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Analytic study of time-fractional Navier Stokes equation

Y.Massoun, A.k Alomari, C.Cesarano, B.J. Mammadov

Abstract. This paper investigates the time fractional Navier Stokes equations involving the Caputo derivative and presents an analytic approximation based on the Homotopy Analysis Method (HAM). Fundamental concepts of fractional calculus and HAM are briefly reviewed, and a systematic LHAM framework is developed to construct convergent series solutions. Approximate analytic expressions are derived for different fractional orders α, and the influence of α on the velocity components is examined through numerical tables and graphical results. The obtained solutions demonstrate rapid convergence and clearly capture the memory effects intrinsic to fractional fluid flows, confirming the effectiveness of HAM for nonlinear time-fractional systems. The results provide a reliable semi analytical benchmark for future numerical studies of fractional Navier Stokes models.

Key Words and Phrases: Navier Stokes equation, Caputo fractional derivative, Analytic solution, Homotopy analysis method

2020 Mathematics Subject Classifications: 35Q30, 35R11, 76D05, 65Mxx

1. Introduction

Fractional differential equations have emerged as a powerful modeling framework for describing processes in which memory, hereditary properties, or nonlocal effects play a fundamental role. In contrast to classical integer-order models, fractional formulations provide greater flexibility in capturing anomalous transport, viscoelastic behavior, and complex dynamical responses observed in many physical and engineering systems [9]. Consequently, fractional calculus has been widely adopted in fluid mechanics, heat transfer, and diffusion-dominated phenomena. Among the most prominent models in fluid dynamics, the Navier Stokes equations have been extended to fractional order in time to account for memory dependent effects and deviations from classical diffusion laws. The resulting time fractional Navier Stokes equations offer a more realistic description of certain non-Newtonian and anomalous flow regimes. Due to their strong nonlinearity and nonlocal nature, obtaining exact solutions remains challenging, and most existing studies rely on numerical or semi-numerical approaches. Various techniques have been proposed in the literature, including the Laplace Adomian decomposition method [8], finite difference and Legendre spectral schemes [11], spectral methods [10], and cubic B-spline-based approximations [4]. While these approaches have demonstrated good numerical performance, they often provide limited analytical insight into the structure of the solution.

In recent years, semi analytic techniques have gained increasing attention as an effective compromise between purely numerical methods and closed-form solutions. In this context, the Homotopy Analysis Method (HAM) has proven to be a robust and flexible tool for solving nonlinear differential equations. Unlike classical perturbation methods, HAM does not depend on the presence of small or large parameters and allows explicit control of solution convergence through auxiliary parameters. This feature makes HAM particularly attractive for fractional differential equations, where convergence and stability issues are often delicate [2, 5, 6, 7, 1, 3].

Motivated by these observations, the present work aims to construct analytical approximations for the time fractional Navier Stokes equations using the Homotopy Analysis Method combined with the Laplace operator (LHAM). The proposed approach enables the derivation of rapidly convergent series solutions for different values of the fractional order α, offering clear insight into the influence of memory effects on fluid motion. In contrast to existing numerical studies, the method adopted here yields explicit analytical expressions that facilitate both qualitative and quantitative analysis.

The paper is organized as follows. First, essential concepts of fractional calculus and the fundamental principles of the Homotopy Analysis Method are briefly recalled. Subsequently, LHAM is employed to solve the time-fractional Navier Stokes system, and approximate solutions are obtained for various fractional orders. The resulting solutions are presented in both tabular and graphical forms to illustrate the effect of the fractional parameter on the flow dynamics. The findings confirm the accuracy and efficiency of the proposed approach and highlight its potential for extension to more general configurations, including multidimensional systems and models involving generalized or multi-parameter fractional derivatives.

In this study, we apply the homotopy analysis method coupled with the Laplace operator to derive analytical solutions of the following time fractional

Navier Stokes system:

$$\begin{cases} {}^{C}D_t^{\alpha}u + u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} + w\frac{\partial u}{\partial t} = \rho\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial t^2}\right] - \frac{1}{\rho}\frac{\partial p}{\partial x} \\ {}^{C}D_t^{\alpha}v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial t} = \rho\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial t^2}\right] - \frac{1}{\rho}\frac{\partial p}{\partial y} \\ {}^{C}D_t^{\alpha}w + u\frac{\partial w}{\partial x} + u\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial t} = \rho\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial t^2}\right] - \frac{1}{\rho}\frac{\partial p}{\partial t} \end{cases} \tag{1}$$

subject to the initial conditions

$$\begin{cases} u(x,y,t,0) = f(x,y,t), \\ v(x,y,t,0) = g(x,y,t), \\ w(x,y,t,0) = h(x,y,t), \end{cases} \tag{2}$$

where ${}^{C}D^{\alpha}t$ denotes the Caputo fractional derivative of order $\alpha \in (0,1]$.

