j| - \varepsilon, \quad 0 < \varepsilon < |a_j|.$$

Then $|a_j| > \beta$, and hence

$$j \in \{m \in J(j, N) : |a_m| > \beta\}.$$

Therefore

$$\#\{m \in J(j, N) : |a_m| > \beta\} \geq 1.$$

Using the definition of $\|a\|_{w\ell_p(J(j, N))}$, we obtain

$$\|a\|_{w\ell_p(J(j, N))} = \sup_{\gamma > 0} \gamma \left(\#\{m \in J(j, N) : |a_m| > \gamma\} \right)^{1/p}$$

$$\begin{aligned}
&\geq \beta \left(\#\{m \in J(j, N) : |a_m| > \beta\} \right)^{1/p} \\
&\geq \beta = |a_j| - \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then

$$\|a\|_{w\ell_p(J(j, N))} \geq |a_j|.$$

This completes the proof.

Lemma 3. *Let $1 \leq p < \infty$ and $\lambda \in \mathbb{R}$. The following properties are valid.*

1. *If $\lambda = 0$, then $w\ell_{p, \lambda}(\mathbb{Z}) = w\ell_p(\mathbb{Z})$ and $\|a\|_{w\ell_{p, \lambda}(\mathbb{Z})} = \|a\|_{w\ell_p(\mathbb{Z})}$.*
2. *If $\lambda < 0$, then $w\ell_{p, \lambda}(\mathbb{Z}) = \{\theta\}$, where θ denotes the zero sequence.*
3. *If $\lambda \geq 1$, then $w\ell_{p, \lambda}(\mathbb{Z}) = \ell_\infty(\mathbb{Z})$ and $\|a\|_{w\ell_{p, \lambda}(\mathbb{Z})} = \|a\|_{\ell_\infty(\mathbb{Z})}$.*
4. *If $0 < \lambda < 1$, then $w\ell_{p, \lambda}(\mathbb{Z})$ is nontrivial.*

Proof.

1. Let $\lambda = 0$. Then

$$\|a\|_{w\ell_{p, 0}(\mathbb{Z})} = \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} \|a\|_{w\ell_p(J(j, N))}.$$

If $a = (a_m)_{m \in \mathbb{Z}} \in w\ell_p(\mathbb{Z})$, then $\|a\|_{w\ell_p(\mathbb{Z})} < \infty$. For every interval $J(j, N)$

$$\|a\|_{w\ell_p(J(j, N))} \leq \sup_{\gamma > 0} \gamma \left(\#\{m \in \mathbb{Z} : |a_m| > \gamma\} \right)^{1/p} = \|a\|_{w\ell_p(\mathbb{Z})}.$$

Hence

$$\|a\|_{w\ell_{p, 0}(\mathbb{Z})} = \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} \|a\|_{w\ell_p(J(j, N))} \leq \|a\|_{w\ell_p(\mathbb{Z})} < \infty.$$

So $a = (a_m)_{m \in \mathbb{Z}} \in w\ell_{p, 0}(\mathbb{Z})$.

Conversely, let us show that $w\ell_{p, 0}(\mathbb{Z}) \subseteq w\ell_p(\mathbb{Z})$. Let $a = (a_m)_{m \in \mathbb{Z}} \in w\ell_{p, 0}(\mathbb{Z})$, then $\|a\|_{w\ell_{p, 0}(\mathbb{Z})} < \infty$.

Taking $j = 0$ we get $J(0, N) = \{-N, \dots, N\}$ and

$$\|a\|_{w\ell_p(J(0, N))} \leq \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} \|a\|_{w\ell_p(J(j, N))} = \|a\|_{w\ell_{p, 0}(\mathbb{Z})}.$$

For each fixed $\gamma > 0$, define

$$E_N(\gamma) := \{m \in J(0, N) : |a_m| > \gamma\}.$$

Then $E_N(\gamma) \subseteq E_{N+1}(\gamma)$ and

$$\bigcup_{N=0}^{\infty} E_N(\gamma) = \{m \in \mathbb{Z} : |a_m| > \gamma\}.$$

Hence, by monotone convergence,

$$\#\{m \in \mathbb{Z} : |a_m| > \gamma\} = \lim_{N \rightarrow \infty} \#E_N(\gamma).$$

Therefore, for each fixed $\gamma > 0$,

$$\gamma \left(\#\{m \in \mathbb{Z} : |a_m| > \gamma\} \right)^{1/p} = \lim_{N \rightarrow \infty} \gamma \left(\#E_N(\gamma) \right)^{1/p}.$$

Since for every $N \in \mathbb{N}_0$,

$$\gamma \left(\#E_N(\gamma) \right)^{1/p} \leq \|a\|_{w\ell_p(J(0, N))},$$

we obtain

$$\gamma \left(\#\{m \in \mathbb{Z} : |a_m| > \gamma\} \right)^{1/p} \leq \sup_{N \in \mathbb{N}_0} \|a\|_{w\ell_p(J(0, N))}.$$

Taking the supremum over $\gamma > 0$ yields

$$\|a\|_{w\ell_p(\mathbb{Z})} \leq \sup_{N \in \mathbb{N}_0} \|a\|_{w\ell_p(J(0, N))} \leq \|a\|_{w\ell_{p,0}(\mathbb{Z})} < \infty.$$

Hence, $a \in w\ell_p(\mathbb{Z})$.

Since both inclusions hold, we obtain

$$w\ell_{p,0}(\mathbb{Z}) = w\ell_p(\mathbb{Z}) \quad \text{and} \quad \|a\|_{w\ell_{p,0}(\mathbb{Z})} = \|a\|_{w\ell_p(\mathbb{Z})}.$$

2. Let $\lambda < 0$. Then $(2N + 1)^{-\lambda/p} \rightarrow \infty$ as $N \rightarrow \infty$.

Suppose $a = (a_m)_{m \in \mathbb{Z}} \in w\ell_{p,\lambda}(\mathbb{Z})$ is nonzero. Then there exists $m_0 \in \mathbb{Z}$ such that $|a_{m_0}| > 0$. For every $N \in \mathbb{N}_0$, according to Proposition 1, we have

$$\|a\|_{w\ell_p(J(m_0, N))} \geq |a_{m_0}|.$$

Thus

$$\begin{aligned}\|a\|_{w\ell_{p,\lambda}(\mathbb{Z})} &\geq (\#J(m_0, N))^{-\lambda/p} \|a\|_{w\ell_p(J(m_0, N))} \\ &\geq (2N+1)^{-\lambda/p} |a_{m_0}| \rightarrow \infty, \quad N \rightarrow \infty.\end{aligned}$$

So no nonzero sequence can belong to $w\ell_{p,\lambda}(\mathbb{Z})$. Hence

$$w\ell_{p,\lambda}(\mathbb{Z}) = \{\theta\}.$$

3. Let $\lambda \geq 1$. Let $a = (a_m)_{m \in \mathbb{Z}} \in \ell_\infty(\mathbb{Z})$, i.e.,

$$\|a\|_{\ell_\infty(\mathbb{Z})} = \sup_{m \in \mathbb{Z}} |a_m| < \infty.$$

Then for any $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$, from (1) it follows that

$$\begin{aligned}\|a\|_{w\ell_p(J(j, N))} &\leq \|a\|_{\ell_p(J(j, N))} = \left(\sum_{m \in J(j, N)} |a_m|^p \right)^{\frac{1}{p}} \\ &\leq ((\#J(j, N)) \|a\|_{\ell_\infty(\mathbb{Z})}^p)^{1/p} = (\#J(j, N))^{1/p} \|a\|_{\ell_\infty(\mathbb{Z})}.\end{aligned}$$

Thus

$$(\#J(j, N))^{-\lambda/p} \|a\|_{w\ell_p(J(j, N))} \leq (\#J(j, N))^{(1-\lambda)/p} \|a\|_{\ell_\infty(\mathbb{Z})}.$$

Since $\lambda \geq 1$, then $(1-\lambda)/p \leq 0$, and thus

$$(\#J(j, N))^{(1-\lambda)/p} = (2N+1)^{(1-\lambda)/p} \leq 1 \quad \text{for all } N \geq 0.$$

