Journal of Contemporary Applied Mathematics
V. 8, No 1, 2018, July
ISSN 2222-5498

On embedding theorems in Sobolev-Morrey type spaces $\bigcap\limits_{i=1}^{2^n} L_{p^i,\varphi,\beta}^{<l^i>}(G)$

Aygun T.Orujova, Rovshan F. Babayev

Abstract. In this paper it is constructed a new generalized Sobolev-Morrey type spaces and untilizing integral representation of weak derivatives of functions defined on n-dimensional domains satisfying flexible φ-horn condition it is proved theorems on differential properties functions from in this spaces is established.

Key Words and Phrases: generalized Sobolev-Morrey type spaces with dominant mixed derivatives, integral representation, embedding theorems.

2010 Mathematics Subject Classifications: Primary 26A33, 46E30, 46E35

---

1. Introduction

In this paper we introduce a generalized Sobolev-Morrey type spaces with dominant mixed derivatives

$$\bigcap_{i=1}^{2^n} L_{p^i,\varphi,\beta}^{<l^i>}(G), \tag{1}$$

where $G \subset R^n$, $1 \le p^i < \infty$, $i = 1, 2, ..., 2^n$, $e_n = \{1, 2, ..., n\}$, $e^i \subseteq e_n$(the number of all possible vectors e^i is equal 2^n); let $e^i = \left(e_1^i, ..., e_n^i\right)$, $e_j^i > 0$ are entire $j \in e^i$; $e_j^i \ge 0$ are entire $j \in e_n \backslash e^i$; $\beta \in [0, 1]^n$; $\varphi(t) = (\varphi_1(t_1), ..., \varphi_n(t_n))$, $\varphi_j(t_j) > 0$ $(t_j > 0,\ j \in e_n)$ by Lebesgue measurable functions $\lim\limits_{t_j \to +0} \varphi_j(t_j) = 0$ and $\lim\limits_{t_j \to +\infty} \varphi_j(t_j) = L \le \infty$. We denote by A the set of all vector functions. Further, using the integral representation method we study differential and differential-difference properties of functions from this spaces.

Definition. Denote by $\bigcap_{i=1}^{2^n} L_{p^i,\varphi,\beta}^{<l^i>}(G)$ the spaces locally summable functions f on G having the weak derivatives $D^{e^i} f$ with the finite norm

$$\|f\|_{\bigcap_{i=1}^{2^n} L_{p^i,\varphi,\beta}^{<l^i>}(G)} = \sum_{i=1}^{2^n} \left\| D^{l^i} f \right\|_{p^i,\varphi,\beta,G}, \tag{2}$$

where

$$\|f\|_{p^i,\varphi,\beta,G}=\|f\|_{L_{p,\varphi,\beta}(G)}=\sup_{\substack{x\in G,\\ t>0}}\left(|\varphi([t]_1)|^{-\beta}\|f\|_{p,G_{\varphi(t)}(x)}\right), \tag{3}$$

$|\varphi([t]_1)|^{-\beta}=\prod_{j\in e_n}\left(\varphi_j\left([t_j]_1\right)\right)^{-\beta_j}$, $[t_j]_1=\min\{1,t_j\}$, $j\in e_n$. For any $x\in R^n$

$$G_{\varphi(t)}(x)=G\bigcap\left\{y:|y_j-x_j|<\frac{1}{2}\varphi_j(t_j),\ j\in e_n\right\}.$$

The space $\bigcap_{i=1}^{2^n}L^{<l^i>}_{p^i,\varphi,\beta}(G)$ in the case when $\beta_j=0$ $(j\in e_n)$ coincides with the $\bigcap_{i=1}^{2^n}L^{l^i}_{p^i,\varphi,\beta}(G)$ introduced and studied [3], in the case $p^i=p$, $l^i=l=(l_1,...,l_n)$, $l_j\in N$, $j\in e_n$ and let $l^e=\left(l_1^2,...,l_n^e\right)$, $l_j^e=l_j$ for $j\in e_n\backslash e$ coincides with the spaces $S^l_{p,\varphi,\beta}W(G)$ introduced and studied [12].

The spaces of such type with different norms were introduced and studied in [1, 4-13] and others.