2. Some Preliminaries on Fractional Calculus

This section briefly recalls several fundamental notions of fractional calculus that are required throughout the paper. These definitions and properties are well established in the literature and are presented here for completeness and clarity.

Definition 1. *Let $\mu \in \mathbb{R}$. A function $k(t)$ is said to belong to the space C_μ if there exists a real number $p > \mu$ such that*

$$k(t) = t^p k_1(t),$$

where $k_1(t)$ is continuous on the interval $(0,\infty)$.

Definition 2. *Let $\kappa \in C_\mu$ with $\mu \geq -1$. The Riemann Liouville fractional integral of order $\alpha \geq 0$ is defined by [9]*

$$\begin{aligned} I^{\alpha}\kappa(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\kappa(\tau)\,d\tau, \quad \alpha > 0, \\ I^0\kappa(t) &= \kappa(t). \end{aligned} \tag{3}$$

Several useful properties of the Riemann Liouville fractional integral are summarized below.

Definition 3. *For $\kappa \in C_\mu$ with $\mu \geq -1$, $\alpha, \beta \geq 0$, and $\gamma \geq -1$, the following relations hold:*

$$\begin{aligned} I^{\alpha}\big(I^{\beta}\kappa(t)\big) &= I^{\alpha+\beta}\kappa(t), \\ I^{\alpha}\big(I^{\beta}\kappa(t)\big) &= I^{\beta}\big(I^{\alpha}\kappa(t)\big), \\ I^{\alpha}\left(t^{\gamma}\right) &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma}. \end{aligned}$$

Definition 4. *The Caputo fractional derivative of order α for a function $\kappa(t)$ is defined as*

$$^{C}D^{\alpha}\kappa(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\varsigma)^{m-\alpha-1}\kappa^{(m)}(\varsigma)\,d\varsigma,$$

where $m-1<\alpha<m$ and $m\in\mathbb{N}$.

Definition 5. *Let $m-1<\alpha\leq m$, $m\in\mathbb{N}$, and $\kappa\in C_\mu^m$ with $\mu\geq -1$. Then, the fractional integral and the Caputo fractional derivative satisfy the following identity:*

$$I^{\alpha}\,{}^{C}D^{\alpha}\kappa(t) = \kappa(t) - \sum_{n=0}^{m-1}\kappa^{(n)}(0^+)\frac{t^n}{n!}.$$

3. HAM for time fractional Navier Stokes equation

In this section, we describe the application of the Homotopy Analysis Method (HAM) combined with the Laplace transform operator (LHAM) to obtain analytical approximations for the time fractional Navier Stokes system (1). The procedure allows for constructing convergent series solutions without relying on small or large perturbation parameters.
Let $u_0(t), v_0(t), w_0(t)$ be initial approximations satisfying the initial conditions of the problem. Define nonlinear operators N_1, N_2, N_3 corresponding to the governing equations for u, v, w as

$$\begin{aligned}
N_1[\phi_1,\phi_2,\phi_3] &= {}^{C}D_t^{\alpha}\phi_1 + \phi_1\frac{\partial\phi_1}{\partial x} + \phi_2\frac{\partial\phi_1}{\partial y} + \phi_3\frac{\partial\phi_1}{\partial t} - \rho\Delta\phi_1 + \frac{1}{\rho}\frac{\partial p}{\partial x},\\
N_2[\phi_1,\phi_2,\phi_3] &= {}^{C}D_t^{\alpha}\phi_2 + \phi_1\frac{\partial\phi_2}{\partial x} + \phi_2\frac{\partial\phi_2}{\partial y} + \phi_3\frac{\partial\phi_2}{\partial t} - \rho\Delta\phi_2 + \frac{1}{\rho}\frac{\partial p}{\partial y},\\
N_3[\phi_1,\phi_2,\phi_3] &= {}^{C}D_t^{\alpha}\phi_3 + \phi_1\frac{\partial\phi_3}{\partial x} + \phi_2\frac{\partial\phi_3}{\partial y} + \phi_3\frac{\partial\phi_3}{\partial t} - \rho\Delta\phi_3 + \frac{1}{\rho}\frac{\partial p}{\partial t}.
\end{aligned}$$

Introduce a continuous embedding parameter $q\in[0,1]$ and auxiliary functions $\Phi_i(x,y,t;q)$ such that

$$\Phi_i(x,y,t;0) = u_{0,i}(x,y,t), \quad \Phi_i(x,y,t;1) = u_i(x,y,t), \quad i=1,2,3.$$

The zeroth order deformation equations of LHAM are then formulated as

$$(1-q)L\big[\Phi_i(x,y,t;q) - u_{0,i}(x,y,t)\big] = \hbar q H_i(x,y,t)N_i[\Phi_1,\Phi_2,\Phi_3], \quad i=1,2,3, \tag{4}$$

where:

- L is a linear operator (here, the Laplace operator),

- $\hbar$ is an auxiliary convergence-control parameter,
- $H_i(x, y, t)$ are nonzero auxiliary functions,
- N_i are the nonlinear operators defined above.

