Journal of Contemporary Applied Mathematics
V. 17, No 1, 2027, January-June
ISSN 2222-5498, E-ISSN 3006-3183
https://doi.org/10.62476/jcam171.12

Statistical Korovkin-Type Theorems on Banach Function Spaces

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Abstract. This study develops statistical Korovkin-type approximation theorems within the framework of Banach function spaces. The main results are established on the shift-invariant subspace X_s of a Banach function space X. Statistical versions of Korovkin-type theorems are formulated for Kantorovich polynomial operators, addressing both rearrangement-invariant and non-rearrangement-invariant Banach function spaces. The analysis leverages the boundedness of the Hardy–Littlewood maximal operator and the behavior of the operators on the finite Korovkin test set $\{1, t, t^2\}$. Applications of the main results are demonstrated in Grand Lebesgue spaces and Morrey-type spaces, illustrating how statistical convergence extends classical approximation theory to more general and non-standard functional settings.

Key Words and Phrases: Korovkin theorems, Banach function spaces, statistical convergence, Kantorovich polynomial operators

2020 Mathematics Subject Classifications: 42E35, 41A36, 46E30, 40A05

1. Introduction

Statistical convergence has become an essential notion in modern analysis, finding applications in summability theory, probability, and approximation. Introduced independently by Fast and Steinhaus in 1951 [14, 31], the notion was later examined by Zygmund [36] in the context of trigonometric series as a form of almost-everywhere convergence. Since then, the subject has continued to develop through the contributions of Schoenberg [29], Salát [28], Fridy [15, 16], Connor [11], Fridy and Orhan [17], and Maddox [27], reinforcing its importance in nonlinear summability and various areas of functional analysis.

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Concurrently, Korovkin-type approximation theorems have emerged as indispensable tools in determining the convergence behavior of positive linear operators. Originally formulated by Korovkin in 1953 [25], the classical result provides sufficient conditions for uniform convergence in spaces like $C[0,1]$. Subsequent generalizations have extended its reach to L^p spaces [3, 12, 26], Morrey spaces [10, 34], and rearrangement-invariant Banach spaces [33, 34]. Foundational contributions by Bohman [7], Dzyadik [13], and Gadzhiev [19] have played a central role in shaping the general Korovkin framework. For a comprehensive treatment of Korovkin-type approximation theory, we refer the reader to the monograph by Altomare and Campiti [2].

A connection between these two areas was established with the statistical version of Korovkin's theorem, introduced by Gadjiev and Orhan [18], where statistical convergence conditions were applied to operator sequences to obtain approximation results. This approach offered a more flexible framework, especially in settings involving stochastic or measure-theoretic structures. Additional progress on statistical convergence was made by Bilalov and Nazarova [4, 5, 6], who examined its behavior in metric and uniform spaces.

In response to the increasing complexity of real-world problems, such as those arising in partial differential equations with irregular data or in variational settings, a variety of function spaces with more adaptable geometric and growth properties have been developed. Examples include variable-summability Lebesgue spaces, Grand Lebesgue spaces, Orlicz-type spaces, and nonseparable Morrey-type spaces [9, 21, 24, 30]. These spaces allow for a more refined description of function behavior than is possible within the classical Lebesgue framework. The problem of approximating sequences of operators to the identity operator in such spaces has been studied in several works [20, 22, 23, 30].

In this context, the recent work of Zeren, Ismailov, and Karacam [35] extended Korovkin-type approximation theory to Banach function spaces by employing shift-invariant subspaces. Motivated by their approach, the present paper establishes statistical Korovkin-type theorems on the shift-invariant subspace $X_s \subset X$, where X denotes a Banach function space and X_s is generated through translation operators.

Our goal is to establish sufficient conditions under which sequences of Kantorovich-type positive linear operators exhibit statistical convergence on X_s, relying on their behavior on the finite Korovkin test set $\{1, t, t^2\}$. This viewpoint is rooted in classical Korovkin-type approximation theory, where convergence can be verified through a small but structurally meaningful collection of test functions. The analysis makes use of boundedness properties of the Hardy–Littlewood maximal operator, a central tool in harmonic analysis that provides control over local averages and oscillatory behavior [32]. Alongside these operator-theoretic meth-

ods, the study also draws on structural aspects of both rearrangement-invariant and non-rearrangement-invariant function spaces, highlighting how symmetry, modular behavior, and local regularity influence convergence. Applications are presented in classical Lebesgue spaces, grand Lebesgue spaces and Morrey-type spaces [8, 33]. These examples illustrate the robustness of the framework across different integrability and regularity settings in statistical approximation.