Let for any $t_j>0$ $(j\in e_n)$, there exists a positive constant $C>0$ such that $|\varphi[t]_1|\le C$, then the embeddings $L_{p,\varphi,\beta}(G)\hookrightarrow L_p(G)$, $\bigcap_{i=1}^{2^n}L^{<l^i>}_{p^i,\varphi,\beta}(G)\hookrightarrow\bigcap_{i=1}^{2^n}L^{<l^i>}_{p^i}(G)$ hold i.e.

$$\|f\|_{p,G}\le c\|f\|_{p,\varphi,\beta,G}\quad\text{and}\quad\|f\|_{\bigcap_{i=1}^{2^n}L^{<l^i>}_{p^i}(G)}\le c\|f\|_{\bigcap_{i=1}^{2^n}L^{<l^i>}_{p^i,\varphi,\beta}(G)} \tag{4}$$

The spaces $L_{p,\varphi,\beta}(G)$ and $\bigcap_{i=1}^{2^n}L^{<l^i>}_{p^i}(G)$ are complete.

Lemma 1.1. Let $G\subset R^n$, $1\le p^i<\infty$ and $f\in\bigcap_{i=1}^{n}L^{<l^i>}_{p^i}(G)$. Then we can construct the sequence $h_s=h_s(x)$ $(s=1,2,...)$ of infinitely differentiable finite in R^n functions for which

$$\lim_{s\to\infty}\|f-h_s\|_{\bigcap_{i=1}^{n}L^{<l^i>}_{p^i}(G)}=0. \tag{5}$$

Proof. Let $G=\bigcup_{\lambda=1}^{M}G^\lambda$. For obtaining equality (5) we estimate the

$$\|f-h_s\|_{\bigcap_{i=1}^{n}L^{<l^i>}_{p^i}(G)}=\sum_{i=1}^{2^n}\left\|D^{le^i}(f-h_s)\right\|_{p^i,G}. \tag{6}$$

The sequence $h_s(x)$ $(s=1,2,...)$ is determined by the equalities

$$h_s(x)=F(x,\varphi(t))|_{t=\frac{1}{s}}=\sum_{\lambda=1}^{M}\eta_\lambda(x)f_{\varphi^\lambda(t)}(x),$$

here the averaging functions are determined as follows:

$$f_{\varphi^\lambda(t)}(x)=\int_{R^n}f\left(x+\varphi^\lambda(t)y\right)K_\lambda(y)\,dy,$$

where $K_\lambda(y) \in C_0^\infty(R^n)$ $(\lambda = 1, 2, ..., M)$, $\operatorname{supp} K_\lambda(\,\cdot\,) \subset [-1,\, 1]$ and

$$\int_{R^n} K_\lambda(y)\, dy = 1,$$

the functions $\eta_\lambda = \eta_\lambda(x)$ $(\lambda = 1, 2, ..., M)$ determine the expansion of a unit in the domain G, i.e.

1. $0 \le \eta_\lambda(x) \le 1$ in R^n;
2. $\eta_\lambda(x) = 0$ in $G \backslash G_\lambda$ for all $\lambda = 1, 2, ..., M$;
3. $\sum_{\lambda=1}^{M} \eta_\lambda(x) = 1$ in G;
4. $|D^\alpha \eta_\lambda(x)| \le C_\lambda$, $C_\lambda = const$ for all $\lambda = 1, 2, ..., M$ and $\alpha \ge 0$.

Obviously,

$$f(x) - h_s(x) = \sum_{\lambda=1}^{M} \eta_\lambda(x) \left(f(x) - f_{\varphi^\lambda(t)}(x) \right).$$

Consequently,

$$\|f(\,\cdot\,) - h_s(\,\cdot\,)\|_{\bigcap_{i=1}^n L_{p^i}^{<l^i>}(G)} \le \sum_{\lambda=1}^{M} \left\| \eta_\lambda(\,\cdot\,) \left(f(\,\cdot\,) - f_{\varphi^\lambda(t)}(\,\cdot\,) \right) \right\|_{\bigcap_{i=1}^n L_{p^i}^{<l^i>}(G)} \le$$

$$\le C \sum_{\lambda=1}^{M} \left\| \left(f(\,\cdot\,) - f_{\varphi^\lambda(t)}(\,\cdot\,) \right) \right\|_{\bigcap_{i=1}^n L_{p^i}^{<l^i>}(G)}. \tag{7}$$