Hence

$$\|a\|_{w\ell_{p,\lambda}(\mathbb{Z})} \leq \|a\|_{\ell_\infty(\mathbb{Z})} < \infty.$$

Therefore,

$$\ell_\infty(\mathbb{Z}) \subseteq w\ell_{p,\lambda}(\mathbb{Z}) \quad \text{and} \quad \|a\|_{w\ell_{p,\lambda}(\mathbb{Z})} \leq \|a\|_{\ell_\infty(\mathbb{Z})}.$$

Let $a = (a_m)_{m \in \mathbb{Z}} \in \ell_{p,\lambda}(\mathbb{Z})$. For any $m \in \mathbb{Z}$ consider $J(m, 0) = \{m\}$. Then by Proposition 1 we obtain

$$\|a\|_{w\ell_p(J(m, 0))} \geq |a_m|.$$

By definition of the discrete weak Morrey norm,

$$\|a\|_{w\ell_{p,\lambda}(\mathbb{Z})} = \sup_{m \in \mathbb{Z}, N \in \mathbb{N}_0} (\#J(m, N))^{-\frac{\lambda}{p}} \|a\|_{w\ell_p(J(m, N))}$$

$$\geq (\#J(m, 0))^{-\frac{\lambda}{p}} \|a\|_{w\ell_p(J(m,0))} \geq |a_m|.$$

So $|a_m| \leq \|a\|_{\ell_{p,\lambda}(\mathbb{Z})}$ for all $m \in \mathbb{Z}$.

Hence

$$\|a\|_{\ell_\infty(\mathbb{Z})} \leq \|a\|_{w\ell_{p,\lambda}(\mathbb{Z})} < \infty.$$

Therefore

$$w\ell_{p,\lambda}(\mathbb{Z}) \subseteq \ell_\infty(\mathbb{Z}) \quad \text{and} \quad \|a\|_{\ell_\infty(\mathbb{Z})} \leq \|a\|_{w\ell_{p,\lambda}(\mathbb{Z})}.$$

Since both inclusions hold, we have

$$w\ell_{p,\lambda}(\mathbb{Z}) = \ell_\infty(\mathbb{Z}) \quad \text{and} \quad \|a\|_{w\ell_{p,\lambda}(\mathbb{Z})} = \|a\|_{\ell_\infty(\mathbb{Z})}.$$

4. Let $0 < \lambda < 1$. Show that $w\ell_{p,\lambda}(\mathbb{Z})$ is not trivial.

Consider the sequence $a = (a_m)_{m \in \mathbb{Z}}$ defined by

$$a_m = \begin{cases} 1, & m = 0, \\ 0, & m \neq 0. \end{cases}$$

The $w\ell_p$ norm of $a = (a_m)_{m \in \mathbb{Z}}$ over the interval $J(j, N)$ is

$$\|a\|_{w\ell_p(J(j,N))} = \sup_{\gamma > 0} \gamma \left(\#\{m \in J(j, N) : |a_m| > \gamma\} \right)^{1/p}.$$

The norm is equal to 0 or 1, depending on whether $0 \in J(j, N)$ or not.

– If $0 \in J(j, N)$ (i.e., $|j| \leq N$), then

$$\begin{aligned} \|a\|_{w\ell_p(J(j,N))} &= \sup_{\gamma > 0} \gamma \left(\#\{m \in J(j, N) : |a_m| > \gamma\} \right)^{1/p} \\ &= \sup_{1 \geq \gamma > 0} \gamma \left(\#\{0\} \right)^{1/p} = 1. \end{aligned}$$

– If $0 \notin J(j, N)$ (i.e., $|j| > N$), then

$$\begin{aligned} \|a\|_{w\ell_p(J(j,N))} &= \sup_{\gamma > 0} \gamma \left(\#\{m \in J(j, N) : |a_m| > \gamma\} \right)^{1/p} \\ &= \sup_{\gamma > 0} \gamma \left(\#\emptyset \right)^{1/p} = 0. \end{aligned}$$

Therefore

$$(\#J(j, N))^{-\lambda/p} \|a\|_{w\ell_p(J(j, N))} = \begin{cases} (2N+1)^{-\lambda/p}, & |j| \leq N, \\ 0, & |j| > N. \end{cases}$$

Hence

$$\|a\|_{w\ell_{p,\lambda}(\mathbb{Z})} = \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} (\#J(j, N))^{-\lambda/p} \|a\|_{w\ell_p(J(j, N))} = \sup_{N \in \mathbb{N}_0} (2N+1)^{-\lambda/p} = 1.$$

Therefore, discrete weak Morrey spaces are nontrivial for $0 < \lambda < 1$:

$$w\ell_{p,\lambda}(\mathbb{Z}) \neq \{\theta\}.$$

Lemma 4. *Let $1 \leq p < \infty$ and $0 \leq \lambda < 1$. Then $(w\ell_{p,\lambda}(\mathbb{Z}), \|a\|_{w\ell_{p,\lambda}(\mathbb{Z})})$ is a quasi-Banach space.*

Proof. Clearly $\|a\|_{w\ell_{p,\lambda}(\mathbb{Z})} \geq 0$ for every $a = (a_m)_{m \in \mathbb{Z}} \in w\ell_{p,\lambda}(\mathbb{Z})$. If $\|a\|_{w\ell_{p,\lambda}(\mathbb{Z})} = 0$ then for every $j \in \mathbb{Z}$, and $N \in \mathbb{N}_0$ we have $\|a\|_{w\ell_p(J(j, N))} = 0$. Thus for every $\gamma > 0$ the set $\{k \in J(j, N) : |a_k| > \gamma\}$ is empty; hence $a_k = 0$ for all k . Therefore, a is a zero sequence.

Conversely, $a = (a_m)_{m \in \mathbb{Z}} = \theta$ implies the quasi-norm is

$$\|a\|_{w\ell_{p,\lambda}(\mathbb{Z})} = \sup_{\gamma > 0} \gamma (\#\{m \in \mathbb{Z} : 0 > \gamma\})^{1/p} = 0.$$

For scalar $\alpha \in \mathbb{R}$, since $\|\alpha a\|_{w\ell_p(J)} = |\alpha| \|a\|_{w\ell_p(J)}$, we get

$$\|\alpha a\|_{w\ell_{p,\lambda}(\mathbb{Z})} = |\alpha| \|a\|_{w\ell_{p,\lambda}(\mathbb{Z})}.$$

Let $a, b \in w\ell_{p,\lambda}(\mathbb{Z})$. For $\gamma > 0$ denote the sets

$$\begin{aligned} E_a(\gamma) &= \{m \in J(j, N) : |a_m| > \gamma\}, \\ E_b(\gamma) &= \{m \in J(j, N) : |b_m| > \gamma\}, \\ E_{a+b}(\gamma) &= \{m \in J(j, N) : |a_m + b_m| > \gamma\}. \end{aligned}$$

Then the following is true

$$\{|a_m + b_m| > \gamma\} \subset \{|a_m| > \gamma/2\} \cup \{|b_m| > \gamma/2\}.$$

So, for every $\gamma > 0$

$$\#E_{a+b}(\gamma) \leq \#E_a(\gamma/2) + \#E_b(\gamma/2).$$

Since the function $u \mapsto u^{1/p}$ is subadditive on $[0, \infty)$ for $p \geq 1$, we have

$$\left(\#E_a(\gamma/2) + \#E_b(\gamma/2)\right)^{1/p} \leq \left(\#E_a(\gamma/2)\right)^{1/p} + \left(\#E_b(\gamma/2)\right)^{1/p}.$$

Hence,

$$\gamma\left(\#E_{a+b}(\gamma)\right)^{1/p} \leq \gamma\left(\#E_a(\gamma/2)\right)^{1/p} + \gamma\left(\#E_b(\gamma/2)\right)^{1/p}.$$

Writing $\gamma = 2(\gamma/2)$, we obtain

$$\gamma\left(\#E_{a+b}(\gamma)\right)^{1/p} \leq 2\left[(\gamma/2)\left(\#E_a(\gamma/2)\right)^{1/p} + (\gamma/2)\left(\#E_b(\gamma/2)\right)^{1/p}\right].$$

Taking the supremum over $\gamma > 0$ yields

$$\|a + b\|_{w\ell_p(J(j,N))} \leq 2(\|a\|_{w\ell_p(J(j,N))} + \|b\|_{w\ell_p(J(j,N))}).$$

Multiplying by $(\#J(j,N))^{-\lambda/p}$ and taking the supremum over $j \in \mathbb{Z}, N \in \mathbb{N}_0$ gives

$$\|a + b\|_{w\ell_{p,\lambda}(\mathbb{Z})} \leq 2(\|a\|_{w\ell_{p,\lambda}(\mathbb{Z})} + \|b\|_{w\ell_{p,\lambda}(\mathbb{Z})}).$$

Hence $\|\cdot\|_{w\ell_{p,\lambda}(\mathbb{Z})}$ is a quasi-norm.