2. Preliminaries and Definitions

Throughout this paper, we let $|\Omega|$ denote the Lebesgue measure of a measurable set $\Omega \subset \mathbb{R}^n$. Given Banach spaces X and Y, we write $\mathrm{B}(X,Y)$ for the space of all bounded linear operators from X to Y, and abbreviate $\mathrm{B}(X) := \mathrm{B}(X,X)$.

Let $(S, \mathcal{A}, \mu)$ be a measure space, where $\mathcal{A}$ is a σ-algebra of subsets of S, and μ is a measure defined on $\mathcal{A}$. Let M denote the set of all measurable functions on S, M_+ the set of all nonnegative measurable functions, and M_0 the set of functions in M that are finite a.e. For a measurable set $E \in \mathcal{A}$, we write χ_E for its characteristic function.

We also set $\mathbb{R}^+ := [0, \infty)$ and let m denote the Lebesgue measure on $\mathbb{R}^+$. The space $C_0^\infty(\Omega)$ stands for the set of infinitely differentiable functions with compact support in an open set $\Omega \subset \mathbb{R}^n$.

We begin by recalling some fundamental facts from the theory of Banach function spaces.

Definition 1. *A functional $\rho : \mathcal{M}_+ \to [0, \infty]$ is called a* function norm *if it satisfies the following properties for all $f, g, f_n \in \mathcal{M}_+$, $\alpha \geq 0$, and measurable sets $E \in \mathcal{A}$:*

1. *$\rho(f) = 0$ if and only if $f = 0$ a.e.; moreover, $\rho(\alpha f) = \alpha\, \rho(f)$ and $\rho(f+g) \leq \rho(f) + \rho(g)$.*
2. *If $g \leq f$ a.e., then $\rho(g) \leq \rho(f)$.*
3. *If $f_n \uparrow f$ a.e., then $\rho(f_n) \uparrow \rho(f)$.*
4. *If $\mu(E) < \infty$, then $\rho(\chi_E) < \infty$.*
5. *If $\mu(E) < \infty$, then there exists a constant $C_E > 0$ such that for all $f \in \mathcal{M}_+$,*

$$\int_E f \, d\mu \leq C_E \, \rho(f),$$

where C_E depends only on E and ρ, but not on f.

Then X consists of all measurable functions f such that $\rho(|f|) < \infty$. Let X be a Banach function space generated by ρ, a Banach space of functions $f \in \mathcal{M}$ equipped with the norm

$$\|f\|_X = \rho(|f|).$$

Definition 2. *Let X be a Banach function space. A function $f \in X$ is said to have an* absolutely continuous norm *if*

$$\|f\chi_{E_n}\|_X \to 0 \quad \text{as } n \to \infty,$$

for every sequence of measurable sets $E_n \in \mathcal{A}$ satisfying $E_{n+1} \subseteq E_n$ for all n and $\bigcap_{n=1}^{\infty} E_n = \varnothing$.

The set of all functions in X having absolutely continuous norm will be denoted by X_a.

2.1. Statistical Convergence in Metric Spaces

Let $A \subset \mathbb{N}$. For each $n \in \mathbb{N}$, we define

$$\delta_n(A) = \frac{1}{n}\sum_{k=1}^{n} \chi_A(k).$$

If the limit

$$\delta(A) = \lim_{n\to\infty} \delta_n(A)$$

exists, then $\delta(A)$ is called the statistical density of the set A.

Let (X, d) be a metric space and let $\{x_n\}_{n\in\mathbb{N}} \subset X$. The sequence $\{x_n\}$ is called statistically convergent to $x \in X$ (denoted st-$\lim_{n\to\infty} x_n = x$) if

$$\forall \varepsilon > 0, \quad \delta(A_\varepsilon) = 0,$$

where

$$A_\varepsilon = \{n \in \mathbb{N} : d(x_n, x) \geq \varepsilon\}.$$

The sequence $\{x_n\}$ is called statistically fundamental (st-fundamental) if

$$\forall \varepsilon > 0, \quad \exists n_\varepsilon \in \mathbb{N} \text{ such that } \delta(A_{n_\varepsilon}) = 0,$$

where

$$A_{n_\varepsilon} = \{n \in \mathbb{N} : d(x_n, x_{n_\varepsilon}) \geq \varepsilon\}.$$

