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Approximation of B-Continuous and B-Differentiable Functions by GBS Operators of Bernstein-Kantorovich Operators of Two Variables

Döne Karahan, Aydın Izgi

Abstract. In this study, the generalized Bernstein-Kantorovich type operators of are introduced and some approximation properties of these operators are studied in the space of continuous functions of two variables on a compact set. The order of approximation using Peetre's K-functional is investigated. The GBS operators of the generalized Bernstein-Kantorovich type operators of two variables are constructed and theorems on approximation of B-continuous and B-differentiable functions with GBS operators are proved. The degree of approximation in terms of the mixed modulus of smoothness is investigated. Lastly, comparisons by some illustrative graphics in Maple for the convergence of the operators to some functions are showed and the error in the approximation by giving numerical examples are estimated.

Key Words and Phrases: Bernstein-Kantorovich operators, B-continuous function, B-differentiable function, GBS operators, mixed modulus of smoothness.

2010 Mathematics Subject Classifications: 41A10, 41A25, 41A36, 41A63

1. Introduction

The research in the present paper is a continuation in the recent article [1] in which generalized Bernstein-Kantorovich operators of function of two variables defined on $\mathbb{A} = [-1,1] \times [-1,1]$ are investigated. In [1], the following problem is tackled:

$$D_{n,m}(f;x,y) = \frac{n+1}{2}\frac{m+1}{2}\sum_{k=0}^{n}\sum_{j=0}^{m}\phi_{n,m}^{k,j}(x,y)\int_{2\frac{k}{n+1}-1}^{2\frac{k+1}{n+1}-1}\int_{2\frac{j}{m+1}-1}^{2\frac{j+1}{m+1}-1} f(t,u)dtdu, \tag{1.1}$$

with

$$\phi_{n,m}^{k,j}(x,y) = \varphi_n^k(x)\varphi_m^j(y) \tag{1.2}$$

and

$$\varphi_n^k(x) = \frac{1}{2^n}\binom{n}{k}(1+x)^k(1-x)^{n-k}, \tag{1.3}$$

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where $f \in C(\mathbb{A})$ for $\mathbb{A} := [-1,1] \times [-1,1]$ and $D_{n,m}(f;x,y)$ $(n,m \in \mathbb{N})$ is positive linear operator.

In [2], for a function f defined on the closed interval $[0,1]$, $B_n(f;x,y)$ Bernstein polynomial of order n of the function f was defined. In [3], Korovkin's theorem arose from the study of the role of Bernstein polynomials in the proof of the Weierstrass approximation theorem (see [4]). Later, the various generalizations of Bernstein polynomials were constructed and approximation properties of these operators in [5]-[12]. It is also considered the bivariate form of these operators, for which the degree of approximation is established in [13]-[19].

The aim of this paper is to get the order of approximation using Peetre's K-functional, construct the GBS operators of Bernstein-Kantorovich type operators of two variables and estimate the degree of approximation in terms of the mixed modulus of smoothness. Then, we prove theorems on approximation of B-continuous and B-differentiable functions with GBS operators of Bernstein-Kantorovich type operators of two variables.

The concepts of B-continuity and B-differentiability were introduced by Karl Bögel ([21]). For more information these notations the reader is referred to [22]. It is easily verified that, under the pointwise operations of scalar multiplication and addition, the set $B(\mathbb{R})$ of B-continuous functions constitutes a real vector space. Not very much appears to be known as far as further algebraic or topological properties of this space are concerned.

2. Preliminaries

Now we establish the following lemmas and theorems which will be useful in the next sections. These lemmas and theorems are proved in [1].

Lemma 2.1. *For $\forall(x,y) \in \mathbb{A}$ and $\forall n,m \in \mathbb{N}$, Bernstein-Kantorovich operators (1.1) are satisfied the following equalities:*

$$D_{n,m}(1;x,y) = 1, \tag{2.1}$$

$$D_{n,m}(t;x,y) = x - \frac{x}{n+1}, \tag{2.2}$$

$$D_{n,m}(u;x,y) = y - \frac{y}{m+1}, \tag{2.3}$$

$$D_{n,m}(t^2+u^2;x,y) = x^2 - \frac{3nx^2 + x^2 - n - \frac{1}{3}}{(n+1)^2} + y^2 - \frac{3my^2 + y^2 - m - \frac{1}{3}}{(m+1)^2}, \tag{2.4}$$

$$D_{n,m}(t^3+u^3;x,y) = x^3 - \frac{x^3 + 6n^2x^3 + 3nx^3 - 3n^2x + 6n + 7nx + 6nx^2}{(n+1)^3}$$
$$+y^3 - \frac{y^3 + 6m^2y^3 + 3my^3 - 3m^2y + 6m + 7my + 6my^2}{(m+1)^3}, \tag{2.5}$$

$$D_{n,m}(t^4+u^4;x,y) = x^4 - \frac{10n^3x^4 - 5n^2x^4 + 10nx^4 - x^4 + 6n^3x^2 + 6nx^2}{(n+1)^4}$$

$$+\frac{10n^2x^2+4nx-3n^2-4n-\frac{1}{5}}{(n+1)^4}$$

$$+y^4-\frac{10m^3y^4-5m^2y^4+10my^4-y^4+6m^3y^2+6my^2}{(m+1)^4}$$

$$+\frac{10m^2y^2+4my-3m^2-4m-\frac{1}{5}}{(m+1)^4}. \tag{2.6}$$

Lemma 2.2. *If the operator $D_{n,m}$ is defined by (1.1), then for all $(x,y)\in\mathbb{A}$ and $\forall n,m\in\mathbb{N}$,*