As much as small for rather small t, as a consequence of continuity of L_p-average functions, belonging to the space $L_p\left(G^\lambda\right)$, from (7) it follows

$$\|f(\,\cdot\,) - h_s(\,\cdot\,)\|_{\bigcap_{i=1}^n L_{p^i}^{<l^i>}(G)} < \varepsilon,$$

in other words,

$$\lim_{s\to\infty} \|f - h_s\|_{\bigcap_{i=1}^n L_{p^i}^{<l^i>}(G)} = 0.$$

Assuming that $\varphi_j(t_j)(j \in e_n)$ are also differentiable on $[0,\, T_j]$ $(j \in e_n)$, we can show that for $f \in \bigcap_{i=1}^n L_{p^i}^{<l^i>}(G)$ determined in n- dimensional domains, satisfying the condition of flexible φ- horn defined in [Naj] it holds the following integral representation $(f_x \in U \subset G)$

$$D^\nu f(x) = \sum_{i=1}^{2^n} (-1)^{|\nu|+|l^i|} \prod_{j \in e_n \backslash e^i} (\varphi_j(T_j))^{l^i_{j-1-\nu_j}} \times$$

$$\times \int_{0^i}^{T^i} \int_{R^n} M_i^{(\nu)} \left(\frac{y}{\varphi(t+T)^i}, \frac{\rho\left(\varphi(t+T)^i, x\right)}{\varphi(t+T)^i}, \rho'\left(\varphi(t+T)^i, x\right) \right) \times$$

$$\times D^{l^i} f(x+y) \prod_{j\in e^i} (\varphi_j(t_j))^{l_j^i-\nu_j-2} \times$$

$$\times \prod_{j\in e^i} \left(\varphi_j'(t_j)\right) dt^i dy, \tag{8}$$

where

$$\int_{0^i}^{T^i} f(x)\, dx^i = \left(\prod_{j\in e^i} \int_0^{T_j} dx_j\right) f(x),$$

i.e. integration is carried out only with respect to the variables x_j whose indices belong to e^i, $(t+T)^i = t_j \left(j\in e^i\right)$; $(t+T)^i = T_j \left(j\in e_n\backslash e^i\right)$; $\nu=(\nu_1,...,\nu_n)$, $\nu_j\geq 0 (j\in e_n)$ are integers. Recall that the flexible φ-horn $x+\bigcup_{\substack{0<t_j\leq T_j\\ j\in e_n}} [\beta(\varphi(t),)+\varphi(T)\theta_I]$ is the support of the representation (8).

Let $M_i(\cdot,y,z)\in C_0^\infty(R^n)$ $(i=1,2,\ldots,2^n)$, be such that

$$S(M_i)\subset I_{\varphi(T)} = \left\{y: |y_j| < \frac{1}{2}\varphi_j(T_j), \quad j=e_n\right\}.$$

and assume that for any $0<T_j\leq 1$ $(j\in e_n)$

$$V = \bigcup_{0<t_j\leq T_j, j\in e_n} \left\{y: \frac{y}{\varphi\left((t+T)^i\right)} \in S(M_i)\right\}.$$

It is clear that $V\subset I_{\varphi(T)}$ and suppose that $U+V\subset G$.

Lemma 1.2. Let $1\leq pi\leq p\leq r\leq\infty$; $0<\eta_j$, $t_j<T_j\leq 1$ $(j\in e_n)$, $\nu=(\nu_1,\nu_2,\ldots,\nu_n)$, $\nu_j\geq 0$ be entire, $(j\in e_n)$; $\Phi\in L_{p,\varphi,\beta}(G)$ and

$$\varepsilon_j = l_j^i - \nu_j - (1-\beta_j p)\left(\frac{1}{p^i}-\frac{1}{p}\right),$$

$$R_\eta^i(x) = \prod_{j=e_n\backslash e'} (\varphi_j(T_j))^{l_j^i-1-\nu_j} \int_{0^i}^{T^i} L_i(x;t,T)\Phi(x+y)$$

$$\times \prod_{j\in e^i} (\varphi_j(t_j))^{l_j^i-\nu_j-2} \prod_{j\in e^i} \varphi_j'(t_j)\, dt^i, \tag{9}$$

$$R_{\eta,T}^i(x) = \prod_{j=e_n\backslash e'} (\varphi_j(T_j))^{l_j^i-\nu_j-2} \int_{0^i}^{T^i} L_i(x;t,T)\Phi(x+y)$$