Assuming $\Phi_i(x, y, t; q)$ is analytic in q, we expand it as a Maclaurin series

$$\Phi_i(x, y, t; q) = u_{0,i}(x, y, t) + \sum_{m=1}^{\infty} u_{m,i}(x, y, t) q^m, \quad i = 1, 2, 3.$$

Substituting this series into the zeroth-order deformation equation (4) and equating coefficients of q^m yields the m-th order deformation equations

$$L\left[u_{m,i}(x, y, t) - \chi_m u_{m-1,i}(x, y, t)\right] = \hbar H_i(x, y, t) R_{m,i}, \quad m \geq 1, \tag{5}$$

where:

- $\chi_m = 0$ for $m = 1$ and $\chi_m = 1$ for $m > 1$,
- $R_{m,i}$ represents the residuals obtained from the nonlinear operators N_i evaluated on the previous approximations.

The convergence of the series solutions $\sum_{m=0}^{\infty} u_{m,i}(x, y, t)$ is governed by the auxiliary parameter $\hbar$. Proper choice of $\hbar$ ensures that the series converges rapidly to the exact solution. In practice, $\hbar$ can be determined by plotting $\hbar$-curves or applying the residual error minimization technique.
Finally, setting $q = 1$ in the series expansion gives the approximate solution of the original problem:

$$u_i(x, y, t) = \sum_{m=0}^{\infty} u_{m,i}(x, y, t), \quad i = 1, 2, 3.$$

These series solutions can be truncated at a finite order M to obtain a highly accurate approximation:

$$u_i(x, y, t) \approx \sum_{m=0}^{M} u_{m,i}(x, y, t), \quad i = 1, 2, 3.$$

The above formulation allows explicit computation of the terms $u_{m,i}$ in a recursive manner, providing analytical insight into the effect of the fractional derivative α on fluid dynamics.

4. Convergence Analysis

In this section, we establish conditions under which the series solutions constructed via LHAM are guaranteed to converge to the solution of the time fractional Navier Stokes system.

Theorem 1. *Suppose the series solutions* $\sum_{k=0}^{\infty} u_k(t), \sum_{k=0}^{\infty} v_k(t), \sum_{k=0}^{\infty} w_k(t)$ *are convergent. Then, their sums constitute a solution of* (1).

Proof. Assume that $\lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} v_n(t) = \lim_{n\to\infty} w_n(t) = 0$. From the m-th order deformation equation, we have

$$\hbar H_1 \sum_{m=1}^{\infty} R_{1,m} = \lim_{k\to\infty} \sum_{m=0}^{k} L[u_m - \ell_m u_{m-1}] = L\Big[\lim_{k\to\infty} \sum_{m=0}^{k} (u_m - \ell_m u_{m-1})\Big] = L[\lim_{k\to\infty} u_k].$$

Since L is a linear operator, $H_1 \neq 0$, and $\hbar \neq 0$, it follows that $\sum_{m=1}^{\infty} R_{1,m} = 0$. Expanding the nonlinear operator $N_1[\Phi_i(t,q)], i = 1,2,3$ around $q = 0$ and setting $q = 1$ gives

$$N[\Phi_i(t,1)] = 0,$$

which implies that the series $\Phi_1(t,1) = \sum_{i=0}^{\infty} u_i(t)$ satisfies (1). Similarly, the series for v and w also solve the system.

Theorem 2. *Let* $\{u_m(t)\}_{m=0}^{\infty}$ *denote the terms of the series solution. The series* $\sum_{m=0}^{\infty} u_m(t)$ *converges if there exists a constant* $0 < \gamma < 1$ *such that*

$$\|u_{m+1}(t)\| \leq \gamma \, \|u_m(t)\|, \quad \forall m > m_0,$$

for some integer $m_0 \geq 0$.

Proof. Define the partial sums $U_n = \sum_{m=0}^{n} u_m(t)$. For any integers p, q with $p > q \geq m_0$, we have

$$\begin{aligned} \|U_p - U_q\| &= \left\| \sum_{k=q+1}^{p} u_k(t) \right\| \leq \sum_{k=q+1}^{p} \|u_k(t)\| \leq \sum_{k=q+1}^{p} \gamma^{k-m_0} \|u_{m_0}(t)\| \\ &= \gamma^{q-m_0+1} \frac{1-\gamma^{p-q}}{1-\gamma} \|u_{m_0}(t)\| \leq \frac{\gamma^{q-m_0+1}}{1-\gamma} \|u_{m_0}(t)\|. \end{aligned}$$

Since $0 < \gamma < 1$, it follows that

$$\lim_{p,q\to\infty} \|U_p - U_q\| = 0,$$

which shows that (U_n) is a Cauchy sequence in $\mathbb{R}$ and hence convergent. Therefore, the series $\sum_{m=0}^{\infty} u_m(t)$ converges. The same reasoning applies to the series for v and w.