$$D_{n,m}((t-x)^2;x,y)=\frac{3x^2-3nx^2+3n+1}{3(n+1)^2}, \tag{2.7}$$

$$D_{n,m}((u-y)^2;x,y)=\frac{3y^2-3my^2+3m+1}{3(m+1)^2}, \tag{2.8}$$

$$\begin{aligned}D_{n,m}((t-x)^4;x,y)=&\frac{n^2x^4+8nx^4+x^4+44nx^2+20n^2x^2+24n^2x+24n^2x^3}{(n+1)^4}\\&+\frac{24nx^3+20nx+2x^2+3n^2+4n+\frac{1}{5}}{(n+1)^4},\end{aligned} \tag{2.9}$$

$$\begin{aligned}D_{n,m}((u-y)^4;x,y)=&\frac{m^2y^4+8my^4+y^4+44my^2+20m^2y^2+24m^2y+24m^2y^3}{(m+1)^4}\\&+\frac{24my^3+20my+2y^2+3m^2+4m+\frac{1}{5}}{(m+1)^4}.\end{aligned} \tag{2.10}$$

Lemma 2.3. *For every fixed $(x_0,y)\in\mathbb{A}$ there exists a positive constant $M_1(x_0)$ such that for $n\in\mathbb{N}$, $D_{n,n}((t-x_0)^4;x_0,y)\le M_1(x_0)n^{-2}$.*

Theorem 2.1. *Let $f\in C(\mathbb{A})$, the the operators $D_{n,m}$ defined by (1.1) converge uniformly to f on $\mathbb{A}\subset\mathbb{R}^2$ as $n,m\to\infty$.*

Theorem 2.2. *For every $f\in C^2(\mathbb{A})$, we have*

$$\lim_{n\to\infty} n\{D_{n,n}(f;x,y)-f(x,y)\}$$

$$=-xf'_x(x,y)-yf'_y(x,y)+\frac{1}{2}\{(1-x^2)f''_{xx}(x,y)+(1-y^2)f''_{yy}(x,y)\}. \tag{2.11}$$

3. Petree's K-functional for $D_{n,m}$ Bernstein-Kantorovich Operators

Let $C^2(\mathbb{A})$ be the space of all functions $f \in C(\mathbb{A})$ such that $\frac{\partial^i f}{\partial x^i}, \frac{\partial^i f}{\partial y^i} \in C(\mathbb{A})$ for $i = 1, 2$ belong to $C^2(\mathbb{A})$. The norm on the space $C^2(\mathbb{A})$ is defined as

$$\|f\|_{C^2(\mathbb{A})} = \|f\|_{C(\mathbb{A})} + \sum_{i=1}^{2} \left(\|\frac{\partial^i f}{\partial x^i}\|_{C(\mathbb{A})} + \|\frac{\partial^i f}{\partial y^i}\|_{C(\mathbb{A})} \right).$$

The Peetre's K-functional of the function $f \in C(\mathbb{A})$ is defined by

$$\mathcal{K}(f;\delta) = \inf_{g \in C^2(\mathbb{A})} \{\|f - g\|_{C(\mathbb{A})} + \delta \|g\|_{C^2(\mathbb{A})}, \quad \delta > 0\}$$

holds for all $\delta > 0$.

Theorem 3.1. *For the function $f \in C(\mathbb{A})$, we have*

$$\|D_{n,m}(f;x,y) - f(x,y)\|_{C(\mathbb{A})} \leq 2\mathcal{K}\left(f; \frac{\delta_{n,m}}{2}\right)$$

where $\delta_{n,m} = \max\left(\frac{1}{n+1}, \frac{1}{m+1}\right)$.

Proof. Let $g \in C^2(\mathbb{A})$. Using the Taylor theorem's, we have

$$\begin{aligned} g(t,u) - g(x,y) &= g(t,y) - g(x,y) + g(t,u) - g(t,y) \\ &= \frac{\partial g(x,y)}{\partial x}(t-x) + \int_x^t (t-\xi)\frac{\partial^2 g(\xi,y)}{\partial^2 \xi^2} d\xi \\ &+ \frac{\partial g(x,y)}{\partial y}(u-y) + \int_y^u (u-\eta)\frac{\partial^2 g(x,\eta)}{\partial^2 \eta^2} d\eta \\ &= \frac{\partial g(x,y)}{\partial x}(t-x) + \int_0^{t-x} (t-x-\xi)\frac{\partial^2 g(\xi+x,y)}{\partial^2 \xi^2} d\xi \\ &+ \frac{\partial g(x,y)}{\partial y}(u-y) + \int_0^{u-y} (u-y-\eta)\frac{\partial^2 g(x,\eta+y)}{\partial^2 \eta^2} d\eta. \end{aligned}$$

Applying the operator $D_{n,m}$ on the above equality, we obtain

$$\begin{aligned} |D_{n,m}(g;x,y) - g(x,y)| \leq & \left|\frac{\partial g(x,y)}{\partial x}\right| |D_{n,m}((t-x);x,y)| \\ & \left|D_{n,m}\left(\int_0^{t-x} (t-x-\xi)\frac{\partial^2 g(\xi+x,y)}{\partial^2 \xi^2} d\xi; x,y\right)\right| \\ & + \left|\frac{\partial g(x,y)}{\partial y}\right| |D_{n,m}((u-y);x,y)| \\ & + \left|D_{n,m}\left(\int_0^{u-y} (u-y-\eta)\frac{\partial^2 g(x,\eta+y)}{\partial^2 \eta^2} d\eta; x,y\right)\right|. \end{aligned}$$