Let us show that $w\ell_{p,\lambda}(\mathbb{Z})$ is a quasi-Banach space. Let $(a^{(n)})_{n \in \mathbb{N}}$ be a Cauchy sequence in $w\ell_{p,\lambda}(\mathbb{Z})$. Then for every $\varepsilon > 0$ there exists N_ε such that for all $n, m \geq N_\varepsilon$,

$$\|a^{(n)} - a^{(m)}\|_{w\ell_{p,\lambda}(\mathbb{Z})} = \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} (\#J(j,N))^{-\lambda/p} \|a^{(n)} - a^{(m)}\|_{w\ell_p(J(j,N))} < \varepsilon.$$

Hence for any interval $J(j,N)$ we obtain

$$\|a^{(n)} - a^{(m)}\|_{w\ell_p(J(j,N))} \leq (\#J(j,N))^{\lambda/p} \varepsilon \quad \text{for all } n, m \geq N_\varepsilon. \quad (6)$$

In particular, taking $N = 0, k \in \mathbb{Z}$ (so $J(k,0) = \{k\}$), by Proposition 1 and (6), we have

$$|a_k^{(n)} - a_k^{(m)}| \leq \|a^{(n)} - a^{(m)}\|_{w\ell_p(J(k,0))} \leq (\#J(k,0))^{\lambda/p} \varepsilon,$$

for all $n, m \geq N_\varepsilon$.

Hence we conclude that for each fixed $k \in \mathbb{Z}$ the scalar sequence $(a_k^{(n)})_n$ is Cauchy in \mathbb{R} and hence converges in \mathbb{R} . Define $a = (a_k)_{k \in \mathbb{Z}}$ by

$$a_k := \lim_{n \rightarrow \infty} a_k^{(n)}, \quad j \in \mathbb{Z}.$$

We now show $a \in w\ell_{p,\lambda}(\mathbb{Z})$ and $a^{(n)} \rightarrow a$ in the weak discrete Morrey quasi-norm.

For $\gamma > 0$, define

$$E(\gamma) := \{k \in J(j, N) : |a_k^{(n)} - a_k| > \gamma\}, \quad E_m(\gamma) := \{k \in J(j, N) : |a_k^{(n)} - a_k^{(m)}| > \gamma\}.$$

Let $k \in E(\gamma)$. Then

$$|a_k^{(n)} - a_k| > \gamma.$$

Set

$$\delta_k := |a_k^{(n)} - a_k| - \gamma > 0.$$

Since $a_k^{(m)} \rightarrow a_k$ as $m \rightarrow \infty$, there exists $m_k \in \mathbb{N}$ such that

$$|a_k^{(m)} - a_k| < \delta_k \quad \text{for all } m \geq m_k.$$

Hence, for all $m \geq m_k$, by the reverse triangle inequality,

$$|a_k^{(n)} - a_k^{(m)}| \geq |a_k^{(n)} - a_k| - |a_k - a_k^{(m)}| > |a_k^{(n)} - a_k| - \delta_k = \gamma.$$

Thus $k \in E_m(\gamma)$ for all $m \geq m_k$.

Since $E(\gamma) \subseteq J(j, N)$ and $J(j, N)$ is finite, the set $E(\gamma)$ is finite. Hence we may define

$$m_0 := \max\{m_i : i \in E(\gamma)\}.$$

Then for every $m \geq m_0$,

$$E(\gamma) \subseteq E_m(\gamma).$$

Therefore,

$$\#E(\gamma) \leq \#E_m(\gamma) \quad \text{for all } m \geq m_0.$$

Multiplying by γ and taking the $1/p$ -power, we obtain

$$\gamma(\#E(\gamma))^{1/p} \leq \gamma(\#E_m(\gamma))^{1/p} \quad \text{for all } m \geq m_0.$$

Taking the supremum over $\gamma > 0$, we get

$$\|a^{(n)} - a\|_{w\ell_p(J(j,N))} \leq \|a^{(n)} - a^{(m)}\|_{w\ell_p(J(j,N))} \quad \text{for all } m \geq m_0.$$

By (6), it follows that

$$\|a^{(n)} - a\|_{w\ell_p(J(j,N))} \leq (\#J(j, N))^{\lambda/p} \varepsilon.$$

Multiplying by $(\#J(j, N))^{-\lambda/p}$ and taking the supremum over $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$, we obtain

$$\|a^{(n)} - a\|_{w\ell_{p,\lambda}(\mathbb{Z})} \leq \varepsilon \quad \text{for all } n \geq N_\varepsilon.$$

Hence $a^{(n)} \rightarrow a$ in $w\ell_{p,\lambda}(\mathbb{Z})$.

Finally, since

$$\|a\|_{w\ell_{p,\lambda}(\mathbb{Z})} \leq 2 \left(\|a - a^{(n)}\|_{w\ell_{p,\lambda}(\mathbb{Z})} + \|a^{(n)}\|_{w\ell_{p,\lambda}(\mathbb{Z})} \right),$$

and the right-hand side is finite for sufficiently large n , we conclude that $a \in w\ell_{p,\lambda}(\mathbb{Z})$.

Therefore every Cauchy sequence converges in $w\ell_{p,\lambda}(\mathbb{Z})$, and thus the space is a quasi-Banach space.

Remark 2. Lemma 4 was proved in [12]. For the sake of completeness, we include the proof in our setting.

Theorem 2. Let $1 \leq p_1 \leq p_2 < \infty$ and $0 \leq \lambda_1, \lambda_2 < 1$. Then

$$\frac{1 - \lambda_1}{p_1} \leq \frac{1 - \lambda_2}{p_2}. \quad (7)$$

is a necessary and sufficient condition for

$$w\ell_{p_1,\lambda_1}(\mathbb{Z}) \hookrightarrow w\ell_{p_2,\lambda_2}(\mathbb{Z}) \quad \text{with} \quad \|a\|_{w\ell_{p_1,\lambda_1}} \leq \|a\|_{w\ell_{p_2,\lambda_2}} \quad (8)$$

for any $a \in w\ell_{p_2,\lambda_2}(\mathbb{Z})$ to hold.