$$\times \prod_{j\in e^i} (\varphi_j(t_j))^{l_j^i-\nu_j-2} \prod_{j\in e^i} \varphi_j'(t_j)\, dt^i, \tag{10}$$

where

$$L_i(x;t,T)=\int_{R^n} M_i^{(\nu)}\left(\frac{y}{\varphi\left((t+T)^i\right)},\frac{\rho\left(\varphi\left((t+T)^i,x\right)\right)}{\varphi\left((t+T)^i\right)},\rho'\left(\varphi\left((t+T)^i\right),x\right)\right)dy.$$

Then for any $\overline{x}\in U$ the following inequalities are true

$$\sup_{\overline{x}\in U}\left\|R^i_\eta\right\|_{q,U_{\psi(\xi)}(\overline{x})}\le C_1\left\|\Phi\right\|_{p^i,\varphi,\beta;G}\times$$

$$\prod_{j\in e_n\setminus e^i}\left(\varphi_j\left(T_j\right)\right)^{l^i_j-\nu_j-(1-\beta_j p^i)\left(\frac{1}{p^i}-\frac{1}{p}\right)}\prod_{j\in e_n}\left(\psi_j\left([\xi]_1\right)\right)^{\beta_j\frac{p^i}{p}}\prod_{j\in e^i}\left(\varphi_j\left(\eta_j\right)\right)^{-\varepsilon^i_j}\ (\varepsilon^i_j>0), \tag{11}$$

$$\sup_{\overline{x}\in U}\left\|R^i_{\eta,T}\right\|_{q,U_{\psi(\xi)}(\overline{x})}\le C_2\left\|\Phi\right\|_{p^i,\varphi,\beta;G}\times$$

$$\prod_{j\in e_n\setminus e^i}\left(\varphi_j\left(T_j\right)\right)^{l^i_j-\nu_j-(1-\beta_j p^i)\left(\frac{1}{p^i}-\frac{1}{p}\right)}\prod_{j\in e_n}\left(\psi_j\left([\xi]_1\right)\right)^{\beta_j\frac{p^i}{p}}\times$$

$$\begin{cases}\prod\limits_{j\in e^i}\left(\varphi_j(T_j)\right)^{-\varepsilon^i_j} & \text{for } \varepsilon^i_j>0,\\ \prod\limits_{j\in e^i}\ln\frac{\varphi_j(T_j)}{\varphi_j(\eta_j)} & \text{for } \varepsilon^i_j=0\\ \prod\limits_{j\in e^i}\left(\varphi_j\left(\eta_j\right)\right)^{-\varepsilon^i_j} & \text{for } \varepsilon^i_j<0,\end{cases} \tag{12}$$

$$\left\|R^i_\eta\right\|_{p,\psi,\beta_1,U}\le C\left\|\Phi\right\|_{p^i,\psi,\beta;G} \tag{13}$$

where $U_{\psi(\xi)}(\overline{x})=\left\{x:|x_j-\overline{x}_j|<\frac{1}{2}\psi_j\left(\xi_j\right),j=e_n\right\}$ and $\psi\in N$, C_1,C_2 are the constants independent of φ, ξ, η and T.

2. Main results

Prove two theorems on the properties of the functions from the space $\bigcap_{i=1}^{2^n} L^{<l^i>}_{p^i,\varphi,\beta}(G)$.

Theorem 2.1. Let $G\subset R^n$ satisfy the condition of flexible φ-horn, $1\le p^i\le p\le\infty$, $\nu=(\nu_1,\nu_2,\ldots,\nu_n)$, $\nu_j\ge 0$ be entire $j\in e_n$, $\varepsilon^i_j>0 (j\in e_n, i=1,2,\ldots,2^n)$ and let $f\in\bigcap_{i=1}^{2^n} L^{<l^i>}_{p^i,\varphi,\beta}(G)$. Then the following embeddings hold

$$D^\nu:\bigcap_{i=1}^{2^n} L^{<l^i>}_{p^i,\varphi,\beta}(G)\to L_{p,\psi,\beta^1}(G)$$

i.e. for $f\in\bigcap_{i=1}^{2^n} L^{<l^i>}_{p^i,\varphi,\beta}(G)$ there exists a generalized derivative $D^\nu f$ and the following inequalities are true

$$\left\|D^\nu f\right\|_{p,G}\le C_1\sum_{i=1}^{2^n}\prod_{j\in e_n}\left(\varphi_j(T_j)\right)^{\varepsilon_j}\left\|D^{l_e}f\right\|_{p^i,\varphi,\beta;G} \tag{14}$$

$$\|D^\nu f\|_{p,\psi,\beta^1;G} \le C_2 \|f\|_{\bigcap\limits_{i=1}^{2^n} L^{<l^i>}_{p^i,\varphi,\beta}(G)}, \ (p^i \le p < \infty), \tag{15}$$