From Lemma 2.1 $D_{n,m}(t-x;x,y) = -\frac{x}{n+1}$ and $D_{n,m}(u-y;x,y) = -\frac{y}{m+1}$, we have

$$\begin{aligned}\|D_{n,m}(g;x,y)-g(x,y)\|_{C(\mathbb{A})} &\leq \frac{1}{n+1}\left\|\frac{\partial g}{\partial x}\right\|_{C(\mathbb{A})} + \frac{1}{m+1}\left\|\frac{\partial g}{\partial y}\right\|_{C(\mathbb{A})} \\ &+ \frac{1}{2}\left\|\frac{\partial^2 g}{\partial x^2}\right\|_{C(\mathbb{A})} |D_{n,m}((t-x)^2;x,y)| \\ &+ \frac{1}{2}\left\|\frac{\partial^2 g}{\partial y^2}\right\|_{C(\mathbb{A})} |D_{n,m}((u-y)^2;x,y)|.\end{aligned}$$

From (2.7) and (2.8), we get

$$\begin{aligned}\|D_{n,m}(g;x,y)-g(x,y)\|_{C(\mathbb{A})} &\leq \frac{1}{n+1}\left\|\frac{\partial g}{\partial x}\right\|_{C(\mathbb{A})} + \frac{1}{m+1}\left\|\frac{\partial g}{\partial y}\right\|_{C(\mathbb{A})} \\ &+ \frac{1}{2(n+1)}\left\|\frac{\partial^2 g}{\partial x^2}\right\|_{C(\mathbb{A})} + \frac{1}{2(m+1)}\left\|\frac{\partial^2 g}{\partial y^2}\right\|_{C(\mathbb{A})} \\ &\leq \max\left(\frac{1}{n+1},\frac{1}{m+1}\right)\left(\left\|\frac{\partial g}{\partial x}\right\|_{C(\mathbb{A})} + \left\|\frac{\partial g}{\partial y}\right\|_{C(\mathbb{A})}\right. \qquad (3.1) \\ &\left.+\left\|\frac{\partial^2 g}{\partial x^2}\right\|_{C(\mathbb{A})} + \left\|\frac{\partial^2 g}{\partial y^2}\right\|_{C(\mathbb{A})}\right) \\ &\leq \delta_{n,m}\|g\|_{C^2(\mathbb{A})}\end{aligned}$$

where $\delta_{n,m} = \max\left(\frac{1}{n+1},\frac{1}{m+1}\right)$. Since $D_{n,m}$ is linear operator, we have

$$\begin{aligned}\|D_{n,m}(f;x,y)-f(x,y)\|_{C(\mathbb{A})} &\leq \|D_{n,m}(f;x,y)-D_{n,m}(g;x,y)\|_{C(\mathbb{A})} \\ &+ \|D_{n,m}(g;x,y)-g(x,y)\|_{C(\mathbb{A})} + \|f-g)\|_{C(\mathbb{A})} \\ &\leq \|(f-g)\|_{C(\mathbb{A})}|D_{n,m}(1;x,y)| \\ &+ \|D_{n,m}(g;x,y)-g(x,y)\|_{C(\mathbb{A})} + \|f-g)\|_{C(\mathbb{A})}.\end{aligned}$$

From (3.1) inequality, we obtain

$$\|D_{n,m}(f;x,y)-f(x,y)\|_{C(\mathbb{A})} \leq 2\left(\|f-g)\|_{C(\mathbb{A})} + \frac{\delta_{n,m}}{2}\|g\|_{C^2(\mathbb{A})}\right).$$

Taking the infimum on the right hand side over all $g \in C^2(\mathbb{A})$, the proof is completed. □

4. Construction of GBS Operator of $D_{n,m}$ Bernstein-Kantorovich Operators

Now, we recall some basic definitions and notations in [21]. Let X and Y be compact real intervals. Let $\Delta_{(x,y)}f[t,u;x,y]$ be mixed difference of f defined by $\Delta_{(x,y)}f[t,u;x,y] = f(x,y)-f(x,u)-f(t,y)+f(t,u)$, $\forall (x,y),(t,u) \in X\times Y$.

A function $f; X \times Y \to \mathbb{R}$ is called a B-continuous (Bögel continuous) at a point $(x_0, y_0) \in X \times Y$ if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta_{(x,y)} f[x_0, y_0; x, y] = 0$$

for any $(x, y) \in X \times Y$. The function $f; X \times Y \to \mathbb{R}$ is B-bounded on $X \times Y$ if there exists $M > 0$ such that

$$|\Delta_{(x,y)} f[t, u; x, y]| \leq M$$

for every $(x, y), (t, u) \in X \times Y$. Also, if $X \times Y$ is a compact subset of $\mathbb{R}^2$ then each B-continuous function is a B-bounded function on $X \times Y \to \mathbb{R}$. In this paper $B_B(\mathbb{A})$ denotes all B-bounded functions on $X \times Y \to \mathbb{R}$. We denote by $C_B(\mathbb{A})$, the space of all B-continuous functions on $X \times Y$. The sup-norm $\|.\|_\infty$ defined on these spaces.

