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On the Padovan Triangle

O. Dişkaya, H. Menken*

Abstract. In the present work, we consider the Padovan numbers. Inspiring of the Hosoya's triangle, we define the Padovan triangle. We give some identities and properties of the Padovan triangle.

Key Words and Phrases: Fibonacci numbers, Padovan numbers, Hosoya's triangle, Padovan triangle.

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1. Introduction

There are so many studies in the literature that are about the special number sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Tribonacci, Padovan, and Perrin (for the details see [3, 8, 9, 10, 11, 13, 14, 18]). The most known of these are the Fibonacci numbers. The other important special numbers are the Padovan numbers. The Padovan sequence is named after Richard Padovan. For more information see [12, 16]. The Padovan sequence $\{P_n\}$ is defined by the third order recurrence

$$P_n = P_{n-2} + P_{n-3}, \quad n \geq 3 \tag{1}$$

with the initial conditions $P_0 = 1, \quad P_1 = 1, \quad P_2 = 1$. The first few members of this sequence are given as follows

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
P_n	1	1	1	2	2	3	4	5	7	9	12	16	21	28	37	49	...

The recurrence (1) involves the characteristic equation

$$x^3 - x - 1 = 0.$$

If its roots are denoted by λ, μ and δ, then the following equalities are hold

$$\lambda + \mu + \delta = 0,$$

*Corresponding author.

$$\lambda\mu + \lambda\delta + \mu\delta = -1,$$
$$\lambda\mu\delta = 1.$$

Moreover, the Binet formula for the Padovan sequence is

$$P_n = a\lambda^n + b\mu^n + c\delta^n \tag{2}$$

where,

$$a = \frac{(\mu - 1)(\delta - 1)}{(\lambda - \mu)(\lambda - \delta)}, b = \frac{(\lambda - 1)(\delta - 1)}{(\mu - \lambda)(\mu - \delta)}, c = \frac{(\lambda - 1)(\mu - 1)}{(\delta - \lambda)(\delta - \mu)}. \tag{3}$$

2. Padovan Triangle

H.Hosoya ([4]) defined a triangular array $\{f_{m,n}\}_{m \geq n \geq 0}$ of positive integers which is called Fibonacci triangle. The Fibonacci or Hosoya's triangle $\{f_{m,n}\}_{m \geq n \geq 0}$ is defined by the two recurrences

(i) $f_{m,n} = f_{m-1,n} + f_{m-2,n}$, $(m \geq 2)$

(ii) $f_{m,n} = f_{m-1,n-1} + f_{m-2,n-2}$, $(m \geq 2)$

with the initial conditions

$$f_{0,0} = 1, f_{1,0} = 1, f_{1,1} = 1, f_{2,1} = 1.$$

Hosoya shows that the $\{f_{m,n}\}$ is two dimensional of the Fibonacci sequence(for the details see [4, 5, 6, 7, 15, 17]). Inspiring of the Fibonacci triangle, we consider a new array by Padovan numbers. The Padovan triangle $\{\Upsilon_n^m\}_{m \geq n \geq 0}$ is defined by the recurrences

(i)

$$\Upsilon_n^m = \Upsilon_{n-2}^m + \Upsilon_{n-3}^m, \quad (m \geq 3) \tag{4}$$

(ii)

$$\Upsilon_n^m = \Upsilon_{n-2}^{m-2} + \Upsilon_{n-3}^{m-3}, \quad (m, n \geq 3) \tag{5}$$

with the initial conditions

$$\begin{aligned} \Upsilon_0^0 = \Upsilon_1^0 = \Upsilon_1^1 = 1, \\ \Upsilon_2^0 = \Upsilon_2^1 = \Upsilon_2^2 = 1, \\ \Upsilon_3^1 = \Upsilon_3^2 = \Upsilon_4^2 = 1. \end{aligned} \tag{6}$$

Here, Υ_n^m denotes the element in row n and column m. The numbers Υ_n^m can be arranged triangularly as in Figure 1 or Figure 2.