In particular, if

$$\varepsilon^i_{j,0} = l^i - \nu_j - (1-\beta_j p^i)\frac{1}{p} > 0, j \in e_n, i = 1,2,\ldots,2^n$$

then $D^\nu f(x)$ is continuous on G, i.e.

$$\sup_{x\in G} |D^\nu f(x)| \le C_1 \sum_{i=1}^{2^n} \prod_{j\in e_n} (\varphi_j(T_j))^{\varepsilon^i_{j,0}} \left\| D^{l^i} f \right\|_{p^i,\varphi,\beta;G} \tag{16}$$

where $0 < T_j \le \min\{1, T_{0j}\}$, T_0 is a fixed number; C_1, C_2 are the constants independent of f, C_1 are independent also on T.

Proof. At first note that in the conditions of our theorem there exists a generalized derivative $D^\nu f$ on G. Indeed, from the condition $\varepsilon^i_j > 0, (j \in e_n)$ it follows that for $f \in \bigcap\limits_{i=1}^{2^n} L^{<l^i>}_{p^i,\varphi,\beta}(G) \to \bigcap\limits_{i=1}^{2^n} L^{<l^i>}_{p^i}(G)$, there exists $D^\nu f \in L_{p^i}(G)$ and for it integral representation (8) with the same kernels is valid.

Based around the Minkowsky inequality, from identities (8) we get

$$\|D^\nu f\|_{p,G} \le \sum_{i=1}^{2^n} \left\|R^i_T\right\|_{p,G}. \tag{17}$$

By means of inequality (11) for $U = G, D^{l^i} f = \Phi\ \eta_j = T_j, (j \in e_n)\ \xi \to \infty$ we get inequality (14). By means of inequality (13) for $\eta_j = T_j, (j \in e_n)$ we get inequality (15).

Now let conditions $\varepsilon^i_j > 0, (j \in e_n, i = 1,2,\ldots,2^n)$ be satisfied, then based around identities (8) from inequality (17) we get

$$\left\|D^\nu f - D^\nu f_{\varphi(T)}\right\|_{\infty,G} \le C \sum_{\varnothing\ne e\subseteq e_n} \prod_{j\in e} (\varphi_j(T_j))^{\varepsilon^i_{j,0}} \left\|D^{l^i} f\right\|_{p^i,\varphi,\beta;G}.$$

As $T_j \to 0$, the left side of this inequality tends to zero, since $f_{\varphi(T)}(x)$ is continuous on G and the convergence on $L_\infty(G)$ coincides with the uniform convergence. Then the limit function $D^\nu f$ is continuous on G. Theorem 1 is proved.

Let γ be an n-dimensional vector.

Theorem 2.2. Let all the conditions of theorem 1 be fulfilled. Then for $\varepsilon^i_j > 0$ $(j \in e_n, i = 1,2,\ldots,2^n)$ the generalized derivatives $D^\nu f$ satisfies on G the generalized Holder condition, i.e. the following inequality is valid:

$$\|\Delta(\gamma, G) D^\nu f\|_{p,G} \le C \|f\|_{\bigcap\limits_{i=1}^{2^n} L^{<l^i>}_{p^i,\varphi,\beta}(G)} \prod_{j\in e_n} (\varphi_j(|\gamma_j|))^{\sigma_j}, \tag{18}$$

where σ_j $(j \in e_n)$ any number, satisfying the inequality:

$$\begin{cases} 0 \le \sigma_j \le 1, & \text{if } \varepsilon^i_j > 1 \\ 0 \le \sigma_j < 1, & \text{if } \varepsilon^i = 1 \\ 0 \le \sigma_j \le \mu_j, & \text{if } \varepsilon^i < 1 \end{cases} \tag{19}$$

If $\varepsilon^i_{j,0} > 0$ $(j \in e_n, i = 1, 2, \ldots, 2^n)$, then

$$\sup_{x \in G} |\Delta(\gamma, G) D^\nu f(x)| \le C \|f\|_{\bigcap_{i=1}^{2^n} L^{<l^i>}_{p^i,\varphi,\beta}(G)} \prod_{j \in e_n} (\varphi_j(|\gamma_j|))^{\sigma^0_j}. \tag{20}$$

where σ^0_j satisfy the same conditions as σ_j, but replaced ε^i_j in $\varepsilon^i_{j,0}$.