Let $R : C_B(\mathbb{A}) \to B_B(\mathbb{A})$ be a linear positive operator then the Generalized Boolean Sum (GBS) operator associated is defined by

$$R_{n,m}(f(t, u); x, y) := D_{n,m}(f(t, y) + f(x, u) - f(t, u); x, y)$$

for every $f \in C_B(\mathbb{A})$, for each $(x, y) \in \mathbb{A}$ and $m, n \in \mathbb{N}$. More precisely, the GBS operator of $D_{n,m}$ Bernstein-Kantorovich Operator is defined as follows;

$$\begin{aligned} R_{n,m}(f; x, y) &= \frac{n+1}{2}\frac{m+1}{2} \sum_{k=0}^{n}\sum_{j=0}^{m} \phi_{n,m}^{k,j}(x, y) \\ &\times \int_{2\frac{k}{n+1}-1}^{2\frac{k+1}{n+1}-1} \int_{2\frac{j}{m+1}-1}^{2\frac{j+1}{m+1}-1} [f(t, y) + f(x, u) - f(t, u)]dtdu \end{aligned} \tag{4.1}$$

where $f \in C_B(\mathbb{A})$ and the operator $R_{n,m}$ is well-defined from the space $C_B(\mathbb{A})$ on itself. The mixed modulus of smoothness of $f \in C_B(\mathbb{A})$ is defined as

$$\omega_{mixed}(f; \delta_1, \delta_2) := sup\{\Delta_{(x,y)} f[t, u; x, y] : |x - t| < \delta_1, \quad |y - u| < \delta_2\}$$

for every $(x, y), (t, u) \in \mathbb{A}$ and for any $\delta_1, \delta_2 > 0$ with $\omega_{mixed} : [0, \infty) \times [0, \infty) \to \mathbb{R}$. The basic properties of ω_{mixed} were obtained by Badea et al. in [20] which are similar to properties of usual modulus of continuity.

The function $f; X \times Y \to \mathbb{R}$ is B-differentiable on $X \times Y$ if the limit

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta_{(x,y)} f[x_0, y_0; x, y]}{(x - x_0)(y - y_0)}$$

exists and is finite. Then, $f : X \times Y \subset \mathbb{R}^2 \to \mathbb{R}$ function is B-differentiable at the point $(x_0, y_0) \in X \times Y$ and is denoted by $\mathcal{D}_{xy} f(x_0, y_0) := \mathcal{D}_B(f; x_0, y_0)$. The space of all B-differentiable functions will be denoted $\mathcal{D}_B(X \times Y)$. In this paper, the space of B-differentiable functions is denoted by $\mathcal{D}_B(\mathbb{A})$ on $\mathbb{A}$.

We shall estimate the rate of convergence of the sequences of the operators (4.1) to $f \in C_B(\mathbb{A})$ using the mixed modulus of smoothness. For this, we use the well-known Shisha–Mond theorem for B-continuous functions established by Badea and Cottin [20].

Theorem 4.1. *For $\forall f \in C_B(\mathbb{A})$ at each point $\forall (x,y) \in \mathbb{A}$, $R_{n,m}$ GBS operator verifies the following inequality*

$$|R_{n,m}(f;x,y) - f(x,y)| \leq 4\omega_{mixed}\left(f;\frac{1}{\sqrt{n+1}},\frac{1}{\sqrt{m+1}}\right). \tag{4.2}$$

Proof. Using the definition of $\omega_{mixed}(f;\delta_1,\delta_2)$ function and for $\lambda_1,\lambda_2>0$

$$\omega_{mixed}(f;\lambda_1\delta_1,\lambda_2\delta_2) \leq (1+\lambda_1)(1+\lambda_2)\omega_{mixed}(f;\delta_1,\delta_2)$$

inequality, we obtain

$$\begin{aligned} |\Delta_{(x,y)}f[t,u;x,y]| &\leq \omega_{mixed}\left(f;|t-x|,|u-y|\right) \\ &\leq \left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|u-y|}{\delta_2}\right)\omega_{mixed}(f;\delta_1,\delta_2) \end{aligned} \tag{4.3}$$

for $\forall (x,y),(t,u) \in \mathbb{A}$ and $\delta_1,\delta_2>0$. From definition of $\Delta_{(x,y)}f[t,u;x,y]$ function, we can write

$$f(x,u)+f(t,y)-f(t,u) = f(x,y) - \Delta_{(x,y)}f[t,u;x,y].$$

Applying the $D_{n,m}$ GBS operator to the above equality and using the (4.1) expression, we obtain

$$R_{n,m}(f;x,y) = f(x,y)D_{n,m}(1;x,y) - D_{n,m}\left(\Delta_{(x,y)}f[t,u;x,y];x,y\right).$$

Since $D_{n,m}(1;x,y)=1$, we get

$$R_{n,m}(f;x,y) - f(x,y) = -D_{n,m}\left(\Delta_{(x,y)}f[t,u;x,y];x,y\right).$$

If we take the absolute value of both sides in this equality, we can write

$$|R_{n,m}(f;x,y) - f(x,y)| \leq D_{n,m}\left(|\Delta_{(x,y)}f[t,u;x,y]|;x,y\right).$$

Considering the inequality (4.3), we get

$$\begin{aligned} &|R_{n,m}(f;x,y) - f(x,y)| \\ &\leq D_{n,m}\left(\Big|\left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|u-y|}{\delta_2}\right)\omega_{mixed}\left(f;\delta_1,\delta_2\right)\Big|;x,y\right). \end{aligned}$$