Proof. Sufficiency. Assume (7) holds and set

$$\alpha := \frac{1 - \lambda_1}{p_1} - \frac{1 - \lambda_2}{p_2} \leq 0.$$

Let $a = (a_m)_{m \in \mathbb{Z}} \in w\ell_{p_2,\lambda_2}(\mathbb{Z})$. Fix $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$. For $\gamma > 0$ set

$$E_\gamma := \{m \in J(j, N) : |a_m| > \gamma\}.$$

Then

$$\gamma (\#E_\gamma)^{\frac{1}{p_1}} = \gamma (\#E_\gamma)^{\frac{1}{p_2}} (\#E_\gamma)^{\frac{1}{p_1} - \frac{1}{p_2}}.$$

Since $\#E_\gamma \leq \#J(j, N)$ and $\frac{1}{p_1} - \frac{1}{p_2} \geq 0$, we have $(\#E_\gamma)^{\frac{1}{p_1} - \frac{1}{p_2}} \leq (\#J(j, N))^{\frac{1}{p_1} - \frac{1}{p_2}}$. Thus for every $\gamma > 0$,

$$\gamma (\#E_\gamma)^{\frac{1}{p_1}} \leq \gamma (\#E_\gamma)^{\frac{1}{p_2}} (\#J(j, N))^{\frac{1}{p_1} - \frac{1}{p_2}}.$$

Taking the supremum over $\gamma > 0$ on both sides yields

$$\|a\|_{w\ell_{p_1}(J(j,N))} \leq (\#J(j, N))^{\frac{1}{p_1} - \frac{1}{p_2}} \|a\|_{w\ell_{p_2}(J(j,N))}. \quad (9)$$

We multiply both sides of (9) by $(\#J(j, N))^{-\frac{\lambda_1}{p_1}}$:

$$(\#J(j, N))^{-\frac{\lambda_1}{p_1}} \|a\|_{w\ell_{p_1}(J(j, N))} \leq (\#J(j, N))^{-\frac{\lambda_1}{p_1} + \frac{1}{p_1} - \frac{1}{p_2}} \|a\|_{w\ell_{p_2}(J(j, N))}.$$

Then combine exponents on the right:

$$-\frac{\lambda_1}{p_1} + \frac{1}{p_1} - \frac{1}{p_2} = \frac{1 - \lambda_1}{p_1} - \frac{1}{p_2} = \underbrace{\left(\frac{1 - \lambda_1}{p_1} - \frac{1 - \lambda_2}{p_2}\right)}_{\alpha} + \left(\frac{1 - \lambda_2}{p_2} - \frac{1}{p_2}\right) = \alpha - \frac{\lambda_2}{p_2}.$$

Therefore

$$(\#J(j, N))^{-\frac{\lambda_1}{p_1}} \|a\|_{w\ell_{p_1}(J(j, N))} \leq (\#J(j, N))^\alpha (\#J(j, N))^{-\frac{\lambda_2}{p_2}} \|a\|_{w\ell_{p_2}(J(j, N))}.$$

Since $\alpha \leq 0$, we have $(\#J(j, N))^\alpha = (2N + 1)^\alpha \leq 1$, and so

$$(\#J(j, N))^{-\frac{\lambda_1}{p_1}} \|a\|_{w\ell_{p_1}(J(j, N))} \leq (\#J(j, N))^{-\frac{\lambda_2}{p_2}} \|a\|_{w\ell_{p_2}(J(j, N))}.$$

Taking the supremum over all $j \in \mathbb{Z}$, $N \in \mathbb{N}_0$ yields

$$\|a\|_{w\ell_{p_1, \lambda_1}} \leq \|a\|_{w\ell_{p_2, \lambda_2}}.$$

This proves the sufficiency of (7).

Necessity. We use proof by contradiction. Suppose (7) fails, i.e.

$$\alpha := \frac{1 - \lambda_1}{p_1} - \frac{1 - \lambda_2}{p_2} > 0.$$

We construct a sequence $a \in w\ell_{p_2, \lambda_2}(\mathbb{Z})$ for which $a \notin w\ell_{p_1, \lambda_1}(\mathbb{Z})$.

Define a sequence $a = (a_m)_{m \in \mathbb{Z}}$ by

$$a_m := \begin{cases} 0, & m = 0, \\ |m|^{-\beta}, & m \neq 0 \end{cases} \quad \text{with} \quad \beta := \frac{1 - \lambda_2}{p_2}$$

Step 1. Show that $a \in w\ell_{p_2, \lambda_2}(\mathbb{Z})$. Fix $j \in \mathbb{Z}$, $N \in \mathbb{N}_0$. For every number $\gamma > 0$ define

$$A(\gamma) := \{m \in J(j, N) : |a_m| > \gamma\}.$$

Since $|a_m| = |m|^{-\beta}$ for $m \neq 0$, the condition $|a_m| > \gamma$ is equivalent to $|m| < \gamma^{-1/\beta}$. Hence

$$\{m \in \mathbb{Z} : |a_m| > \gamma\} = \{m \in \mathbb{Z} : 0 < |m| < \gamma^{-1/\beta}\},$$

so its cardinality satisfies

$$\#\{m \in \mathbb{Z} : |a_m| > \gamma\} \leq 2\gamma^{-1/\beta}.$$

Since

$$A(\gamma) = \{m \in J(j, N) : |a_m| > \gamma\} = J(j, N) \cap \{m \in \mathbb{Z} : |a_m| > \gamma\},$$

then

$$\#A(\gamma) \leq \min\{2N + 1, 2\gamma^{-1/\beta}\}. \quad (10)$$

We want to estimate

$$\|a\|_{w\ell_{p_2}(J(j, N))} = \sup_{\gamma > 0} \gamma(\#A(\gamma))^{1/p_2}.$$

Now consider two cases:

Case 1: $2\gamma^{-1/\beta} \geq 2N + 1$, equivalently

$$\gamma \leq \left(\frac{2}{2N + 1}\right)^\beta.$$

Then from (10) we obtain $\#A(\gamma) \leq 2N + 1$. Hence

$$\gamma(\#A(\gamma))^{1/p_2} \leq \gamma(2N + 1)^{1/p_2}.$$

The supremum over $\left(\frac{2}{2N + 1}\right)^\beta \geq \gamma > 0$ is

$$\begin{aligned} \sup_{\left(\frac{2}{2N + 1}\right)^\beta \geq \gamma > 0} \gamma(\#A(\gamma))^{1/p_2} &\leq \sup_{\left(\frac{2}{2N + 1}\right)^\beta \geq \gamma > 0} \gamma(2N + 1)^{1/p_2} \\ &\leq \left(\frac{2}{2N + 1}\right)^\beta (2N + 1)^{1/p_2} = 2^\beta (2N + 1)^{1/p_2 - \beta}. \end{aligned} \quad (11)$$

Case 2: $2\gamma^{-1/\beta} < 2N + 1$, equivalently

$$\gamma > \left(\frac{2}{2N + 1}\right)^\beta.$$

Then from (10) we have

$$\#A(\gamma) \leq 2\gamma^{-1/\beta},$$

so

$$\gamma(\#A(\gamma))^{1/p_2} \leq \gamma(2\gamma^{-1/\beta})^{1/p_2} = 2^{1/p_2} \gamma^{1 - \frac{1}{\beta p_2}}.$$

By our choice we obtain

$$1 - \frac{1}{\beta p_2} = 1 - \frac{1}{\frac{1-\lambda_2}{p_2} p_2} = 1 - \frac{1}{1-\lambda_2} < 0.$$

Thus the function $\gamma \mapsto \gamma^{1-\frac{1}{\beta p_2}}$ is decreasing, and so

$$\sup_{\gamma > (\frac{2}{2N+1})^\beta} \gamma (\#A(\gamma))^{1/p_2} \leq 2^{1/p_2} \left(\frac{2}{2N+1} \right)^{\beta(1-\frac{1}{\beta p_2})} = 2^\beta (2N+1)^{1/p_2-\beta}. \quad (12)$$

From (11) and (12) it follows that

$$\begin{aligned} \sup_{\gamma > 0} \gamma (\#A(\gamma))^{1/p_2} &\leq \max \left\{ \sup_{(\frac{2}{2N+1})^\beta \geq \gamma > 0} \gamma (\#A(\gamma))^{1/p_2}, \sup_{\gamma > (\frac{2}{2N+1})^\beta} \gamma (\#A(\gamma))^{1/p_2} \right\} \\ &= 2^\beta (2N+1)^{1/p_2-\beta}. \end{aligned}$$

Multiplying by $(\#J(j, N))^{-\lambda_2/p_2} = (2N+1)^{-\lambda_2/p_2}$ we obtain

$$\begin{aligned} (\#J(j, N))^{-\lambda_2/p_2} \sup_{\gamma > 0} \gamma (\#A(\gamma))^{1/p_2} &\leq (2N+1)^{-\lambda_2/p_2} 2^\beta (2N+1)^{1/p_2-\beta} \\ &= 2^\beta (2N+1)^{1/p_2-\beta-\lambda_2/p_2} = 2^\beta (2N+1)^{1/p_2-\frac{1-\lambda_2}{p_2}-\lambda_2/p_2} = 2^\beta (2N+1)^0 = 2^\beta. \end{aligned}$$

Taking the supremum over all $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$ we conclude

$$\|a\|_{w\ell_{p_2, \lambda_2}} = \sup_{j \in \mathbb{Z}, N \in \mathbb{N}_0} (\#J(j, N))^{-\lambda_2/p_2} \|a\|_{w\ell_{p_2}(J(j, N))} \leq 2^\beta < \infty.$$

Therefore $a \in w\ell_{p_2, \lambda_2}(\mathbb{Z})$.