Proof. According to lemma 8.6 from [2] there exists a domain

$$G_u \subset G \left(u_j = \zeta_j k\left(x\right), \, \zeta_j > 0, j \in e_n, \; k\left(x\right) = \rho\left(x, \partial G\right), \; x \in G\right)$$

and assume that $|\gamma| < u$, then for any $x \in G_u$ the segment connecting the points $x, x+\gamma$ is contained in G. Consequently, for all the points of this segment, identies (8) with the same kernels are valid. After same transformations, from (8) we get

$$\begin{aligned}
|\Delta\left(\gamma, G\right) D^\nu f\left(x\right)| \le C_1 \sum_{i=1}^{2^n} \prod_{j \in e_n \setminus e^i} \left(\varphi_j\left(T_j\right)\right)^{l^i_j - 1 - \nu_j} \\
\times \int_{0^i}^{|\gamma^i_1|} \cdots \int_{0^i}^{|\gamma^i_n|} \prod_{j \in e^i} \left(\varphi_j\left(t_j\right)\right)^{l^i_j - \nu_j - 2} \prod_{j \in e^i} \varphi'_j(t_j) dt_j \\
\times \int_{R^n} \left| M^{(\nu)}_i \left(\frac{y}{\varphi(t+T)^i}, \frac{\rho\left(\varphi(t+T)^i, x\right)}{\varphi(t+T)^i}, \rho'\left(\varphi(t+T)^i, x\right) \right) \right| \\
\times \left| \Delta\left(\gamma, G\right) D^{l^i} f\left(x+y\right) \right| dy + C_2 \sum_{i=1}^{2^n} \prod_{j \in e_n \setminus e^i} \left(\varphi_j\left(T_j\right)\right)^{l^i_j - 2 - \nu_j} \\
\times \prod_{j \in e_n} \varphi_j(|\gamma_j|) \int_{|\gamma^i_1|}^{T^i_1} \cdots \int_{|\gamma^i_n|}^{T^i_n} \prod_{j \in e^i} \left(\varphi_j\left(t_j\right)\right)^{l^i_j - \nu_j - 3} \prod_{j \in e^i} \varphi'_j(t_j) dt^i_j \\
\times \int_{R^n} \left| M^{(\nu+1)}_i \left(\frac{y}{\varphi(t+T)^i}, \frac{\rho\left(\varphi(t+T)^i, x\right)}{\varphi(t+T)^i}, \rho'\left(\varphi(t+T)^i, x\right) \right) \right| \\
\times \int_0^1 \cdots \int_0^1 \left| D^{l^i} f(x + y + \gamma_1 u_1 + \ldots + \gamma_n u_n) \right| dy du \\
= C_1 \sum_{i=1}^{2^n} A^i_\gamma(x,t) + C_2 \sum_{i=1}^{2^n} A^i_{\gamma,T}(x,t),
\end{aligned} \tag{21}$$

where $0 < T_j \leq \{1, T_{0,j}\}$, $j \in e_n$. We also assume that $|\gamma_j| < T_j (j \in e_n)$. Consequently, $|\gamma_j| < \min(u_j, T_j)(j \in e_n)$. If $x \in G \setminus G_u$ then by definition

$$\Delta(\gamma, G) D^\nu f(x) = 0.$$

Based around (19) we have

$$\|\Delta(\gamma, G) D^\nu f\|_{p,G} \leq C_1 \sum_{i=1}^{2^n} \left\|A_\gamma^i(\cdot, \gamma)\right\|_{p,G_u} +$$

$$+C_2 \sum_{i=1}^{2^n} \left\|A_{\gamma,T}^i(\cdot, \gamma)\right\|_{p,G_u}. \tag{22}$$