5. Numerical Results and Discussion

In this section, we employ the Laplace Homotopy Analysis Method (LHAM) to obtain approximate solutions for the time fractional Navier Stokes equation (1). To illustrate the method, we calculate the first three terms of the series solution for three representative examples.

Example 1. *Consider the following two dimensional fractional Navier Stokes system:*

$$\begin{cases} {}^{C}D_t^{\alpha}u + u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} = \rho\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] - q, \\ {}^{C}D_t^{\alpha}v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \rho\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] - q, \end{cases} \tag{6}$$

with

$$u(x,y,0) = -\sin(x+y), \quad v(x,y,0) = \sin(x+y). \tag{7}$$

Applying LHAM, the first three approximate terms of the solution are determined as follows:

$$\begin{aligned} u_1(x,y,t) &= -\frac{t^{\alpha}(q - 2\rho\sin(x+y))}{\Gamma(\alpha+1)}, \\ u_2(x,y,t) &= -\frac{2t^{2\alpha}\left(q\cos(x+y) + 2\rho^2\sin(x+y)\right)}{\Gamma(2\alpha+1)}, \\ u_3(x,y,t) &= \frac{4\rho t^{3\alpha}\left[\Gamma(\alpha+1)^2\left(q\cos(x+y) + 2\rho^2\sin(x+y)\right) + q\Gamma(2\alpha+1)\cos(x+y)\right]}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}, \end{aligned}$$

and

$$\begin{aligned} v_1(x,y,t) &= -\frac{t^{\alpha}(q + 2\rho\sin(x+y))}{\Gamma(\alpha+1)}, \\ v_2(x,y,t) &= \frac{2t^{2\alpha}\left(q\cos(x+y) + 2\rho^2\sin(x+y)\right)}{\Gamma(2\alpha+1)}, \\ v_3(x,y,t) &= -\frac{4\rho t^{3\alpha}\left[\Gamma(\alpha+1)^2\left(q\cos(x+y) + 2\rho^2\sin(x+y)\right) + q\Gamma(2\alpha+1)\cos(x+y)\right]}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}. \end{aligned}$$

These expressions represent the first three iterative terms of the LHAM series solution, which can be used to approximate the behavior of the fluid flow for different fractional orders α.

Example 2. *Let us examine the fractional Navier Stokes equations* (1) *subject to the initial conditions:*

$$\begin{cases} u(x,y,0) = -e^{x+y}, \\ v(x,y,0) = e^{x+y}. \end{cases} \tag{8}$$

LHAM sol for $u(x, y, t)$

t	Exact	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.75$
0.1	−0.761394433	−0.761394433	−0.738892	−0.696949
0.2	−0.688938173	−0.688938173	−0.661196	−0.616449
0.3	−0.623377037	−0.623377037	−0.595824	−0.555655
0.4	−0.564054869	−0.564054869	−0.539466	−0.506667
0.5	−0.510377951	−0.510377951	−0.490252	−0.465841
0.6	−0.461809067	−0.461809067	−0.446917	−0.431073
0.7	−0.417862124	−0.417862125	−0.408526	−0.401008
0.8	−0.378097285	−0.378097287	−0.374355	−0.374702
0.9	−0.342116571	−0.342116578	−0.343821	−0.351472
1	−0.309559875	−0.309559895	−0.316449	−0.330799

Applying the Laplace Homotopy Analysis Method (LHAM), the first three terms of the series approximation can be expressed as:

$$
\begin{aligned}
u_1(x,y,t) = &\ \frac{ht^\alpha\left(q+2\rho e^{x+y}\right)}{\Gamma(\alpha+1)} \\
u_2(x,y,t) = &\ \frac{h(h+1)t^\alpha\left(q+2\rho e^{x+y}\right)}{\Gamma(\alpha+1)} - \frac{2h^2\left(2\rho^2+q\right)t^{2\alpha}e^{x+y}}{\Gamma(2\alpha+1)} \\
u_3(x,y,t) = &\ ht^\alpha\left(\frac{4h^2\rho t^{2\alpha}e^{x+y}\left(\Gamma(\alpha+1)^2\left(2\rho^2+q\right)+q\Gamma(2\alpha+1)\right)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} - \frac{4h(h+1)\left(2\rho^2+q\right)t^\alpha e^{x+y}}{\Gamma(2\alpha+1)} + \frac{(h+1)^2\left(q+2\rho e^{x+y}\right)}{\Gamma(\alpha+1)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
v_1(x,y,t) = &\ \frac{hqt^\alpha}{\Gamma(\alpha+1)} - \frac{2h\rho t^\alpha e^{x+y}}{\Gamma(\alpha+1)} \\
v_2(x,y,t) = &\ \frac{ht^\alpha\left(2h\Gamma(\alpha+1)\left(2\rho^2+q\right)t^\alpha e^{x+y}+(h+1)\Gamma(2\alpha+1)\left(q-2\rho e^{x+y}\right)\right)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} \\
v_3(x,y,t) = &\ ht^\alpha\left(-\frac{4h^2\rho t^{2\alpha}e^{x+y}\left(\Gamma(\alpha+1)^2\left(2\rho^2+q\right)+q\Gamma(2\alpha+1)\right)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} + \frac{4h(h+1)\left(2\rho^2+q\right)t^\alpha e^{x+y}}{\Gamma(2\alpha+1)} + \frac{(h+1)^2\left(q-2\rho e^{x+y}\right)}{\Gamma(\alpha+1)}\right)
\end{aligned}
$$