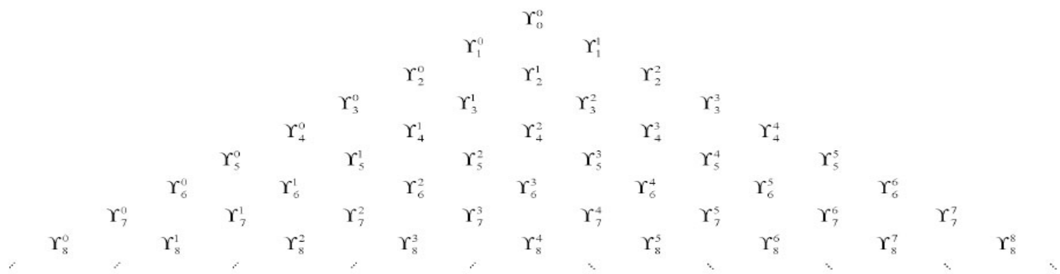

Figure 1: Binomial Padovan Triangle

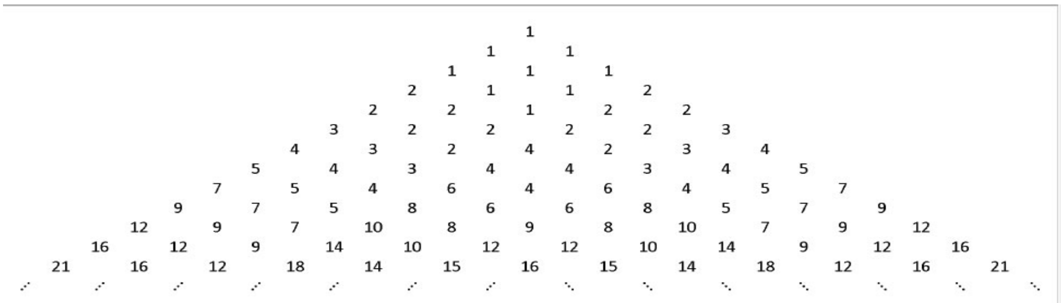

Figure 2: Padovan Triangle numbers

From the relation (4) we write

$$\Upsilon_n^0 = \Upsilon_{n-2}^0 + \Upsilon_{n-3}^0,$$

and by $\Upsilon_0^0 = 1 = P_0$, $\Upsilon_1^0 = 1 = P_1$, $\Upsilon_2^0 = 1 = P_2$, we conclude

$$\Upsilon_n^0 = P_n.$$

Likewise, since $\Upsilon_n^n = \Upsilon_{n-2}^n + \Upsilon_{n-3}^n$ it follows that

$$\Upsilon_n^n = P_n.$$

Similarly, we can show that

$$\Upsilon_n^0 = \Upsilon_n^n = P_n,$$

or

$$\Upsilon_n^1 = \Upsilon_n^{n-1} = P_{n-1}.$$

Successive applications of the recurrence (4) give interesting patterns

$$\Upsilon_n^m = \Upsilon_{n-2}^m + \Upsilon_{n-3}^m$$

$$
\begin{aligned}
&= \Upsilon_{n-3}^m + \Upsilon_{n-4}^m + \Upsilon_{n-5}^m \\
&= \Upsilon_{n-4}^m + 2\Upsilon_{n-5}^m + \Upsilon_{n-6}^m \\
&= 2\Upsilon_{n-5}^m + 2\Upsilon_{n-6}^m + \Upsilon_{n-7}^m \\
&= 2\Upsilon_{n-6}^m + 3\Upsilon_{n-7}^m + 2\Upsilon_{n-8}^m \\
&= \dots
\end{aligned}
$$

Continuing like this, we find a close link between Υ_n^m and Padovan numbers:

$$\Upsilon_n^m = P_{k-2}\Upsilon_{n-k}^m + P_{k-1}\Upsilon_{n-k-1}^m + P_{k-3}\Upsilon_{n-k-2}^m$$

where, $2 \leq k \leq n - m - 2$.
In particular, let $k = n - m - 2$. Then, we have

$$
\begin{aligned}
\Upsilon_n^m &= P_{n-m-4}\Upsilon_{m+2}^m + P_{n-m-3}\Upsilon_{m+1}^m + P_{n-m-5}\Upsilon_m^m \\
&= P_{n-m-4}P_m + P_{n-m-3}P_m + P_{n-m-5}P_m \\
&= (P_{n-m-4} + P_{n-m-3} + P_{n-m-5})P_m \\
&= (P_{n-m-2} + P_{n-m-3})P_m \\
&= P_{n-m}P_m. \qquad (7)
\end{aligned}
$$

Thus, every element in the array is a product of two Padovan numbers. For example,

$$\Upsilon_6^4 = P_{6-4}P_4 = P_2P_4 = 1.2 = 2,$$

$$\Upsilon_9^6 = P_{9-6}P_6 = P_3P_6 = 2.4 = 8.$$

Since $\Upsilon_n^m = \Upsilon_n^{n-m}$, it follows from equation (7) that

$$\Upsilon_n^m = \Upsilon_n^{n-m} = P_{n-m}P_m.$$

Let $n = 2r$ and $m = r$.The equality (7) yields

$$\Upsilon_{2r}^r = P_rP_r = P_r^2.$$

Thus, Υ_{2r}^r is the square of a Padovan number. In other words, the elements along the vertical line through the middle are Padovan squares. For example,

$$\Upsilon_6^3 = P_3^2 = 4,$$

$$\Upsilon_{10}^5 = P_5^2 = 9.$$

.