Step 2. Let us show that $a \notin w\ell_{p_1, \lambda_1}(\mathbb{Z})$. Consider $j = 0$ and $N \in \mathbb{N}_0$. For $\gamma > 0$ set

$$A_0(\gamma) := \{m \in J(0, N) : |a_m| > \gamma\}.$$

Since $|a_m| = |m|^{-\beta}$ for $m \neq 0$,

$$A_0(\gamma) = \{m \in J(0, N) : |m|^{-\beta} > \gamma\}.$$

Choose $\gamma_N := (N+1)^{-\beta}$. Then

$$A_0(\gamma_N) = \{m \in J(0, N) : 0 < |m| < (N+1)\},$$

and therefore

$$\#A_0(\gamma_N) = 2N - 2.$$

Consequently

$$\gamma_N(\#A_0(\gamma_N))^{1/p_1} = (N+1)^{-\beta}(2N)^{1/p_1} = \frac{(2N)^{1/p_1}}{(N+1)^\beta}.$$

Hence, for any $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$ we obtain

$$\|a\|_{w\ell_p(J(j,N))} = \sup_{\gamma>0} \gamma \left(\#\{m \in J(j,N) : |a_m| > \gamma\}^{1/p_1} \right) \geq \frac{(2N)^{1/p_1}}{(N+1)^\beta}.$$

Multiplying both sides by $(\#J(j,N))^{-\lambda_1/p_1} = (2N+1)^{-\lambda_1/p_1}$ we obtain

$$\begin{aligned} (\#J(j,N))^{-\lambda_1/p_1} \|a\|_{w\ell_p(J(j,N))} &\geq (2N+1)^{-\lambda_1/p_1} \frac{(2N)^{1/p_1}}{(N+1)^\beta} \\ &= \frac{(2N)^{1/p_1}}{(N+1)^\beta (2N+1)^{\lambda_1/p_1}}. \end{aligned}$$

Since

$$\frac{1}{p_1} - \beta - \frac{\lambda_1}{p_1} = \frac{1}{p_1} - \frac{1-\lambda_2}{p_2} - \frac{\lambda_1}{p_1} = \frac{1-\lambda_1}{p_1} - \frac{1-\lambda_2}{p_2} = \alpha > 0,$$

then taking the supremum over all $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$ gives $\|a\|_{w\ell_{p_1,\lambda_1}(\mathbb{Z})} = \infty$. This contradicts (8). Thus (7) must hold.

Combining sufficiency and necessity, we conclude that (7) is the necessary and sufficient condition for (8) to be valid.

Corollary 2. *Let $1 \leq p < \infty$. Then $0 \leq \lambda_2 \leq \lambda_1 < 1$ is a necessary and sufficient condition for*

$$w\ell_{p,\lambda_1}(\mathbb{Z}) \hookrightarrow w\ell_{p,\lambda_2}(\mathbb{Z}) \text{ with } \|a\|_{w\ell_{p,\lambda_1}(\mathbb{Z})} \leq \|a\|_{w\ell_{p,\lambda_2}(\mathbb{Z})}$$

for any $a \in w\ell_{p,\lambda_2}(\mathbb{Z})$ to hold.

4. Discrete Hilbert transform

In this section the main result is strong type boundedness of the discrete Hilbert transform on discrete Morrey space $\ell_{p,\lambda}(\mathbb{Z})$ for $1 < p < \infty$, and weak-type $(1,1)$ boundedness of the Hilbert transform from discrete Morrey space $\ell_{1,\lambda}(\mathbb{Z})$ to weak discrete Morrey space $w\ell_{1,\lambda}(\mathbb{Z})$.

For any sequence $a = (a_n)_{n \in \mathbb{Z}}$, the discrete Hilbert transform H is defined by

$$\mathcal{H}a_n = \frac{1}{\pi} \sum_{m \in \mathbb{Z}, m \neq n} \frac{a_m}{n - m}.$$

This operator was introduced by D. Hilbert in the first decade of 20th century.

We first present the theorems from [4]

Theorem 3. [4] *Let $a = (a_n)_{n \in \mathbb{Z}} \in \ell_p(\mathbb{Z})$ with $1 < p < \infty$. Then*

$$\|\mathcal{H}a\|_{\ell_p(\mathbb{Z})} \leq \cot\left(\frac{\pi}{2p^*}\right) \|a\|_{\ell_p(\mathbb{Z})},$$

where $p^* := \max\left\{p, \frac{p}{p-1}\right\}$. Consequently, $\|\mathcal{H}\|_{\ell_p(\mathbb{Z}) \rightarrow \ell_p(\mathbb{Z})} \leq \cot\left(\frac{\pi}{2p^*}\right)$.

Theorem 4. [4] *Let $a = (a_n)_{n \in \mathbb{Z}} \in \ell_1(\mathbb{Z})$. Then*

$$\|\mathcal{H}a\|_{w\ell_1(\mathbb{Z})} \leq D \|a\|_{\ell_1(\mathbb{Z})},$$

where $D := \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \dots}$. Consequently, $\|\mathcal{H}\|_{\ell_1(\mathbb{Z}) \rightarrow w\ell_1(\mathbb{Z})} \leq D$.

The following theorem, which provides local estimates for the discrete Hilbert transform, serves as a fundamental tool in establishing our main results. We note that this local estimate is a discrete analogue of the corresponding result obtained by the second author in [6]; see also [8].

Theorem 5. *Let $1 \leq p < \infty$.*

(i) *If $1 < p < \infty$, then for $a = (a_n)_{n \in \mathbb{Z}} \in \ell_p^{\text{loc}}(\mathbb{Z})$ and $(j, N) \in \mathbb{Z} \times \mathbb{N}_0$, then*

$$\|\mathcal{H}a\|_{\ell_p(J(j, N))} \leq C_p (\#J(j, N))^{1/p} \sum_{n > N} \frac{\|a\|_{\ell_p(J(j, n))}}{(\#J(j, n))^{1 + \frac{1}{p}}},$$

where $C_p = \frac{2 \cdot 3^{\frac{1}{p}}}{p} \cot\left(\frac{\pi}{2p^*}\right) + 2^{\frac{1}{p}} \frac{60}{\pi}$.

(ii) *Moreover, if $p = 1$, then for $a = (a_n)_{n \in \mathbb{Z}} \in \ell_1^{\text{loc}}(\mathbb{Z})$ and $(j, N) \in \mathbb{Z} \times \mathbb{N}_0$, then*

$$\|\mathcal{H}a\|_{w\ell_1(J(j, N))} \leq C_1 \#J(j, N) \sum_{n > N} \frac{\|a\|_{\ell_1(J(j, n))}}{(\#J(j, n))^2},$$

where $C_1 = 6D + \frac{120}{\pi}$.