Define the set

$$\mathcal{K} = \{A \subset \mathbb{N} : \delta(A) = 1\}.$$

Theorem 1 ([4]). *Let (X, d) be a metric space and let $\{x_n\}_{n\in\mathbb{N}} \subset X$. The following conditions are equivalent:*

(i) $st\text{-}\lim_{n\to\infty} x_n = x$;

(ii) The sequence $\{x_n\}$ is statistically fundamental;

(iii) There exists a sequence $\{y_n\} \subset X$ such that $\lim_{n\to\infty} y_n = x$ and $\{n \in \mathbb{N} : x_n = y_n\} \in \mathcal{K}$.

Corollary 1 ([5]). *Let (X, d) be a metric space and let the sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ be such that*

$$st\text{-}\lim_{n\to\infty} x_n = x.$$

Then there exists a strictly increasing subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} x_{n_k} = x,$$

and the set $\{n_k : k \in \mathbb{N}\} \in \mathcal{K}$

Moreover, the following elementary fact shows that the family K is closed under finite intersections.

Lemma 1 ([5]). *If $A, B \in \mathcal{K}$, then $A \cap B \in \mathcal{K}$.*

2.2. Shift-Invariance and the Subspace X_s

Let X be a Banach function space over the interval $[0, 1]$. For every $\delta \in \mathbb{R}$ define the right-shift operator $T_\delta : X \to X$ as

$$T_\delta f(x) := \begin{cases} f(x+\delta), & \text{if } x+\delta \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

We define $X_s \subset X$ to be the closure in X of all functions $f \in X$ for which

$$\lim_{\delta\to 0} \|T_\delta f - f\|_X = 0.$$

This subspace collects functions exhibiting strong translation-invariance and will form the analytic foundation for our approximation results. This formulation follows the framework introduced in [35].

Theorem 2 ([35]). *Let X be a Banach function space and $1 \in X_a$. Then the set $C_0^\infty([0,1])$ is dense in X_s.*

We next recall a fundamental tool from harmonic analysis, namely the Hardy–Littlewood maximal operator, which provides essential control over the local oscillation and pointwise behavior of measurable functions. For a locally integrable function f on $[0,1]$, it is defined by

$$(Mf)(x) = \sup_{x \in I \subset [0,1]} \frac{1}{|I|} \int_I |f(t)|\, dt.$$

Theorem 3 ([32]). *Let X be a rearrangement-invariant Banach function space. Then the Hardy–Littlewood maximal operator M is bounded on X if and only if the upper Boyd index of X satisfies $\beta_X < 1$.*

2.3. Classical and Statistical Korovkin Theorems

We now recall the classical Korovkin theorem and its statistical version. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators on $C([0,1])$.

If

$$\lim_{n\to\infty} \|L_n g - g\|_\infty = 0 \quad \forall g \in \{1, t, t^2\},$$

then

$$\lim_{n\to\infty} \|L_n f - f\|_\infty = 0 \quad \forall f \in C([0,1]).$$

In the statistical setting, the analogous result holds with the limit replaced by statistical convergence, provided the convergence occurs on the same test set.

Theorem 4 ([19]). *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators acting on the space $C([0,1])$ and taking values in a normed linear space.*

Assume that for the Korovkin test set $\{1, t, t^2\}$, the operators converge statistically:

$$st\text{-}\lim_{n\to\infty} \|L_n g - g\|_\infty = 0, \quad \text{for all } g \in \{1, t, t^2\}.$$

Then, statistical convergence holds uniformly for all $f \in C([0,1])$:

$$st\text{-}\lim_{n\to\infty} \|L_n f - f\|_\infty = 0, \quad \forall f \in C([0,1]).$$

Theorem 5 ([35]). *Let X be a Banach function space such that $1 \in X_a$, and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive bounded linear operators in X_s satisfying the condition*

$$\lim_{n\to\infty} L_n g = g \quad \text{in } C([0,1]) \text{ for all } g \in \{1, t, t^2\}.$$

Then the relation

$$\lim_{n\to\infty} L_n f = f \quad \text{in } X \text{ for all } f \in X_s$$

holds if and only if

$$\sup_n \|L_n\|_{B(X_s)} = c < +\infty.$$

3. Main Results

In the following section, we formulate and demonstrate the statistical counterpart of the classical Korovkin-type approximation theorem, specifically within the framework of Banach function spaces.