Theorem 2.1. *The Binet-like formula for the Padovan triangle is*

$$\Upsilon_n^m = a_m\lambda^n + b_m\mu^n + c_m\delta^n$$

where

$$a_m = a^2 + ab\mu^{2m}\delta^m + ac\mu^m\delta^{2m},$$
$$b_m = b^2 + ab\lambda^{2m}\delta^m + bc\lambda^m\delta^{2m},$$
$$c_m = c^2 + ac\lambda^{2m}\mu^m + bc\lambda^m\mu^{2m},$$

and a, b *and* c *are defined in (3).*

Proof. By using the relations above, we can write that

$$\begin{aligned}\Upsilon_n^m = P_{n-m}P_m &= (a\lambda^{n-m} + b\mu^{n-m} + c\delta^{n-m})(a\lambda^m + b\mu^m + c\delta^m)\\ &= a^2\lambda^n + ab\lambda^{n-m}\mu^m + ac\lambda^{n-m}\delta^m\\ &+ ab\lambda^m\mu^{n-m} + b^2\mu^n + bc\mu^{n-m}\delta^m\\ &+ ac\lambda^m\delta^{n-m} + bc\mu^m\delta^{n-m} + c^2\delta^n\end{aligned}$$

$$\begin{aligned}&= (a^2 + ab\mu^{2m}\delta^m + ac\mu^m\delta^{2m})\lambda^n\\ &+ (b^2 + ab\lambda^{2m}\delta^m + bc\lambda^m\delta^{2m})\mu^n\\ &+ (c^2 + ac\lambda^{2m}\mu^m + bc\lambda^m\mu^{2m})\delta^n\\ &= a_m\lambda^n + b_m\mu^n + c_m\delta^n.\end{aligned}$$

□

Using the similar techniques in [1, 2] we can prove the following result.

Theorem 2.2. *The generating function for the partial sum of the Padovan triangle numbers is*

$$G_\Upsilon(x) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\Upsilon_n^m\right)x^n = \frac{1 + 2x + x^2}{(1 - x^2 - x^3)^2}.$$

Proof. Let

$$\begin{aligned}G_\Upsilon(x) &= \sum_{n=0}^{\infty}\left(\Upsilon_n^0 + \Upsilon_n^1 + \Upsilon_n^2 \cdots + \Upsilon_n^{n-1} + \Upsilon_n^n\right)x^n\\ &= \Upsilon_0^0 + \left(\Upsilon_1^0 + \Upsilon_1^1\right)x + \left(\Upsilon_2^0 + \Upsilon_2^1 + \Upsilon_2^2\right)x^2 + \cdots + \left(\Upsilon_n^0 + \Upsilon_n^1 + \cdots + \Upsilon_n^n\right)x^n + \ldots\end{aligned}$$

be generating function for the partial sum of the Padovan triangle numbers. Multiplying every side of this function with $-2x^2$, $-2x^3$, x^4, $2x^5$, x^6, respectively, such as

$$\begin{aligned}-2x^2G_\Upsilon(x) &= -2\Upsilon_0^0x^2 - 2\left(\Upsilon_1^0 + \Upsilon_1^1\right)x^3 - 2\left(\Upsilon_2^0 + \Upsilon_2^1 + \Upsilon_2^2\right)x^4 + \ldots\\ &- 2\left(\Upsilon_n^0 + \Upsilon_n^1 + \cdots + \Upsilon_n^n\right)x^{n+2} + \ldots\end{aligned}$$

$$-2x^3G_\Upsilon(x) = -2\Upsilon_0^0x^3 - 2\left(\Upsilon_1^0 + \Upsilon_1^1\right)x^4 - 2\left(\Upsilon_2^0 + \Upsilon_2^1 + \Upsilon_2^2\right)x^5 + \ldots$$

$$- 2\left(\Upsilon_n^0 + \Upsilon_n^1 + \dots + \Upsilon_n^n\right) x^{n+3} + \dots$$