Proof. Let $a = (a_k)_{k \in \mathbb{Z}} \in \ell_p^{\text{loc}}(\mathbb{Z})$. For $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$, we write a as

$$a = a^{(1)} + a^{(2)}, \quad a^{(1)} = a \cdot \chi_{J(j, 2N+1)}, \quad a^{(2)} = a \cdot \chi_{J(j, 2N+1)^C},$$

where $J(j, 2N+1)^C = \mathbb{Z} \setminus J(j, 2N+1)$. In other words

$$a_m = a_m^{(1)} + a_m^{(2)}, \quad a_m^{(1)} = a_m \cdot \chi_{J(j, 2N+1)}, \quad a_m^{(2)} = a_m \cdot \chi_{J(j, 2N+1)^C}.$$

Then by linearity of \mathcal{H} , for every $k \in \mathbb{Z}$,

$$\begin{aligned} \mathcal{H}a_k &= \frac{1}{\pi} \sum_{m \in \mathbb{Z}, m \neq k} \frac{a_m}{k-m} \\ &= \frac{1}{\pi} \sum_{m \in \mathbb{Z}, m \neq k} \frac{a_m^{(1)}}{k-m} + \frac{1}{\pi} \sum_{m \in \mathbb{Z}, m \neq k} \frac{a_m^{(2)}}{k-m} = \mathcal{H}a_k^{(1)} + \mathcal{H}a_k^{(2)}. \end{aligned}$$

Hence by the triangle inequality for all $1 \leq p < \infty$ the following inequality

$$\|\mathcal{H}a\|_{\ell_p(J(j, N))} \leq \|\mathcal{H}a^{(1)}\|_{\ell_p(J(j, N))} + \|\mathcal{H}a^{(2)}\|_{\ell_p(J(j, N))}$$

is valid.

Step 1. Since $\#J(j, n) = 2n + 1$, for $1 \leq p < \infty$,

$$\sum_{n \geq 2N+1} \frac{1}{(\#J(j, n))^{1+\frac{1}{p}}} = \sum_{n \geq 2N+1} \frac{1}{(2n+1)^{1+\frac{1}{p}}} \geq \int_{2N+1}^{\infty} \frac{dx}{(2x+1)^{1+\frac{1}{p}}} = \frac{p}{2} (4N+3)^{-\frac{1}{p}}.$$

Thus

$$\sum_{n \geq 2N+1} \frac{1}{(\#J(j, n))^{1+\frac{1}{p}}} \geq \frac{p}{2(\#J(j, 2N+1))^{\frac{1}{p}}}.$$

Using monotonicity $\|a\|_{\ell_p(J(j, n))} \geq \|a\|_{\ell_p(J(j, 2N+1))}$ for $n \geq 2N+1$, we obtain

$$\begin{aligned} \sum_{n > N} \frac{\|a\|_{\ell_p(J(j, n))}}{(\#J(j, n))^{1+\frac{1}{p}}} &\geq \|a\|_{\ell_p(J(j, 2N+1))} \sum_{n \geq 2N+1} \frac{1}{(\#J(j, n))^{1+\frac{1}{p}}} \\ &\geq \frac{p}{2(\#J(j, 2N+1))^{\frac{1}{p}}} \|a\|_{\ell_p(J(j, 2N+1))}. \end{aligned}$$

Therefore

$$\|a\|_{\ell_p(J(j, 2N+1))} \leq \frac{2}{p} (\#J(j, 2N+1))^{\frac{1}{p}} \sum_{n > N} \frac{\|a\|_{\ell_p(J(j, n))}}{(\#J(j, n))^{1+\frac{1}{p}}}. \quad (13)$$

Since $\#J(j, 2N+1) = 4N+3 \leq 3(2N+1) = 3\#J(j, N)$, we have $(\#J(j, 2N+1))^{1/p} \leq 3^{1/p}(\#J(j, N))^{1/p}$. Then from (13) we obtain

$$\|a\|_{\ell_p(J(j, 2N+1))} \leq \frac{2 \cdot 3^{\frac{1}{p}}}{p} (\#J(j, N))^{1/p} \sum_{n>N} \frac{\|a\|_{\ell_p(J(j, n))}}{(\#J(j, n))^{1+\frac{1}{p}}}. \quad (14)$$

Step 2. Estimation for $\mathcal{H}a^{(1)}$.

Case $1 < p < \infty$. By the $\ell_p(\mathbb{Z})$ boundedness of \mathcal{H} in Theorem 3 we have

$$\|\mathcal{H}a^{(1)}\|_{\ell_p(J(j, N))} \leq \|\mathcal{H}a^{(1)}\|_{\ell_p(\mathbb{Z})} \leq \cot\left(\frac{\pi}{2p^*}\right) \|a^{(1)}\|_{\ell_p(\mathbb{Z})} = \cot\left(\frac{\pi}{2p^*}\right) \|a\|_{\ell_p(J(j, 2N+1))}.$$

Combining this inequality with (14), we obtain

$$\|\mathcal{H}a^{(1)}\|_{\ell_p(J(j, N))} \leq \frac{2 \cdot 3^{\frac{1}{p}}}{p} \cot\left(\frac{\pi}{2p^*}\right) (\#J(j, N))^{\frac{1}{p}} \sum_{n>N} \frac{\|a\|_{\ell_p(J(j, n))}}{(\#J(j, n))^{1+\frac{1}{p}}}. \quad (15)$$

Case $p = 1$. By the weak-type $(1, 1)$ boundedness of \mathcal{H} in Theorem 4 we have

$$\|\mathcal{H}a^{(1)}\|_{w\ell_1(J(j, N))} \leq \|\mathcal{H}a^{(1)}\|_{w\ell_1(\mathbb{Z})} \leq D \|a^{(1)}\|_{\ell_1(\mathbb{Z})} = D \|a\|_{\ell_1(J(j, 2N+1))}.$$

Now applying (14) with $p = 1$, we obtain

$$\|\mathcal{H}a^{(1)}\|_{w\ell_1(J(j, N))} \leq 6D (\#J(j, N)) \sum_{n>N} \frac{\|a\|_{\ell_1(J(j, n))}}{(\#J(j, n))^2}. \quad (16)$$

Step 3. Estimation for $\mathcal{H}a^{(2)}$.

For $k \in J(j, N)$ and $m \in J(j, 2N+1)^c$ we have

$$\frac{1}{2}|m-j| \leq |k-m| \leq \frac{3}{2}|m-j|,$$

hence

$$|\mathcal{H}a_k^{(2)}| \leq \sum_{m \in \mathbb{Z} \setminus \{k\}} \frac{|a_m^{(2)}|}{|k-m|} = \sum_{m \in J(j, 2N+1)^c} \frac{|a_m|}{|k-m|} \leq 2 \sum_{m \in J(j, 2N+1)^c} \frac{|a_m|}{|m-j|}.$$

Let

$$A_r(j, N) := \{m \in \mathbb{Z} : 2^r(2N+1) < |m-j| \leq 2^{r+1}(2N+1)\}, \quad r \in \mathbb{N}_0.$$

Then $|m - j| > 2^r(2N + 1)$ for $m \in A_r(j, N)$, so

$$\begin{aligned}
|\mathcal{H}a_k^{(2)}| &\leq \frac{2}{\pi} \sum_{r=0}^{\infty} \left[\sum_{m \in A_r(j, N)} \frac{|a_m|}{|m - j|} \right] \leq \frac{2}{\pi} \sum_{r=0}^{\infty} \frac{1}{2^r(2N + 1)} \sum_{m \in A_r(j, N)} |a_m| \\
&= \frac{2}{\pi} \sum_{r=0}^{\infty} \frac{1}{2^r(2N + 1)} \sum_{m \in J(j, 2^{r+1}(2N+1))} |a_m| \\
&= \frac{2}{\pi} \sum_{r=0}^{\infty} \frac{1}{2^r(2N + 1)} \sum_{m \in J(j, R_r(N))} |a_m|. \tag{17}
\end{aligned}$$

where $R_r(N) := 2^{r+1}(2N + 1)$.