Theorem 6. *Let X be a Banach function space such that $1 \in X_a$, and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive bounded linear operators in X_s satisfying*

$$st\text{-}\lim_{n\to\infty} L_n g = g \quad \text{in } C([0,1]) \text{ for all } g \in \{1, t, t^2\}.$$

If

$$\sup_n \|L_n\|_{B(X_s)} = c < +\infty,$$

then

$$st\text{-}\lim_{n\to\infty} L_n f = f \quad \text{in } X \text{ for all } f \in X_s.$$

Proof. From

$$\text{st-}\lim_{n\to\infty} L_n g_i = g_i$$

where $g_i(t) = t^i$, $i = 0, 1, 2$, by Corollary 1, it follows that there exist subsequences

$\{n_k^{(i)}\}_{k\in\mathbb{N}} \subset \mathbb{N}$, such that

$$\lim_{k\to\infty} \|L_{n_k^{(i)}} g_i - g_i\|_\infty = 0, \quad i = 0, 1, 2.$$

Let $\{n_k\} = \bigcap_{i=0}^{2} \{n_k^{(i)}\}$. It follows directly from Lemma 1 that $\delta(\{n_k\}) = 1$. Obviously,

$$\lim_{k\to\infty} \|L_{n_k} g_i - g_i\|_\infty = 0, \quad i = 0, 1, 2.$$

Then, by Theorem 5, we have

$$\lim_{k\to\infty} \|L_{n_k} f - f\|_X = 0, \quad \forall f \in X_s.$$

Consequently, by Theorem 1, for all $f \in X_s$, the sequence $\{L_n f\}_{n\in\mathbb{N}}$ statistically converges to f in X.

The theorem is proved.

Let X_s be a Banach function space defined on the interval $[-\pi, \pi]$. To introduce its 2π-periodic counterpart, we begin with some notation.

We denote by

$$L^0(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \text{ measurable}\}$$

the space of all real-valued measurable functions on $\mathbb{R}$. A function $f \in L^0(\mathbb{R})$ is called 2π-periodic if

$$f(x + 2\pi) = f(x) \quad \text{for a.e. } x \in \mathbb{R}.$$

The class of all such functions is denoted by

$$L^0_{2\pi}(\mathbb{R}) := \{f \in L^0(\mathbb{R}) : f(\cdot + 2\pi) = f(\cdot)\}.$$

Given the Banach function space X_s on $[-\pi, \pi]$, we define its associated 2π-periodic version by

$$X^s_{2\pi} := \{f \in L^0_{2\pi}(\mathbb{R}) : f|_{[-\pi,\pi]} \in X_s\}.$$

In what follows, we establish a statistical trigonometric Korovkin-type approximation result within the framework of Banach function spaces. We shall use the following trigonometric analogue of Theorem 5. Let $X^s_{2\pi}$ be the periodic shift-invariant subspace. If

$$\lim_{n\to\infty} L_n g = g \quad \text{in } C_{2\pi}(\mathbb{R})$$

for all $g \in \{1, \sin x, \cos x\}$, and

$$\sup_n \|L_n\|_{B(X^s_{2\pi})} < +\infty,$$

then

$$\lim_{n\to\infty} L_n f = f \quad \text{in } X$$

for every $f \in X^s_{2\pi}$.

Theorem 7. *Let X be a Banach function space such that $1 \in X_a$, and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive bounded linear operators acting on the periodic Banach space $X^s_{2\pi}$.*

Assume that

$$st\text{-}\lim_{n\to\infty} L_n g = g \quad \textit{in } C_{2\pi}(\mathbb{R}) \textit{ for all } g \in \{1, \sin x, \cos x\},$$

and

$$\sup_{n\in\mathbb{N}} \|L_n\|_{B(X^s_{2\pi})} = c < +\infty.$$

Then

$$st\text{-}\lim_{n\to\infty} L_n f = f \quad \textit{in } X \textit{ for all } f \in X^s_{2\pi}.$$