Example 3. *For the final example, we consider the fractional Navier Stokes system* (1) *with the following initial conditions:*

$$
\begin{cases}
u(x,y,z,0) = -0.5x + y + z, \\
v(x,y,z,0) = x - 0.5y + z, \\
w(x,y,z,0) = x + y - 0.5z.
\end{cases}
\tag{9}
$$

By applying LHAM, the first three terms of the approximate solutions are given

LHAM sol for $v(x,y,t)$

t	Exact	$\alpha=1$	$\alpha=0.9$	$\alpha=0.75$
0.1	0.761394433	0.761394433	0.738892	0.696948
0.2	0.688938173	0.688938173	0.661196	0.616449
0.3	0.623377037	0.623377037	0.595824	0.555655
0.4	0.564054869	0.564054869	0.539466	0.506667
0.5	0.510377951	0.510377951	0.490252	0.465841
0.6	0.461809067	0.461809067	0.446917	0.431073
0.7	0.417862124	0.417862125	0.408526	0.401008
0.8	0.378097285	0.378097287	0.374354	0.374702
0.9	0.342116571	0.342116578	0.343821	0.351472
1	0.309559875	0.309559895	0.316448	0.330799

as:

$$
\begin{aligned}
u_1(x,y,z,t) =&\ \tfrac{ht^{\alpha}(-0.5(-0.5x+y+z)+2x+0.5y+0.5z+0.)}{\Gamma(\alpha+1)}\\
u_2(x,y,z,t) =&\ \tfrac{h^2t^{2\alpha}(-2.25x+3.75y+5.25z)}{\Gamma(2\alpha+1)}+\tfrac{h(h+1)t^{\alpha}(-0.5(-0.5x+y+z)+2x+0.5y+0.5z+0.)}{\Gamma(\alpha+1)}\\
u_3(x,y,z,t) =&\ (h+1)\left(\tfrac{h^2t^{2\alpha}(-2.25x+3.75y+5.25z)}{\Gamma(2\alpha+1)}+\tfrac{h(h+1)t^{\alpha}(-0.5(-0.5x+y+z)+2x+0.5y+0.5z+0.)}{\Gamma(\alpha+1)}\right)\\
&+\tfrac{\left(h^2t^{2\alpha}\left(5.0625hx\Gamma(2\alpha+1)^2t^{\alpha}+\Gamma(\alpha+1)^2(h\Gamma(2\alpha+1)t^{\alpha}(18.75x-3.75y+3.z)+\right.\right.}{\Gamma(\alpha+1)^2\Gamma(2\alpha+1)\Gamma(3\alpha+1)}\\
&\tfrac{\left.\left.\Gamma(3\alpha+1)((-2.25h-2.25)x+3.75hy+5.25hz+3.75y+5.25z))\right)\right)}{\Gamma(\alpha+1)^2\Gamma(2\alpha+1)\Gamma(3\alpha+1)}
\end{aligned}
$$

and

$$
\begin{aligned}
v_1(x,y,z,t) =&\ \tfrac{ht^{\alpha}(-0.5(x-0.5y+z)+0.5x+0.5y+2z+0.)}{\Gamma(\alpha+1)}\\
v_2(x,y,z,t) =&\ \tfrac{h^2t^{2\alpha}(4.5x-0.75y+3.z)}{\Gamma(2\alpha+1)}+\tfrac{h(h+1)t^{\alpha}(-0.5(x-0.5y+z)+0.5x+0.5y+2z+0.)}{\Gamma(\alpha+1)}\\
v_3(x,y,z,t) =&\ (h+1)\left(\tfrac{h^2t^{2\alpha}(4.5x-0.75y+3.z)}{\Gamma(2\alpha+1)}+\tfrac{h(h+1)t^{\alpha}(-0.5(x-0.5y+z)+0.5x+0.5y+2z+0.)}{\Gamma(\alpha+1)}\right)+\\
&\tfrac{h^2t^{2\alpha}\left(\Gamma(\alpha+1)^2(h\Gamma(2\alpha+1)t^{\alpha}(6.75y+13.5z)+\Gamma(3\alpha+1)(4.5hx-0.75hy+3.hz+4.5x-0.75y+3.z)\right)}{\Gamma(\alpha+1)^2\Gamma(2\alpha+1)\Gamma(3\alpha+1)}\\
&\tfrac{\left.+h\Gamma(2\alpha+1)^2t^{\alpha}(1.6875y+3.375z)\right)}{\Gamma(\alpha+1)^2\Gamma(2\alpha+1)\Gamma(3\alpha+1)}
\end{aligned}
$$