Applying the Cauchy-Schwartz inequality to right side of this inequality, we obtain

$$\begin{aligned} |R_{n,m}(f;x,y) - f(x,y)| \leq &\Bigg[D_{n,m}(1;x,y) + \frac{1}{\delta_1}\sqrt{D_{n,m}((t-x)^2;x,y)} \\ &+\frac{1}{\delta_2}\sqrt{D_{n,m}((u-y)^2;x,y)} \\ &+\frac{1}{\delta_1\delta_2}\sqrt{D_{n,m}((t-x)^2;x,y)D_{n,m}((u-y)^2;x,y)}\Bigg] \\ &\times \omega_{mixed}\left(f;\delta_1,\delta_2\right). \end{aligned} \tag{4.4}$$

From (2.7) and (2.8) equalities in the Lemma 2.2, for all $(x, y) \in \mathbb{A}$, we obtain

$$D_{n,m}((t-x)^2; x, y) \leq \frac{1}{n+1}$$

and

$$D_{n,m}((u-y)^2; x, y) \leq \frac{1}{m+1}.$$

Choosing the $\delta_1 = \frac{1}{\sqrt{n+1}}$ and $\delta_2 = \frac{1}{\sqrt{m+1}}$, from (4.4), we reach the desired inequality. □

The following theorem is an important theorem related to order of approximation for the B-differentiable functions was proved by [25].

Theorem 4.2. *Let $f \in \mathcal{D}_B(\mathbb{A})$ and $\mathcal{D}_B f \in B_B(\mathbb{A})$. Then for all $(x, y) \in \mathbb{A}$, the following inequality is true:*

$$\begin{aligned} |R_{n,m}(f; x, y) - f(x, y)| \leq \frac{7}{\sqrt{(n+1)(m+1)}} & (\|\mathcal{D}_B f\|_\infty \\ & + \omega_{mixed}\left(\mathcal{D}_B f; \frac{1}{\sqrt{n+1}}, \frac{1}{\sqrt{m+1}}\right)\bigg). \end{aligned} \tag{4.5}$$

Proof. Since $f \in \mathcal{D}_B(\mathbb{A})$, for $x < \xi < t$ and $y < \eta < u$, we have

$$\Delta_{(x,y)} f[t, u; x, y] = (t-x)(u-y)\mathcal{D}_B(\xi, \eta). \tag{4.6}$$

It is clear that,

$$\Delta_{(x,y)} f[\xi, \eta; x, y] = f(x, y) - f(x, \eta) - f(\xi, y) + f(\xi, \eta).$$

Applying B-differentiate to both sides of this equality, we obtain

$$\mathcal{D}_B f(\xi, \eta) = \Delta_{(x,y)} \mathcal{D}_f[\xi, \eta; x, y] + \mathcal{D}_B f(\xi, y) + \mathcal{D}_B f(x, \eta) - \mathcal{D}_B f(x, y).$$

Since $\mathcal{D}_B f \in B_B(\mathbb{A})$, by (4.6), we can write

$$\begin{aligned} & |D_{n,m}\left(\Delta_{(x,y)} f[t, u; x, y]; x, y\right)| \\ & = |D_{n,m}\left((t-x)(u-y)\mathcal{D}_B f(\xi, \eta); x, y\right)| \\ & \leq D_{n,m}\left(|t-x||u-y|\left|\Delta_{(x,y)} \mathcal{D}_B f(\xi, \eta)\right|; x, y\right) \\ & + D_{n,m}\left(|t-x||u-y|\left(|\mathcal{D}_B f(\xi, y)| + |\mathcal{D}_B f(x, \eta)| + |\mathcal{D}_B f(x, y)|\right); x, y\right) \\ & \leq D_{n,m}\left(|t-x||u-y|\omega_{mixed}\left(\mathcal{D}_B f; |\xi - x|, |\eta - y|; x, y\right)\right) \\ & + 3\|\mathcal{D}_B f\|_\infty D_{n,m}\left(|t-x||u-y|; x, y\right). \end{aligned}$$

Since the mixed modulus of smoothness ω_{mixed} is non-decreasing, we have

$$\begin{aligned} & \omega_{mixed}\left(\mathcal{D}_B f; |\xi - x|, |\eta - y|; x, y\right) \\ & \leq \omega_{mixed}\left(\mathcal{D}_B f; |t-x|, |u-y|; x, y\right) \\ & \leq \left(1 + \frac{|t-x|}{\delta_1}\right)\left(1 + \frac{|u-y|}{\delta_2}\right)\omega_{mixed}(\mathcal{D}_B f; \delta_1, \delta_2). \end{aligned}$$

Using the above inequality and the linearity of $D_{n,m}$ operator, we have

$$\begin{aligned}|R_{n,m}(f;x,y)-f(x,y)| &= |D_{n,m}\left(\Delta_{(x,y)}f[t,u;x,y];x,y\right)| \\ &\leq [D_{n,m}\left(|t-x||u-y|;x,y\right) \\ &+\frac{1}{\delta_2}D_{n,m}\left(|t-x|(u-y)^2;x,y\right) \\ &+\frac{1}{\delta_1\delta_2}D_{n,m}\left((t-x)^2(u-y)^2;x,y\right)\Big]\,\omega_{mixed}(\mathcal{D}_Bf;\delta_1,\delta_2) \\ &+3\|\mathcal{D}_Bf\|_\infty D_{n,m}\left(|t-x||u-y|;x,y\right).\end{aligned}$$