Proof. Since

$$\text{st-}\lim_{n\to\infty} L_n g = g, \quad \forall g \in \{1, \sin x, \cos x\},$$

by Corollary 1 and Lemma 1, there exists a subsequence

$$\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N},$$

such that

$$\lim_{k\to\infty} \|L_{n_k} g - g\|_\infty = 0, \quad \forall g \in \{1, \sin x, \cos x\}.$$

Consequently, by the trigonometric analogue of Theorem 5, we obtain

$$\lim_{k\to\infty} \|L_{n_k} f - f\|_X = 0, \qquad f \in X^s_{2\pi}.$$

Thus, by Theorem 1, for every $f \in X^s_{2\pi}$, the sequence $\{L_n f\}_{n\in\mathbb{N}}$ statistically converges to f in X.

The theorem is proved.

4. Statistical Korovkin-Type Theorems in Banach Function Spaces with Regard to Kantorovich Polynomials

Kantorovich polynomials are defined for all $f \in L^1(0,1)$ by the formula:

$$K_n(f)(x) = (n+1)\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt, \quad 0 \le x \le 1.$$

These operators are positive and linear, and play a central role in Korovkin-type approximation theory. The following results explore their behavior in Banach function spaces.

4.1. Statistical Approximation in Rearrangement-Invariant Banach Function Spaces

We now extend classical Korovkin-type approximation results to the statistical convergence setting in the framework of rearrangement invariant Banach function spaces, employing Kantorovich polynomial operators.

Theorem 8. Let X be a rearrangement-invariant Banach function space with $\mathbf{1} \in X_a$, and suppose that the upper Boyd index satisfies $\beta_X < 1$. Then, for every $f \in X_s$, the sequence $\{K_n f\}$ converges statistically to f in X, that is,

$$\text{st-}\lim_{n\to\infty} K_n f = f.$$

Proof. Since X is rearrangement-invariant and $\beta_X < 1$, it follows from Theorem 3 that the Hardy–Littlewood maximal operator M is bounded in X. Therefore, the Kantorovich operators $\{K_n\}$, which are positive and dominated by M, are uniformly bounded:

$$\sup_n \|K_n\|_{B(X_s)} < +\infty.$$

It is known that the sequence $\{K_n\}$ satisfies statistical Korovkin convergence on the test set $\{1, t, t^2\}$, that is,

$$\text{st-}\lim_{n\to\infty} K_n g = g \quad \text{in } C([0,1]) \quad \text{for all } g \in \{1, t, t^2\}.$$

Then, by the Statistical Korovkin-type theorem in Banach function spaces (Theorem 6), it follows that

$$\text{st-}\lim_{n\to\infty} K_n f = f \quad \text{in } X, \quad \forall f \in X_s.$$

The theorem is proved.

As an application of Theorem 8, we consider the Grand Lebesgue space. Let

$$X = L^{p)}(0,1), \qquad 1 < p < \infty,$$

be the Grand Lebesgue space of measurable functions on $(0,1)$ equipped with the norm

$$\|f\|_{L^{p)}(0,1)} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(0,1)}.$$

It is well known that $L^{p)}(0,1)$ is a rearrangement-invariant Banach function space and that its Boyd index satisfies $\beta_{L^{p)}} = 1/p < 1$. Consequently, the Hardy–Littlewood maximal operator M is bounded on $L^{p)}(0,1)$. Since in addition the shift-invariant subspace of $L^{p)}(0,1)$ coincides with $G^{p)}(0,1)$ (see [33]), all assumptions of Theorem 8 are fulfilled. Therefore, we obtain the statistical Korovkin-type approximation

$$\text{st-}\lim_{n\to\infty} K_n f = f \quad \text{in } L^{p)}(0,1), \quad \forall f \in G^{p)}(0,1).$$

4.2. Statistical Approximation in Non-Rearrangement-Invariant Banach Function Spaces

In this section, we consider the statistical Korovkin-type approximation behavior of Kantorovich polynomial operators in general Banach function spaces which are not necessarily rearrangement-invariant.