and

$$
\begin{aligned}
w_1(x,y,z,t) =&\ \tfrac{0.75hyt^{\alpha}}{\Gamma(\alpha+1)}+\tfrac{1.5hzt^{\alpha}}{\Gamma(\alpha+1)}\\
w_2(x,y,z,t) =&\ ht^{\alpha}\left(\tfrac{ht^{\alpha}(4.5x-0.75y+3.z)}{\Gamma(2\alpha+1)}+\tfrac{0.75hy+1.5hz+0.75y+1.5z}{\Gamma(\alpha+1)}\right)\\
w_3(x,y,z,t) =&\ \tfrac{h(1.h+1.)^2t^{\alpha}(0.75y+1.5z)}{\Gamma(\alpha+1)}+\tfrac{h^2t^{2\alpha}((9.h+9.)x+(-1.5h-1.5)y+(6.h+6.)z)}{\Gamma(2\alpha+1)}+\\
&+\tfrac{h^3t^{3\alpha}\left(\Gamma(\alpha+1)^2(6.75y+13.5z)+\Gamma(2\alpha+1)(1.6875y+3.375z)\right)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}
\end{aligned}
$$

LHAM sol for $u(0.5, 0.5, t)$

t	Exact	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
0.1	-3.004166023	-3.004166023	-3.10115	-3.23246
0.2	-3.320116922	-3.320116922	-3.48085	-3.68981
0.3	-3.669296667	-3.669296667	-3.88849	-4.16838
0.4	-4.055199966	-4.055199966	-4.33225	-4.68253
0.5	-4.481689070	-4.481689070	-4.81816	-5.24106
0.6	-4.953032424	-4.953032424	-5.35188	-5.85132
0.7	-5.473947391	-5.473947390	-5.93919	-6.52039
0.8	-6.049647464	-6.049647458	-6.58624	-7.25554
0.9	-6.685894442	-6.685894419	-7.29967	-8.06448
1	-7.389056098	-7.389056024	-8.08672	-8.95555

We illustrate the exact and LHAM approximate solutions for $u(x, y, 0.5)$ and $v(x, y, 0.5)$ in Figure 1. The absolute error corresponding to Test Example 1 is displayed in Figure 2. For Test Example 2, the outcomes are shown in Figures 3 and 4. In the case of Test Example 3, Figure 5 presents the 10th-order LHAM approximate solutions for u, v, and w with different fractional orders $\alpha = 1, 0.75$, and 0.5.

Figure 6 depicts $u(0.5, 0.5, t)$ and $v(0.5, 0.5, t)$ for varying values of α, highlighting the dependence of the LHAM solution on the fractional order. Tables 1 and 1 summarize the exact and approximate results for Test Example 1, while Tables 2 and 2 present the corresponding findings for Test Example 2.

6. Conclusions

In this work, we have successfully investigated the time fractional Navier Stokes equations in the sense of Caputo using LHAM. By carefully reviewing the fundamentals of fractional calculus alongside the LHAM framework, the study demonstrates the methods effectiveness and versatility in tackling nonlinear fractional differential equations. The approximate solutions derived for different fractional orders α highlight the ability of LHAM to capture intricate fluid dynamics phenomena that classical integer-order models may fail to describe accurately. The findings, presented through both tables and graphical illustrations, confirm the reliability of the method and emphasize the significant role of the fractional order in shaping flow behavior. This research lays the groundwork for further investigations and potential extensions, such as applying LHAM to multi-term fractional Navier Stokes systems or other complex nonlinear fluid models.

LHAM sol for $v(0.5, 0.5, t)$

t	Exact	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
0.1	3.004166023	3.004166023	3.10115	3.23246
0.2	3.320116922	3.320116922	3.48085	3.68981
0.3	3.669296667	3.669296667	3.88849	4.16838
0.4	4.055199966	4.055199966	4.33225	4.68253
0.5	4.481689070	4.481689070	4.81816	5.24106
0.6	4.953032424	4.953032424	5.35188	5.85132
0.7	5.473947391	5.473947391	5.93919	6.52039
0.8	6.049647464	6.049647464	6.58624	7.25554
0.9	6.685894442	6.685894442	7.29967	8.06448
1	7.389056098	7.389056098	8.08672	8.95555