Applying the Cauchy-Schwartz inequality to right side of the above equality, we obtain

$$\begin{aligned}|R_{n,m}(f;x,y)-f(x,y)| &\leq \Big[\sqrt{D_{n,m}\left((t-x)^2(u-y)^2;x,y\right)} \\ &+\frac{1}{\delta_2}\sqrt{D_{n,m}\left((t-x)^2(u-y)^4;x,y\right)} \\ &+\frac{1}{\delta_1\delta_2}D_{n,m}\left((t-x)^4(u-y)^4;x,y\right)\Big]\,\omega_{mixed}(\mathcal{D}_Bf;\delta_1,\delta_2) \\ &+3\|\mathcal{D}_Bf\|_\infty\sqrt{D_{n,m}\left((t-x)^2(u-y)^2;x,y\right)}.\end{aligned}$$

From $D_{n,m}\left((t-x)^2;x,y\right) \leq \frac{1}{n+1}$ and $D_{n,m}\left((u-y)^2;x,y\right) \leq \frac{1}{m+1}$ inequalities, for $(x,y),(t,u)\in\mathbb{A}$, $a,b\in\{1,2\}$, we get

$$D_{n,m}\left((t-x)^{2a}(u-y)^{2b};x,y\right)=D_{n,m}\left((t-x)^{2a};x,y\right)D_{n,m}\left((u-y)^{2b};x,y\right).$$

Choosing $\delta_1=\frac{1}{\sqrt{n+1}}$ and $\delta_2=\frac{1}{\sqrt{m+1}}$, we have

$$\begin{aligned}|R_{n,m}(f;x,y)-f(x,y)| \leq \frac{7}{\sqrt{(n+1)(m+1)}}&\left(\|\mathcal{D}_Bf\|_\infty\right. \\ &\left.+\omega_{mixed}\left(\mathcal{D}_Bf;\frac{1}{\sqrt{n+1}},\frac{1}{\sqrt{m+1}}\right)\right).\end{aligned}$$

So the proof is completed. □

In order to improve the measure of smoothness, the mixed K-functional is introduced in [23]. For $f\in C_B(\mathbb{A})$, we can define the mixed K-functional by

$$\begin{aligned}\mathcal{K}_{mixed}(f;\delta_1,\delta_2) = \inf_{g_1,g_2,h}&\{\|f-g_1-g_2-h\|_\infty+t_1\|\mathcal{D}_B^{2,0}g_1\|_\infty \\ &+t_2\|\mathcal{D}_B^{0,2}g_2\|_\infty+t_1t_2\|\mathcal{D}_B^{2,2}h\|_\infty\}\end{aligned} \tag{4.7}$$

where $g_1\in C_B^{2,0}$, $g_2\in C_B^{0,2}$, $h\in C_B^{2,2}$ and for $0\leq a,b\leq 2$, the space $C_B^{a,b}$ denotes the space of the functions $f\in C_B(\mathbb{A})$ with continuous mixed partial derivatives $\mathcal{D}_B^{r,s}f$, $0\leq r\leq a$, $0\leq s\leq b$. The partial derivatives of the function $f\in C_B(\mathbb{A})$ are following:

$$\mathcal{D}_xf(t,u):=\mathcal{D}_B^{1,0}(f;t,u)=\lim_{x\to t}\frac{\Delta_xf\left([t,x];u\right)}{(x-t)}$$

and

$$\mathcal{D}_y f(t,u) := \mathcal{D}_B^{0,1}(f;t,u) = \lim_{y\to u} \frac{\Delta_y f\left(t;[u,y]\right)}{(y-u)}$$

where $\Delta_x f\left([t,x];u\right) = f(x,u) - f(t,u)$ and $\Delta_y f\left(t;[u,y]\right) = f(t,y) - f(t,u)$. The second order partial derivatives are analogous to the ordinary derivatives. For example, the derivative of $\mathcal{D}_x f(t,u)$ with respect to the variable y at point (t,u) is defined by

$$\mathcal{D}_y\mathcal{D}_x f(t,u) := \mathcal{D}_B^{0,1}\mathcal{D}_B^{1,0}(f;t,u) = \lim_{y\to u} \frac{\Delta_y(\mathcal{D}_x f)\left(t;[u,y]\right)}{(y-u)}.$$

Now we give the order of approximation of $R_{n,m}$ operator to the function $f \in C_B(\mathbb{A})$ function in terms of the mixed K-functional.

Theorem 4.3. *For $\forall f \in C_B(\mathbb{A})$, $R_{n,m}$ GBS operator verifies the following inequality:*

$$|R_{n,m}(f;x,y) - f(x,y)| \le 2\mathcal{K}_{mixed}\left(f; \frac{1}{2(n+1)}, \frac{1}{2(m+1)}\right).$$

Proof. From Taylor formula for $g_1 \in C_B^{2,0}$, we can write (see [21])

$$g_1(t,u) = g_1(x,y) + (t-x)\mathcal{D}_B^{1,0}g_1(x,y) + \int_x^t (t-\xi)\mathcal{D}_B^{2,0}g_1(\xi,y)d\xi.$$

Applying $R_{n,m}$ GBS operator to the both sides of the above equality, for $R_{n,m}((t-x);x,y) = 0$, we have

$$R_{n,m}(g_1;x,y) = g_1(x,y) + R_{n,m}\left(\int_x^t (t-\xi)\mathcal{D}_B^{2,0}g_1(\xi,y)d\xi;x,y\right).$$