Theorem 9. Let X be a Banach function space such that $\mathbf{1} \in X_a$, and suppose that the Hardy–Littlewood maximal operator M is bounded on X. Then, for every $f \in X_s$, the Kantorovich operators satisfy

$$\text{st-}\lim_{n\to\infty} K_n f = f \quad \text{in } X.$$

Proof. By the result of [35], if the Hardy–Littlewood maximal operator M is bounded on X, then the Kantorovich operators $\{K_n\}$ converge in norm to the identity on X_s, that is,

$$\lim_{n\to\infty} K_n f = f \quad \text{in } X, \quad \forall f \in X_s.$$

In particular, this implies that

$$\lim_{n\to\infty} K_n g = g \quad \text{in } X, \quad \text{for all } g \in \{1, t, t^2\}.$$

Since norm convergence implies statistical convergence, we obtain

$$\text{st-}\lim_{n\to\infty} K_n g = g \quad \text{in } C([0,1]) \quad \text{for all } g \in \{1, t, t^2\}.$$

Moreover, the operators K_n are positive and uniformly bounded in X_s, as they are dominated by the maximal operator M. Therefore, by the Statistical Korovkin-type theorem in Banach function spaces Theorem 6, we conclude that

$$\text{st-}\lim_{n\to\infty} K_n f = f \quad \text{in } X, \quad \forall f \in X_s.$$

The theorem is proved.

As an illustrative example, we now turn to the case of Morrey spaces.

Let $1 < p \le p_0 < +\infty$, and let $X = M^{p,p_0}(0,1)$ be the Morrey space of measurable functions $f \in L^p(0,1)$ equipped with the norm

$$\|f\|_{M^{p,p_0}(0,1)} = \sup_{I\subset[0,1]} |I|^{\frac{1}{p_0}-\frac{1}{p}} \left(\int_I |f(t)|^p \, dt\right)^{1/p},$$

where the supremum is taken over all closed intervals $I \subset [0,1]$ with $|I| > 0$.

It is known that the Hardy–Littlewood maximal operator M is bounded on $M^{p,p_0}(0,1)$ [8, 10]. Then for every $f \in M_s^{p,p_0}(0,1)$, the sequence of Kantorovich polynomial operators satisfies

$$\text{st-}\lim_{n\to\infty} K_n f = f \quad \text{in } M^{p,p_0}(0,1).$$

The Morrey space $M^{p,p_0}(0,1)$ is a Banach function space that is not rearrangement-invariant. However, it satisfies the assumptions of Theorem 9, namely: $\mathbf{1} \in X_a$, the Hardy–Littlewood maximal operator M is bounded on X, and the Kantorovich operators K_n are uniformly bounded and converge on the test set $\{1, t, t^2\}$. Therefore, by applying Theorem 9, we conclude that statistical convergence holds in $M_s^{p,p_0}(0,1)$.

5. Conclusion and Future Directions

In this study, statistical Korovkin-type approximation theorems were established in the setting of Banach function spaces. By introducing the shift-invariant subspace X_s, we obtained a unified framework that reduces the verification of statistical convergence for positive linear operators to their behavior on the

minimal Korovkin test set $\{1, t, t^2\}$. This extends the classical Korovkin theory—traditionally confined to $C[0,1]$ or L^p-type settings—to functional environments that may lack rearrangement invariance, smooth density, or standard modular structures.

A central component of our approach is the boundedness of the Hardy–Littlewood maximal operator, which provides uniform control of the operator sequence and ensures the validity of the statistical Korovkin theorem on the entire subspace X_s. The results apply to several nonclassical spaces, including Grand Lebesgue spaces $L^{p)}(0,1)$ and Morrey-type spaces, where the classical Korovkin method is not directly applicable. In these situations, our findings give new statistical approximation principles for Kantorovich-type operators and, potentially, for broader classes of positive linear operators dominated by maximal-type operators.

Future research may examine weighted versions of Grand Lebesgue and Morrey spaces, statistical modular convergence in Musielak–Orlicz and variable exponent spaces, and the behavior of nonlinear or sublinear approximation schemes. Extensions to settings with non-doubling measures or to operators associated with fractional or localized maximal functions would further expand the reach of the statistical Korovkin framework presented here.

In conclusion, the statistical approximation approach developed in this work refines classical techniques and offers a flexible methodology for connecting abstract operator theory with modern function space analysis.

Acknowledgements.

Competing Interests. The authors declare that there are no conflicts of interest associated with this publication.

Authors' Contributions. All authors have equally contributed to the conceptual development, analysis, and composition of the manuscript.

Funding. This work was supported by the Azerbaijan Science Foundation, Grant No. AEF-MGC-2025-1(54)-20/03/1-M-03.

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Received 06 December 2025
Accepted 19 June 2026