By means of inequality (11), for $D^{l_i} f = \Phi$, $\eta = |\gamma_j|$ $(j \in e_n)$ we get

$$\left\|A_{|\gamma|}^i(\cdot, \gamma)\right\|_{p,G_u} \leq C_1 \left\|D^{l^i} f\right\|_{p^i,\varphi,\beta;G} \prod_{j \in e_n} (\varphi_j(|\gamma_j|))^{\varepsilon_j^i}. \tag{23}$$

and by means of inequality (12) for $D^{l^i} f = \Phi$, $\eta_j = |\gamma_j|$, $j \in e_n$ we get

$$\left\|A_{|\gamma|,T}(\cdot, \gamma)\right\|_{p,G_u} \leq C_2 \left\|D^{l^i} f\right\|_{p^i,\varphi,\beta;G} \prod_{j \in e_n} \varphi_j(|\gamma_j|) \prod_{j \in e_n} (\varphi_j(|\gamma_j|))^{\delta_j}. \tag{24}$$

From inequalities (22) - (24) we get the required inequality.

Now suppose that $|\gamma_j| \geq \min(u_j, T_j)$, $(j \in e_n)$. Then

$$\|\Delta(\gamma, G) D^\nu f\|_{p,G} \leq 2 \|D^\nu f\|_{p,G} \leq C(u, T) \|D^\nu f\|_{p,G} \prod_{j \in e_n} (\varphi_j(|\gamma_j|))^{\delta_j}.$$

Estimating for $\|D^\nu f\|_{p,G}$ by means of inequality (14)), in this case we get estimation (18).

Theorem Theorem 2 is proved.

References

[1] Akbulut A. , Eroglu A. and Najafov A.M.: *Some Embedding Theorems on the Nikolskii-Morrey Type Spaces*, Advances in Analysis, 1(1), 18-26 (2016).

[2] Besov O.V. , Il'yin V.P., Nicolskii S. M. : *Integral representations of functions and embeddings theorems*, M. Nauka, 1996, 480 p.

[3] Dzhabrailov A. D.: *Families of spaces of functions whose mixed derivatives satisfy a multiple-integral Holder condition*, Trudy Mat. Inst. Steklov., 117, 139-158 (1972).

[4] Guliyev V.S., Mustafayev R.Ch. : *Boundedness of the anisotropic maximal and anisotropic singular integral operators in generalized Morrey spaces*, Acta Mathematica Sinica-English series, 27 (12), 2361-2370 (2011).

[5] Guliyev V.S. , Sawano Y.: *Linear and sublinear operators on Generalized Morrey spaces with non-doubling measures*, Publicationes Mathematicae Debrecen, 83 (3), 1-17 (2013).

[6] Morrey C.B. : *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., 43, 126-166 (1938).

[7] Nakai E. : *Hardy-Littlewood maximal, singular integral operator and Riesz potentials on generalized Morrey spaces*, Math. Nach. 166, 95-103 (1994).

[8] Najafov A. M.: Spaces with function parameters with mixed derivatives, Dokt. tezis, 1996, 126 p.

[9] Nadzhafov A. M.: Embedding theorems in the Sobolev-Morrey type spaces $S_{p,a,\varkappa,\tau}W(G)$ with dominant mixed derivatives , Siberian Mathematical Journal, 47(3), 505-516 (2006).

[10] Najafov A.M. : *Some properties of functions from the intersection of Besov-Morrey type spaces with dominant mixed derivatives*, Proc. of A. Razmadze Math. Inst. 139, 71-82 (2005).

[11] Najafov A.M.: *The embedding theorems of space $W^l_{p,\varphi,\beta}(G)$*, Math. Aeterna, 3(4), 299 - 308 (2013).

[12] Alik M. Najafov, Nilufer R. Rustamova *On properties functions from Sobolev-Morrey type spaces with dominant mixed derivatives*, Transactions of NAS of Azerbaijan, Issue Mathematics, 2017

[13] Najafov A.M., Babayev R.F., Some properties of functions from generalized Sobolev-Morrey type spaces, Math. Aeterna, 2017.

Aygun T. Orujova
Institute of Mathematics and Mechanics of NAS of Azerbaijan
*E-mail:*orudjova.aygun@gmail.com

Rovshan F. Babayev
Ganja State University, Azerbaijan
*E-mail:*rovsanbaba@box.az