From definition of $R_{n,m}$ GBS operator, we get

$$\begin{aligned}
\left|R_{n,m}(g_1;x,y) - g_1(x,y)\right| &= \left|D_{n,m}\left(\int_x^t (t-\xi)\left[\mathcal{D}_B^{2,0}g_1(\xi,y) - \mathcal{D}_B^{2,0}g_1(\xi,u)\right]d\xi;x,y\right)\right| \\
&\le D_{n,m}\left(\left|\int_x^t |t-\xi|\left|\mathcal{D}_B^{2,0}g_1(\xi,y) - \mathcal{D}_B^{2,0}g_1(\xi,u)\right|d\xi;x,y\right|\right) \\
&\le \left\|\mathcal{D}_B^{2,0}g_1\right\|_\infty D_{n,m}\left((t-x)^2;x,y\right) \\
&\le \frac{1}{n+1}\left\|\mathcal{D}_B^{2,0}g_1\right\|_\infty.
\end{aligned}$$

Similarly, for $g_2 \in C_B^{0,2}$ function, we have

$$\begin{aligned}
\left|R_{n,m}(g_2;x,y) - g_2(x,y)\right| &= \left|D_{n,m}\left(\int_y^u (u-\eta)\left[\mathcal{D}_B^{0,2}g_2(x,\eta) - \mathcal{D}_B^{0,2}g_2(x,\eta)\right]d\eta;x,y\right)\right| \\
&\le D_{n,m}\left(\left|\int_y^u |u-\eta|\left|\mathcal{D}_B^{0,2}g_2(x,\eta) - \mathcal{D}_B^{0,2}g_2(t,\eta)\right|d\eta;x,y\right|\right) \\
&\le \left\|\mathcal{D}_B^{0,2}g_2\right\|_\infty D_{n,m}\left((u-y)^2;x,y\right) \\
&\le \frac{1}{m+1}\left\|\mathcal{D}_B^{0,2}g_2\right\|_\infty.
\end{aligned}$$

Now, for $h \in C_B^{2,2}$ function, we can write

$$\begin{aligned} h(t,u) &= h(x,y) + (t-x)\mathcal{D}_B^{1,0}h(x,y) + (u-y)\mathcal{D}_B^{0,1}h(x,y) \\ &+ (t-x)(u-y)\mathcal{D}_B^{1,1}h(x,y) + \int_x^t (t-\xi)\mathcal{D}_B^{2,0}h(\xi,y)d\xi \\ &+ \int_y^u (u-\eta)\mathcal{D}_B^{0,2}h(x,\eta)d\eta + \int_x^t (u-y)(t-\xi)\mathcal{D}_B^{2,1}h(\xi,y)d\xi \\ &+ \int_y^u (t-x)(u-\eta)\mathcal{D}_B^{1,2}h(x,\eta)d\eta + \int_x^t \int_y^u (t-\xi)(u-\eta)\mathcal{D}_B^{2,2}h(\xi,\eta)d\eta d\xi. \end{aligned}$$

Taking into account the definition of the $R_{n,m}$ GBS operator and by using $R_{n,m}((t-x);x,y)=0$ and $R_{n,m}((u-y);x,y)=0$, we obtain

$$\begin{aligned} \left|R_{n,m}(h;x,y)-h(x,y)\right| &= \left|D_{n,m}\left(\int_x^t \int_y^u (t-\xi)(u-\eta)\mathcal{D}_B^{2,2}h(\xi,\eta)d\eta d\xi;x,y\right)\right| \\ &\leq D_{n,m}\left(\left|\int_x^t \int_y^u (t-\xi)(u-\eta)\mathcal{D}_B^{2,2}h(\xi,\eta)d\eta d\xi\right|;x,y\right) \\ &\leq D_{n,m}\left(\int_x^t \int_y^u |t-\xi||u-\eta|\left|\mathcal{D}_B^{2,2}h(\xi,\eta)\right|d\eta d\xi;x,y\right) \\ &\leq \frac{1}{4}\left\|\mathcal{D}_B^{2,2}h\right\|_\infty D_{n,m}\left((t-x)^2(u-y)^2;x,y\right) \\ &\leq \frac{1}{4}\frac{1}{(n+1)(m+1)}\left\|\mathcal{D}_B^{2,2}h\right\|_\infty. \end{aligned}$$

So, for $f \in C_B(\mathbb{A})$, we get

$$\begin{aligned} \left|R_{n,m}(f;x,y)-f(x,y)\right| &\leq \left|(f-g_1-g_2-h)(xx,y)\right| + \left|(g_1-R_{n,m}g_1)(x,y)\right| \\ &+ \left|(g_2-R_{n,m}g_2)(x,y)\right| + \left|(h-R_{n,m}h)(x,y)\right| \\ &+ \left|R_{n,m}\left((f-g_1-g_2-h);x,y\right)\right| \\ &\leq 2\|f-g_1-g_2-h\|_\infty + \frac{1}{n+1}\left\|\mathcal{D}_B^{2,0}g_1\right\|_\infty \\ &+ \frac{1}{m+1}\left\|\mathcal{D}_B^{0,2}g_2\right\|_\infty + \frac{1}{4}\frac{1}{(n+1)(m+1)}\left\|\mathcal{D}_B^{2,2}h\right\|_\infty \\ &\leq 2\left(\|f-g_1-g_2-h\|_\infty + \frac{1}{2(n+1)}\left\|\mathcal{D}_B^{2,0}g_1\right\|_\infty \right. \\ &\left. + \frac{1}{2(m+1)}\left\|\mathcal{D}_B^{0,2}g_2\right\|_\infty + \frac{1}{4}\frac{1}{(n+1)(m+1)}\left\|\mathcal{D}_B^{2,2}h\right\|_\infty\right). \end{aligned}$$

Taking the infimum over all $g_1 \in C_B^{2,0}$, $g_2 \in C_B^{0,2}$ and $h \in C_B^{2,2}$ functions, the desired result is obtained. □

5. Numerical examples for $D_{n,m}$ and $R_{n,m}$ operators

Example 1. For $n = 5$, $m = 5$, the comparison of convergence of $D_{n,m}(f; x, y)$ Bernstein-Kantorovich operator (yellow) and $R_{n,m}(f; x, y)$ GBS operator (red) to $f(x, y) = (1-x^2-y^2)e^{x^2-y^2}$ function (blue) is illustrated in Fig.1.