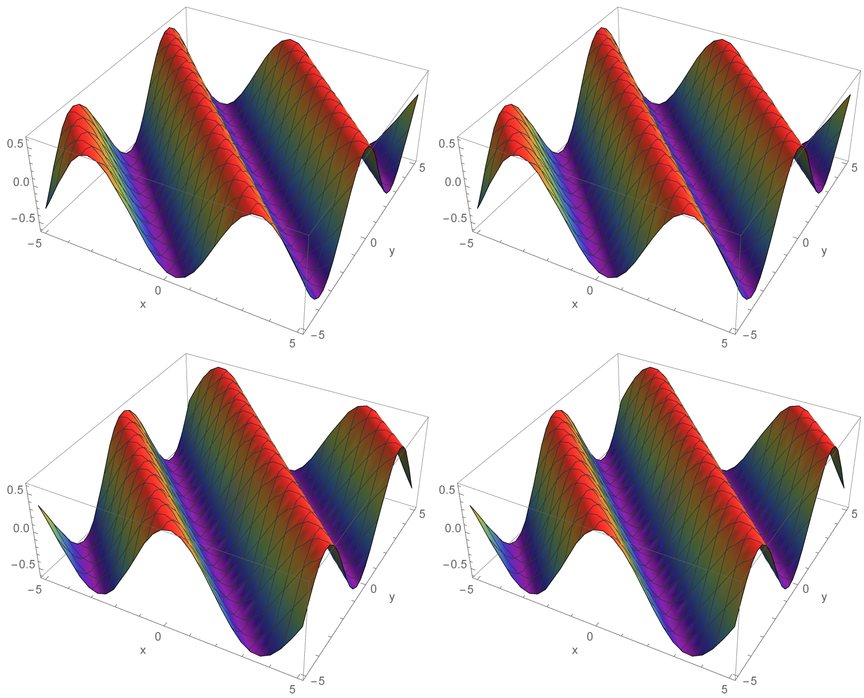

Figure 1: The exact and approximate solution for Example 1 of u (right), and v (left).

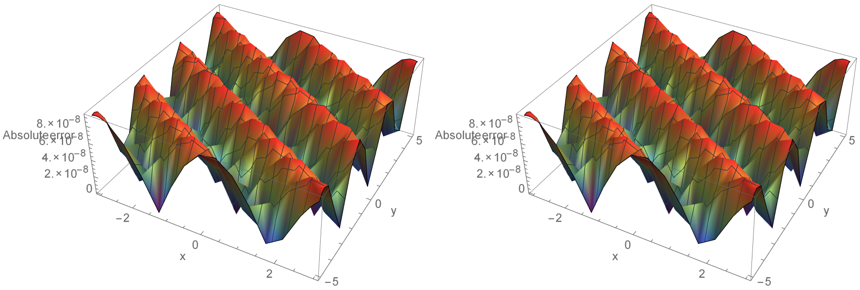

Figure 2: The absolute error for Example 2 of u (right), and v (left).

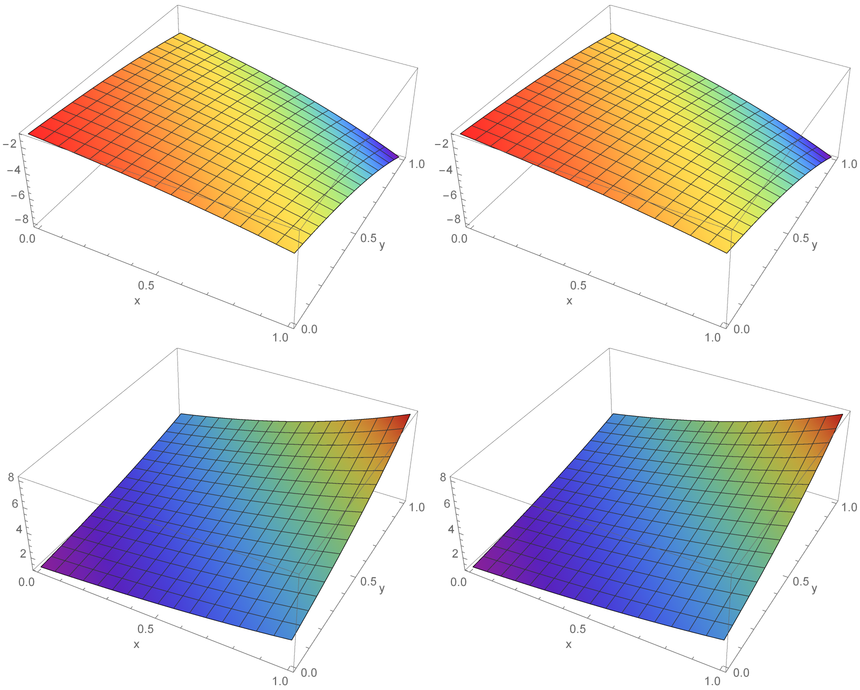

Figure 3: The exact and approximate solution for Example 2 of $u(x, y, 0.1)$, and $v(x, y, 0.1)$.

