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On the Mathematical Expectation of the Renewal-Reward Process

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Abstract. In this paper the mathematical expectation of the renewal-reward process is investigated and some results for the renewal function are generalized to the mathematical expectation of the renewal-reward process. Finiteness of the mathematical expectation of the renewal-reward process is proved. Also, an inequality that gives us an upper bound for the renewal function is generalized for the mathematical expectation of the renewal-reward process.

Key Words and Phrases: Renewal process, renewal function, renewal-reward process, mathematical expectation
2010 Mathematics Subject Classifications: 60G20, 60G50, 60K05

1. Introduction

Renewal-reward processes occur in various stochastic optimization models, particularly in Markov and semi-Markov decision processes (see, for example, [1-5], [8]).

In [7] there were obtained asymptotic expansions for the mathematical expectation and variance of the renewal-reward process. These are the generalizations of the analogical formulas for the renewal processes. Also, in [9] there was obtained an asymptotic expansion for the covariance function of the rewards of the multivariate renewal-reward process. In [2] the remainder terms of the asymptotic expansions in [7] and [9] were sharpened from $o(1)$ to $o(t^{-k})$.

The results that obtained for the renewal-reward processes as the generalization of the renewal processes make it necessary to generalize other formulas for the renewal processes to the renewal-reward processes.

Our aim is to investigate the renewal-reward process and to generalize some inequalities proved for the renewal function to these processes. First, let us introduce some notations and theorems from literatures.

Consider a sequence of independent and identically distributed random variables $\{T_i, i = 1, 2, \ldots\}$ with common distribution function $F(t) = P\{T < t\}$. Define

$$S_n = \sum_{i=1}^{n} T_i, \;\; n = 1, 2, \ldots,$$

$$N(t) = \max\{n : S_n \le t\}, \tag{1}$$

$$H(t) = E\{N(t)\}. \tag{2}$$

(1) is called a renewal process and (2) is called a renewal function.

Define

$$\mu_k = E\{T^k\} = \int_0^\infty t^k dF(t), \ \ k \ge 1.$$

The next theorem is about finiteness of the renewal function (see, for example, [10]).

Theorem 1. *$H(t) < \infty$ for all $0 \le t < \infty$.*

The following inequality has been proved for the renewal function (see [6]).

Theorem 2. *Let $F(t)$ be an arbitrary distribution function. Then for all $t \ge 0$*

$$H(t) \le 1 + \frac{2t}{m_0}. \tag{3}$$

where m_0 is the median of distribution F.

2. Main results

Main purpose of this paper is to generalize Theorem 1 and Theorem 2 to mathematical expectation of the renewal-reward. For this, first, let us introduce some notations.

Consider a sequence of independent random vectors $\{(T_i, X_i),\ i = 1, 2, \ldots\}$, where (T_i, X_i), $i \ge 1$, are identically distributed. Assume that $\{T_i,\ i = 1, 2, \ldots\}$ is a renewal sequence. Consider the process

$$C(t) = \sum_{i=1}^{N(t)} X_i, \ \ t \ge 0. \tag{4}$$

The process C is called a renewal-reward process, and is a generalization of a renewal process.

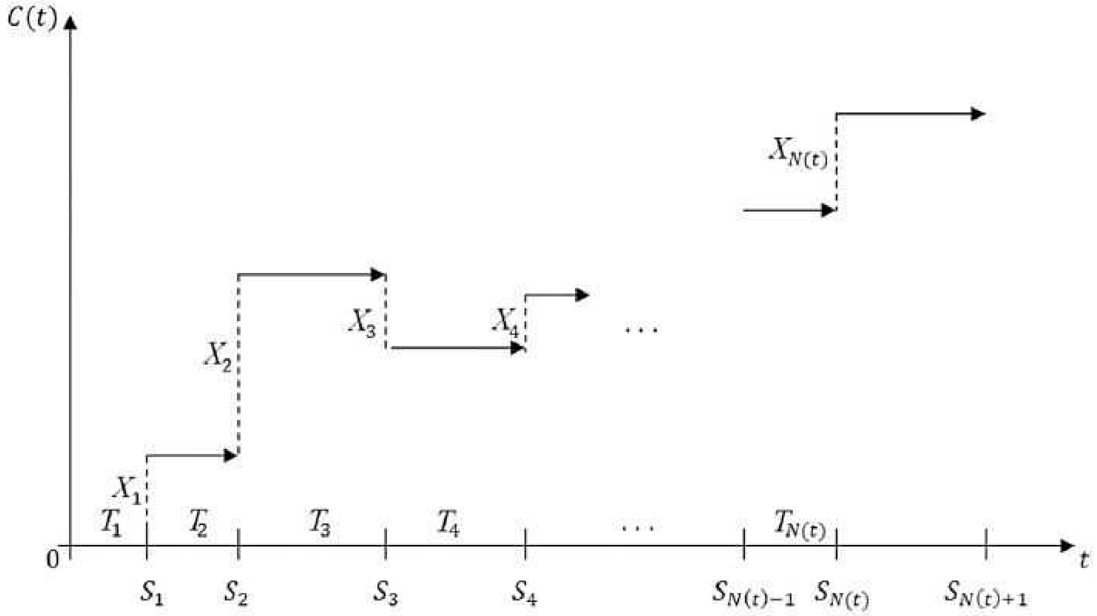

Figure 1: A trajectory of the process $C(t)$.

Example 1 ([10]). Consider a system that can be in two states: on or off. Initially it is on and it remains on for a time Z_1; it then goes off and remains off for a time Y_1; it then goes on for a time Z_2; then off for a time Y_2; then on, and so forth.