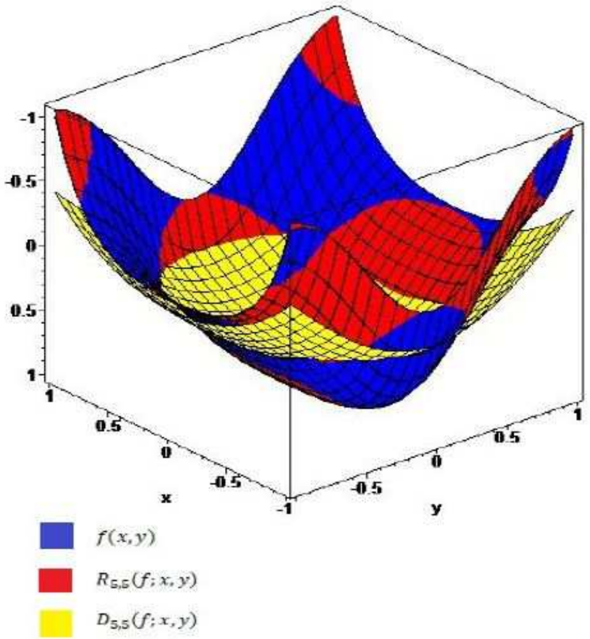

Figure 1: The convergence of the $D_{n,m}(f; x, y)$ (yellow) and $R_{n,m}(f; x, y)$ GBS operator (red) to $f(x, y) = (1 - x^2 - y^2)e^{x^2-y^2}$ (blue).

Example 2. For $n = 5$, $m = 5$, the comparison of convergence of $D_{n,m}(f; x, y)$ Bernstein-Kantorovich operator (yellow) and $R_{n,m}(f; x, y)$ GBS operator (red) to $f(x, y) = (1+x+y)cos(x + y)$ function (blue) is illustrated in Fig.2.

Now we investigate the ratio of rate of convergences of $R_{n,m}(f; x, y)$ GBS operator and $D_{n,m}(f; x, y)$ Bernstein-Kantorovich operator to f function. Since we observe that the rate of convergence of $R_{n,m}(f; x, y)$ is better than the operator $D_{n,m}(f; x, y)$, the following limit is true:

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \frac{\left|R_{n,m}(f; x, y) - f(x, y)\right|}{\left|D_{n,m}(f; x, y) - f(x, y)\right|} = 0.$$

Table 1. is a table of numeric values of $\frac{|R_{n,m}(f;x,y)-f(x,y)|}{|D_{n,m}(f;x,y)-f(x,y)|}$.

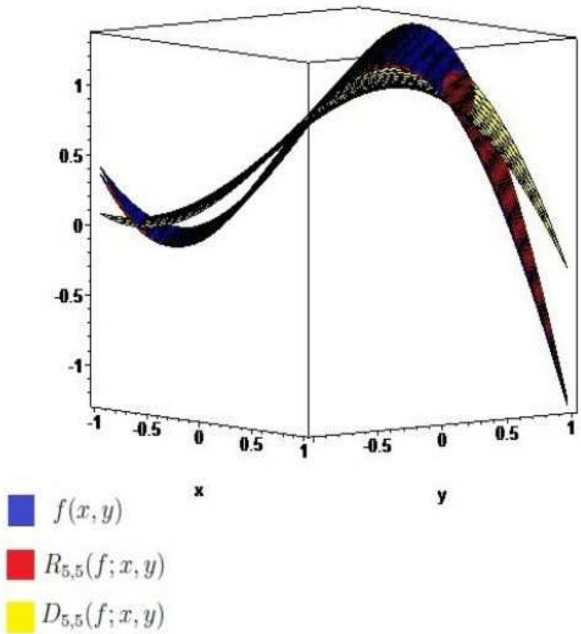

Figure 2: The convergence of the $D_{n,m}(f;x,y)$ (yellow) and $R_{n,m}(f;x,y)$ GBS operator (red) to $f(x,y) = (1+x+y)\cos(x+y)$ (blue).

Table 1: The error rate of $\frac{|R_{n,m}(f;x,y)-f(x,y)|}{|D_{n,m}(f;x,y)-f(x,y)|}$ for $f(x,y) = (1+x+y)cos(x+y)$.

$n=m$	$\frac{\lvert R_{n,m}(f;x,y)-f(x,y)\rvert}{\lvert D_{n,m}(f;x,y)-f(x,y)\rvert}$	
	$x=y=-0.9$	$x=y=0.9$
2	1.34657587	0.1734633994
5	0.2981842558	0.06920811806
10	0.1287749154	0.03370763366
50	0.02296071511	0.0064696935599
100	0.01138421001	0.003212264255
200	0.005401342461	0.001600276959
500	0.006587724028	0.0002066270426

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Döne Karahan
Science and Letter Faculty, Mathematics Department, Harran University Sanlıurfa TURKEY
E-mail: dkarahan@harran.edu.tr

Aydın Izgi
Science and Letter Faculty, Mathematics Department, Harran University Sanlıurfa TURKEY
*E-mail:*aydinizgi@yahoo.com

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