The following estimation is true:

$$\#J(j, R_r(N)) = 2R_r(N) + 1 = 2^{r+2}(2N + 1) + 1 \leq 5 \cdot 2^r(2N + 1) \tag{18}$$

Case $1 < p < \infty$. By Hölder inequality,

$$\begin{aligned}
\sum_{m \in J(j, R_r(N))} |a_m| &\leq \|a\|_{\ell_p(J(j, R_r(N)))} \|1\|_{\ell_{p'}(J(j, R_r(N)))} \\
&= \|a\|_{\ell_p(J(j, R_r(N)))} (\#J(j, R_r(N)))^{\frac{1}{p'}}.
\end{aligned}$$

Since

$$\frac{(\#J(j, R_r(N)))^{\frac{1}{p'}}}{2^r(2N + 1)} \leq 5 \frac{(\#J(j, R_r(N)))^{\frac{1}{p'}}}{\#J(j, R_r(N))} = 5 (\#J(j, R_r(N)))^{-\frac{1}{p'}}.$$

then, for every $k \in J(j, N)$,

$$|\mathcal{H}a_k^{(2)}| \leq \frac{10}{\pi} \sum_{r=0}^{\infty} \frac{\|a\|_{\ell_p(J(j, R_r(N)))}}{(\#J(j, R_r(N)))^{\frac{1}{p}}}. \tag{19}$$

For each $r \in \mathbb{N}_0$ let

$$I_r(N) := \{n \in \mathbb{N} : R_r(N) \leq n < 2R_r(N)\}.$$

For $n \in I_r(N)$ we have $\|a\|_{\ell_p(J(j, n))} \geq \|a\|_{\ell_p(J(j, R_r(N)))}$, $\#I_r(N) \geq R_r(N)$ and $\#J \leq \#J(j, 2R_r(N)) = 2 \cdot (2R_r(N)) + 1 \leq 2(2R_r(N) + 1) = 2(\#J(j, R_r(N)))$, so

$$\sum_{n \in I_r(N)} \frac{\|a\|_{\ell_p(J(j, n))}}{(\#J(j, n))^{1+\frac{1}{p}}} \geq \#I_r(N) \cdot \frac{\|a\|_{\ell_p(J(j, R_r(N)))}}{(\#J(j, 2R_r(N)))^{1+\frac{1}{p}}} \geq R_r(N) \cdot \frac{\|a\|_{\ell_p(J(j, R_r(N)))}}{(2(\#J(j, R_r(N))))^{1+\frac{1}{p}}}.$$

Since

$$\begin{aligned} R_r(N) \cdot \frac{\|a\|_{\ell_p(J(j, R_r(N)))}}{(2 \#J(j, R_r(N)))^{1+\frac{1}{p}}} &= \frac{R_r(N)}{2R_r(N) + 1} \cdot \frac{\|a\|_{\ell_p(J(j, R_r(N)))}}{2^{1+\frac{1}{p}} (\#J(j, R_r(N)))^{\frac{1}{p}}} \\ &\geq \frac{1}{3} \cdot \frac{\|a\|_{\ell_p(J(j, R_r(N)))}}{2^{1+\frac{1}{p}} (\#J(j, R_r(N)))^{\frac{1}{p}}}. \end{aligned}$$

Then

$$\frac{\|a\|_{\ell_p(J(j, R_r(N)))}}{(\#J(j, R_r(N)))^{\frac{1}{p}}} \leq 6 \cdot 2^{\frac{1}{p}} \sum_{n \in I_r(N)} \frac{\|a\|_{\ell_p(J(j, n))}}{(\#J(j, n))^{1+\frac{1}{p}}} \quad (20)$$

Summing over $r \in \mathbb{N}_0$ (the sets $I_r(N)$ are disjoint and cover $\{n > 2N + 1\}$) gives

$$\begin{aligned} \sum_{r=0}^{\infty} \frac{\|a\|_{\ell_p(J(j, R_r(N)))}}{(\#J(j, R_r(N)))^{\frac{1}{p}}} &\leq 6 \cdot 2^{\frac{1}{p}} \sum_{r=0}^{\infty} \sum_{n \in I_r(N)} \frac{\|a\|_{\ell_p(J(j, n))}}{(\#J(j, n))^{1+\frac{1}{p}}} \\ &= 6 \cdot 2^{\frac{1}{p}} \sum_{r=0}^{\infty} \sum_{n \in \mathbb{N}} \chi_{I_r(N)}(n) \frac{\|a\|_{\ell_p(J(j, n))}}{(\#J(j, n))^{1+\frac{1}{p}}} \\ &= 6 \cdot 2^{\frac{1}{p}} \sum_{n \in \mathbb{N}} \frac{\|a\|_{\ell_p(J(j, n))}}{(\#J(j, n))^{1+\frac{1}{p}}} \sum_{r=0}^{\infty} \chi_{I_r(N)}(n) \\ &= 6 \cdot 2^{\frac{1}{p}} \sum_{n \in \mathbb{N}} \frac{\|a\|_{\ell_p(J(j, n))}}{(\#J(j, n))^{1+\frac{1}{p}}} \chi_{\cup_{r \geq 0} I_r(N)}(n) \\ &\leq 6 \cdot 2^{\frac{1}{p}} \sum_{n > 2N+1} \frac{\|a\|_{\ell_p(J(j, n))}}{(\#J(j, n))^{1+\frac{1}{p}}} \leq 6 \cdot 2^{\frac{1}{p}} \sum_{n > N} \frac{\|a\|_{\ell_p(J(j, n))}}{(\#J(j, n))^{1+\frac{1}{p}}}. \quad (21) \end{aligned}$$

We insert this into (19),

$$|\mathcal{H}a_k^{(2)}| \leq 2^{\frac{1}{p}} \frac{60}{\pi} \sum_{n > N} \frac{\|a\|_{\ell_p(J(j, n))}}{(\#J(j, n))^{1+\frac{1}{p}}}.$$

Taking the ℓ_p norm over $k \in J(j, N)$ (the bound is independent of k) yields

$$\|\mathcal{H}a^{(2)}\|_{\ell_p(J(j, N))} \leq 2^{\frac{1}{p}} \frac{60}{\pi} (\#J(j, N))^{\frac{1}{p}} \sum_{n > N} \frac{\|a\|_{\ell_p(J(j, n))}}{(\#J(j, n))^{1+\frac{1}{p}}}. \quad (22)$$

Case $p = 1$. For $k \in J(j, N)$, from (17) and (18) it follows that

$$|\mathcal{H}a_k^{(2)}| \leq \frac{2}{\pi} \sum_{r=0}^{\infty} \frac{1}{2^r (2N+1)} \sum_{m \in J(j, R_r(N))} |a_m|$$

$$\begin{aligned}
 &= \frac{2}{\pi} \sum_{r=0}^{\infty} \frac{1}{2^r(2N+1)} \|a\|_{\ell_1(J(j,R_r(N)))} \\
 &\leq \frac{10}{\pi} \sum_{r=0}^{\infty} \frac{\|a\|_{\ell_1(J(j,R_r(N)))}}{\#J(j,R_r(N))}.
 \end{aligned}$$

It is obvious that (20) and (21) work for $p = 1$. Then

$$|\mathcal{H}a_k^{(2)}| \leq \frac{120}{\pi} \sum_{n>N} \frac{\|a\|_{\ell_1(J(j,n))}}{(\#J(j,n))^2}.$$

Therefore

$$\|\mathcal{H}a^{(2)}\|_{\ell_1(J(j,N))} \leq \frac{120}{\pi} (\#J(j,N)) \sum_{n>N} \frac{\|a\|_{\ell_1(J(j,n))}}{(\#J(j,n))^2},$$

and since $\|b\|_{w\ell_1(J)} \leq \|b\|_{\ell_1(J)}$ for any sequence $b = (b_m)_{m \in J} \in \ell_p(J)$,

$$\|H(a^{(2)})\|_{w\ell_1(J(j,N))} \leq \frac{120}{\pi} (\#J(j,N)) \sum_{n>N} \frac{\|a\|_{\ell_1(J(j,n))}}{(\#J(j,n))^2}. \tag{23}$$

Step 4. Conclusion.