We suppose that the random vectors (Z_n, Y_n), $n \geq 1$, are independent and identically distributes. Hence, both the sequence of random variables $\{Z_n\}$ and the sequence $\{Y_n\}$ are independent and identically distributed; but we allow Z_n and Y_n to be dependent. In other words, each time the process goes on everything starts over again, but when it goes off we allow the length of the off time to depend on the previous on time.

Suppose that we earn at a rate of per unit time when system is on (and thus the reward for a cycle equals the on time of that cycle). Then the total reward earned by t is just the total on time in $[0, t]$, and thus with probability 1 ([10] pp. 114-115)

$$\frac{\text{amount of on time in } [0,t]}{t} \to \frac{E[Z]}{E[Z] + E[Y]}.$$

Thus when the cycle distribution is non-lattice the limiting probability of the system being on is equal to the long-run probability of time it is on.

Define

$$\lambda_s = E\{X^s\} = \int_0^\infty E\{X^s \mid T = t\} dF(t), \;\; s \geq 1,$$

whenever this expectations exist. By *existence of an expectation* $E\{g(T, X)\}$ we mean that $E\{|g(T, X)|\} < \infty$.

Theorem 3. *If* $\lambda_1 = E\{X\} < \infty$, *then* $|D(t)| < \infty$ *for all* $t \geq 0$.

Proof. By the definition

$$D(t) = E\{C(t)\} = E\left\{ \sum_{i=1}^{N(t)} X_i \right\}.$$

Using properties of mathematical expectation it can be written

$$|D(t)| \leq E\{|C(t)|\} = E\left\{ \left| \sum_{i=1}^{N(t)} X_i \right| \right\} \leq E\left\{ \sum_{i=1}^{N(t)} |X_i| \right\}.$$

As $(N(t) + 1)$ is a stopping time for $|X_1|, |X_2|, \ldots$, then by Wald's identity

$$|D(t)| \leq E\left\{ \sum_{i=1}^{N(t)+1} |X_i| \right\} - E\left\{ \left|X_{N(t)+1}\right| \right\} =$$

$$= E\{N(t) + 1\} \cdot E\{|X|\} - E\left\{ \left|X_{N(t)+1}\right| \right\}. \tag{5}$$

Taking into account $E\{N(t)\} = H(t) < \infty$, $E\{X\} = \lambda_1 < \infty$ and $E\left\{ \left|X_{N(t)+1}\right| \right\} \geq 0$, from (5) can be obtained:

$$|D(t)| \leq (H(t) + 1) \cdot E\{|X|\} - E\left\{ \left|X_{N(t)+1}\right| \right\} \leq H(t)E\{|X|\} < \infty.$$

This completes the proof of Theorem 3.

The next theorem is the generalization of Theorem 2 for the renewal-reward processes.

Theorem 4. *If* $\lambda_1 = E\{X\} < \infty$*, then for all* $t \geq 0$

$$|D(t)| \leq \left(2 + \frac{2t}{m_0}\right) E\{|X|\}, \tag{6}$$

where m_0 *is the median of distribution* F.

Proof. Taking into account $E\{N(t)\} = H(t)$, $E\{|X|\} < \infty$, $E\left\{ \left|X_{N(t)}\right| \right\} \geq 0$, and Theorem 2 from (5) can be obtained:

$$|D(t)| \leq (H(t) + 1) \cdot E\{|X|\} - E\left\{ \left|X_{N(t)+1}\right| \right\} \leq$$

$$\leq (H(t) + 1)E\{|X|\} \leq \left(2 + \frac{2t}{m_0}\right) E\{|X|\}.$$

This completes the proof of Theorem 4.

Example 2. Assume that T has an exponential distribution with the parameter $\alpha > 0$, so, $F(t) = 1 - e^{-\alpha t}$, $\mu_1 = 1/\alpha$, $\mu_2 = 2/\alpha^2$ and $m_0 = \ln 2/\alpha$. Let us take $X = e^{-T}$. Since, $X > 0$, then $E\{|X|\} = E\{X\} = \lambda_1$ and

$$\lambda_1 = \int_0^\infty e^{-t} dF(t) = \alpha \int_0^\infty e^{-(\alpha+1)t} dt = \frac{\alpha}{\alpha+1} < \infty.$$

The conditions of Theorem 4 are satisfied. So, it can be written:

$$|D(t)| \le \frac{\alpha}{\alpha+1}\left(2 + \frac{2\alpha}{\ln 2} t\right) = \frac{2\alpha^2}{(\alpha+1)\ln 2} t + \frac{2\alpha}{\alpha+1}.$$

Example 3. Assume that T has a log-normal distribution with the parameters μ and $\sigma^2 > 0$, then $\mu_n = \exp\left(n\mu + \frac{n^2\sigma^2}{2}\right)$ and $m_0 = e^\mu$. Let us take $X = \frac{1}{T}$. It is clear that, X has a log-normal distribution with parameters $-\mu$ and $\sigma^2 > 0$. Since, $X > 0$, then

$$E\{|X|\} = E\{X\} = \lambda_1 = e^{-\mu + \frac{\sigma^2}{2}}$$

The conditions of Theorem 4 are satisfied. So, it can be written:

$$|D(t)| \le e^{-\mu + \frac{\sigma^2}{2}} (2 + \frac{2t}{e^\mu}) = 2e^{-2\mu + \frac{\sigma^2}{2}} t + 2e^{-\mu + \frac{\sigma^2}{2}}.$$

3. Conclusion

Renewal-reward processes occur in various stochastic optimization models, particularly in Markov and semi-Markov decision. Thus, generalization results for renewal processes to renewal-reward processes can help investigation of such models.

In this study, finiteness of mathematical expectation of the renewal-reward process is proved. Also, an inequality that gives us an upper bound for the renewal function is generalized for the mathematical expectation of the renewal-reward process.

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