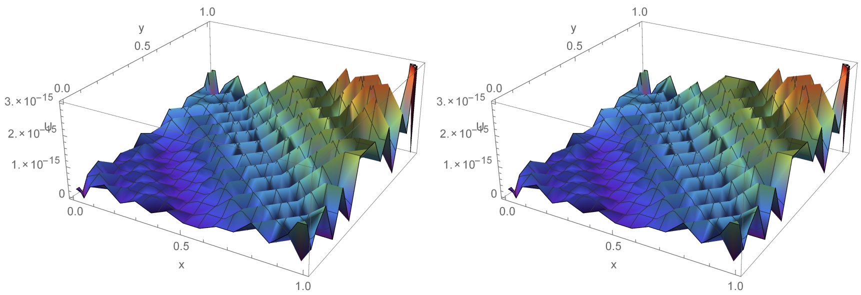

Figure 4: The absolute error for Example 2 of $u(x, y, 0.1)$, and $v(x, y, 0.1)$.

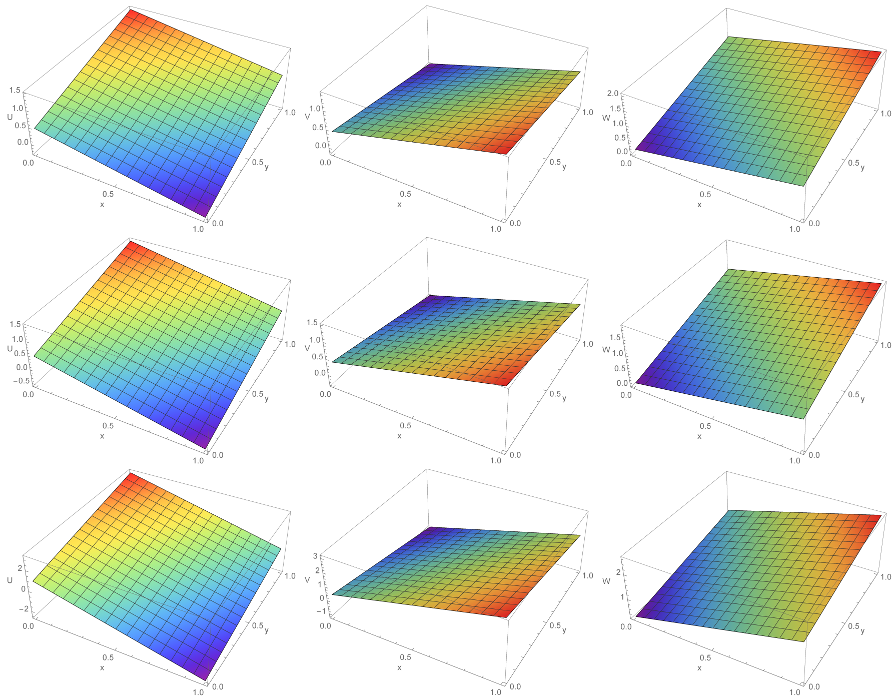

Figure 5: The LHAM solution for Example 3 of u, v, and w for $\alpha = 1$, $\alpha = 0.75$, and $\alpha = 0.5$.

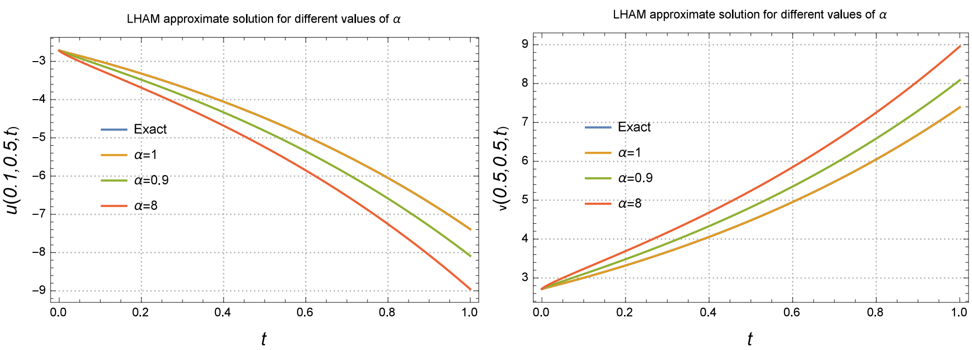

Figure 6: The approximate solution for different values of α in Example 2 for u, and v.

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Y.Massoun
Department of Mathematics, University of Algiers, Benyoucef Benkhedda, 2 Rue Didouche Mourad,16000, Algeria.
E-mail: youcef.ens@hotmail.fr

A.K Alomari
Department of Mathemtics, Faculty of Science, Islamic University of Madinah, 42351, Madinah, Saudi Arabia.
E-mail: abdomari2008@yahoo.com

C.Cesarano
Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy
E-mail: clemente.cesarano@uninettuno.it

B.J. Mammadov
Depatment of Mathematics and Computer Science, Baku Engineering University University Khirdalan city, Hasan Aliyev str., 120 AZ0101,Absheron, Azerbaijan
*E-mail:*bmemmedov1@beu.edu.az

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