If $1 < p < \infty$, combine (15) and (22) to get

$$\|\mathcal{H}a\|_{\ell_p(J(j,N))} \leq \left(\frac{2 \cdot 3^{\frac{1}{p}}}{p} \cot\left(\frac{\pi}{2p^*}\right) + 2^{\frac{1}{p}} \frac{60}{\pi} \right) (\#J(j,N))^{1/p} \sum_{n>N} \frac{\|a\|_{\ell_p(J(j,n))}}{(\#J(j,n))^{1+\frac{1}{p}}}.$$

If $p = 1$, combine (16) and (23) to obtain

$$\|\mathcal{H}a\|_{w\ell_1(J(j,N))} \leq \left(6D + \frac{120}{\pi} \right) (\#J(j,N)) \sum_{n>N} \frac{\|a\|_{\ell_1(J(j,n))}}{(\#J(j,n))^2}.$$

The proof was completed.

The following theorem is our main result in this section regarding the boundedness of the discrete Hilbert transform on the discrete Morrey space over \mathbb{Z} .

Theorem 6. *Let $1 \leq p < \infty$ and $0 \leq \lambda < 1$. Then, for $p > 1$ the discrete Hilbert transform \mathcal{H} is bounded on $\ell_{p,\lambda}(\mathbb{Z})$. Moreover,*

$$\|\mathcal{H}a\|_{\ell_{p,\lambda}(\mathbb{Z})} \leq \frac{pC_p}{2(1-\lambda)} \|a\|_{\ell_{p,\lambda}(\mathbb{Z})},$$

where $C_p = \frac{2 \cdot 3^{\frac{1}{p}}}{p} \cot\left(\frac{\pi}{2p^*}\right) + 2^{\frac{1}{p}} \frac{60}{\pi}$.

Also, for $p = 1$ the discrete Hilbert transform \mathcal{H} is bounded from $\ell_{1,\lambda}(\mathbb{Z})$ to $w\ell_{1,\lambda}(\mathbb{Z})$. Moreover,

$$\|\mathcal{H}a\|_{w\ell_{1,\lambda}(\mathbb{Z})} \leq \frac{C_1}{2(1-\lambda)} \|a\|_{\ell_{1,\lambda}(\mathbb{Z})},$$

where $C_1 = 6D + \frac{120}{\pi}$.

Proof.

Case 1: $1 < p < \infty$. Let $a \in \ell_{p,\lambda}(\mathbb{Z})$. Fix $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$. By Theorem 5(i),

$$\|\mathcal{H}a\|_{\ell_p(J(j,N))} \leq C_p (\#J(j,N))^{\frac{1}{p}} \sum_{n>N} \frac{\|a\|_{\ell_p(J(j,n))}}{(\#J(j,n))^{1+\frac{1}{p}}}.$$

We multiply both sides by $(\#J(j,N))^{-\frac{\lambda}{p}}$ to obtain

$$(\#J(j,N))^{-\frac{\lambda}{p}} \|\mathcal{H}a\|_{\ell_p(J(j,N))} \leq C_p (\#J(j,N))^{\frac{1-\lambda}{p}} \sum_{n>N} \|a\|_{\ell_p(J(j,n))} (\#J(j,n))^{-1-\frac{1}{p}}.$$

Using the definition of the discrete Morrey norm we get $\|a\|_{\ell_p(J(j,n))} \leq (\#J(j,n))^{\frac{\lambda}{p}} \|a\|_{\ell_{p,\lambda}(\mathbb{Z})}$, and therefore

$$(\#J(j,N))^{-\frac{\lambda}{p}} \|\mathcal{H}a\|_{\ell_p(J(j,N))} \leq C_p \|a\|_{\ell_{p,\lambda}(\mathbb{Z})} (\#J(j,N))^{\frac{1-\lambda}{p}} \sum_{n>N} (\#J(j,n))^{-1-\frac{1-\lambda}{p}}.$$

Then the sum becomes

$$\begin{aligned} \sum_{n>N} (\#J(j,n))^{-1-\frac{1-\lambda}{p}} &= \sum_{n>N} (2n+1)^{-1-\frac{1-\lambda}{p}} \leq \int_N^\infty (2x+1)^{-1-\frac{1-\lambda}{p}} dx \\ &= \frac{p C_p}{2(1-\lambda)} (2N+1)^{-\alpha} = \frac{p C_p}{2(1-\lambda)} (\#J(j,N))^{-\alpha}. \end{aligned}$$

Hence

$$(\#J(j,N))^{-\frac{\lambda}{p}} \|\mathcal{H}a\|_{\ell_p(J(j,N))} \leq \frac{p C_p}{2(1-\lambda)} \|a\|_{\ell_{p,\lambda}(\mathbb{Z})}.$$

Taking the supremum for any $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$ yields

$$\|\mathcal{H}a\|_{\ell_{p,\lambda}(\mathbb{Z})} \leq \frac{p C_p}{2(1-\lambda)} \|a\|_{\ell_{p,\lambda}(\mathbb{Z})},$$

which proves the boundedness $\mathcal{H} : \ell_{p,\lambda}(\mathbb{Z}) \rightarrow \ell_{p,\lambda}(\mathbb{Z})$ for $1 < p < \infty$.

Case 2: $p = 1$. Let $a \in \ell_{1,\lambda}(\mathbb{Z})$ and fix $j \in \mathbb{Z}$ and $N \in \mathbb{N}$. By Theorem 5(ii),

$$\|\mathcal{H}a\|_{w\ell_1(J(j,N))} \leq C_1 (\#J(j,N)) \sum_{n>N} \frac{1}{(\#J(j,n))^2} \|a\|_{\ell_1(J(j,n))}.$$

Multiply by $(\#J(j,N))^{-\lambda}$:

$$(\#J(j,N))^{-\lambda} \|\mathcal{H}a\|_{w\ell_1(J(j,N))} \leq C_1 (\#J(j,N))^{1-\lambda} \sum_{n>N} (\#J(j,n))^{-2} \|a\|_{\ell_1(J(j,n))}.$$

Using $\|a\|_{\ell_1(J(j,n))} \leq (\#J(j,n))^\lambda \|a\|_{\ell_{1,\lambda}(\mathbb{Z})}$ gives

$$(\#J(j,N))^{-\lambda} \|\mathcal{H}a\|_{w\ell_1(J(j,N))} \leq C_1 \|a\|_{\ell_{1,\lambda}(\mathbb{Z})} (\#J(j,N))^{1-\lambda} \sum_{n>N} (\#J(j,n))^{-2+\lambda}.$$

Then

$$\begin{aligned} \sum_{n>N} (\#J(j,n))^{-2+\lambda} &= \sum_{n>N} (2n+1)^{-2+\lambda} \leq \int_N^\infty (2x+1)^{-2+\lambda} dx \\ &= \frac{1}{2(1-\lambda)} (2N+1)^{-1+\lambda} = \frac{1}{2(1-\lambda)} (\#J(j,N))^{-1+\lambda}. \end{aligned}$$

Therefore

$$(\#J(j,N))^{-\lambda} \|\mathcal{H}a\|_{w\ell_1(J(j,N))} \leq \frac{C_1}{2(1-\lambda)} \|a\|_{\ell_{1,\lambda}(\mathbb{Z})} = \frac{C_1}{2(1-\lambda)} \|a\|_{\ell_{1,\lambda}(\mathbb{Z})}.$$

Taking the supremum for any $j \in \mathbb{Z}$ and $N \in \mathbb{N}_0$ yields

$$\|\mathcal{H}a\|_{w\ell_{1,\lambda}} \leq \frac{C_1}{2(1-\lambda)} \|a\|_{\ell_{1,\lambda}(\mathbb{Z})}.$$

Hence $\mathcal{H} : \ell_{1,\lambda}(\mathbb{Z}) \rightarrow w\ell_{1,\lambda}(\mathbb{Z})$ is bounded.

The proof was completed.

Remark 3. *The boundedness of the discrete Hilbert transform on discrete Morrey spaces over \mathbb{Z} has been previously investigated in [2]. In contrast, our work establishes this boundedness via local estimation theorems for the Hilbert transform. Furthermore, we obtain weak-type boundedness of the discrete Hilbert transform on discrete Morrey spaces over \mathbb{Z} and derive explicit—and sharper—constants in the corresponding inequalities.*

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