$$\begin{aligned} x^4 G_\Upsilon(x) &= \Upsilon_0^0 x^4 + \left(\Upsilon_1^0 + \Upsilon_1^1\right) x^5 + \left(\Upsilon_2^0 + \Upsilon_2^1 + \Upsilon_2^2\right) x^6 + \dots \\ &+ \left(\Upsilon_n^0 + \Upsilon_n^1 + \dots + \Upsilon_n^n\right) x^{n+4} + \dots \end{aligned}$$

$$\begin{aligned} 2x^5 G_\Upsilon(x) &= 2\Upsilon_0^0 x^5 + 2\left(\Upsilon_1^0 + \Upsilon_1^1\right) x^6 + 2\left(\Upsilon_2^0 + \Upsilon_2^1 + \Upsilon_2^2\right) x^7 + \dots \\ &+ 2\left(\Upsilon_n^0 + \Upsilon_n^1 + \dots + \Upsilon_n^n\right) x^{n+5} + \dots \end{aligned}$$

$$\begin{aligned} x^6 G_\Upsilon(x) &= \Upsilon_0^0 x^6 + \left(\Upsilon_1^0 + \Upsilon_1^1\right) x^7 + \left(\Upsilon_2^0 + \Upsilon_2^1 + \Upsilon_2^2\right) x^8 + \dots \\ &+ \left(\Upsilon_n^0 + \Upsilon_n^1 + \dots + \Upsilon_n^n\right) x^{n+6} + \dots \end{aligned}$$

Then, we have

$$\begin{aligned} (1 - x^2 - x^3)^2 G_\Upsilon(x) &= \Upsilon_0^0 + \left(\Upsilon_1^0 + \Upsilon_1^1\right) x + \left(\Upsilon_2^0 + \Upsilon_2^1 + \Upsilon_2^2 - 2\Upsilon_0^0\right) x^2 \\ &+ (\Upsilon_3^0 + \Upsilon_3^1 + \Upsilon_3^2 + \Upsilon_3^3 - 2\Upsilon_1^0 - 2\Upsilon_1^1 - 2\Upsilon_0^0) x^3 + \dots \\ &+ (-2\Upsilon_n^0 - 2\Upsilon_n^1 - \dots - 2\Upsilon_n^n - \dots - 2\Upsilon_{n-2}^0 - 2\Upsilon_{n-2}^1 - \dots \\ &- 2\Upsilon_{n-2}^n + \dots + \Upsilon_{n-3}^0 + \Upsilon_{n-3}^1 + \dots + \Upsilon_{n-3}^n + \dots \\ &+ \Upsilon_{n-4}^0 + \Upsilon_{n-4}^1 + \dots + \Upsilon_{n-4}^n + \dots + 2\Upsilon_{n-5}^0 + 2\Upsilon_{n-5}^1 + \dots \\ &+ 2\Upsilon_{n-5}^n + \dots + \Upsilon_{n-6}^0 + \Upsilon_{n-6}^1 + \dots + \Upsilon_{n-6}^n) x^n + \dots \end{aligned}$$

Now, using (6) and

$$\sum_{m=0}^{n} \Upsilon_n^m - 2\sum_{m=0}^{n-2} \Upsilon_{n-2}^m - 2\sum_{m=0}^{n-3} \Upsilon_{n-3}^m + \sum_{m=0}^{n-4} \Upsilon_{n-4}^m + 2\sum_{m=0}^{n-5} \Upsilon_{n-5}^m + \sum_{m=0}^{n-6} \Upsilon_{n-6}^m = 0,$$

we obtain that

$$G_\Upsilon(x) = \frac{1 + 2x + x^2}{(1 - x^2 - x^3)^2}.$$

Thus, the proof is completed. □

3. Some Identities of the Padovan Triangle

The properties of various configurations within the triangular array for the Fibonacci triangle were investigated in [4]. Here, we examine similar properties for the Padovan triangle.

Proposition 3.1. *The following relations are valid*

1. $\Upsilon_n^m = \Upsilon_{n+6}^{m+3} + \Upsilon_{n+2}^{m+1} - \left(\Upsilon_{n+4}^{m+1} + \Upsilon_{n+4}^{m+3}\right)$,
2. $\Upsilon_{n-2}^{m-1}\Upsilon_{n+2}^{m+1} = \Upsilon_n^{m-1}\Upsilon_n^{m+1}$,
3. $\Upsilon_{n+6}^{m+3} = \Upsilon_{n+2}^{m+1} + \Upsilon_{n+1}^{m+1} + \Upsilon_{n+1}^{m} + \Upsilon_n^m$,
4. $\Upsilon_{2n}^n = \Upsilon_n^0\Upsilon_n^n$.

Proof. 1. Using (7) and Figure 2 we can illustrated the Figure 3: By Figure 2 and (7)

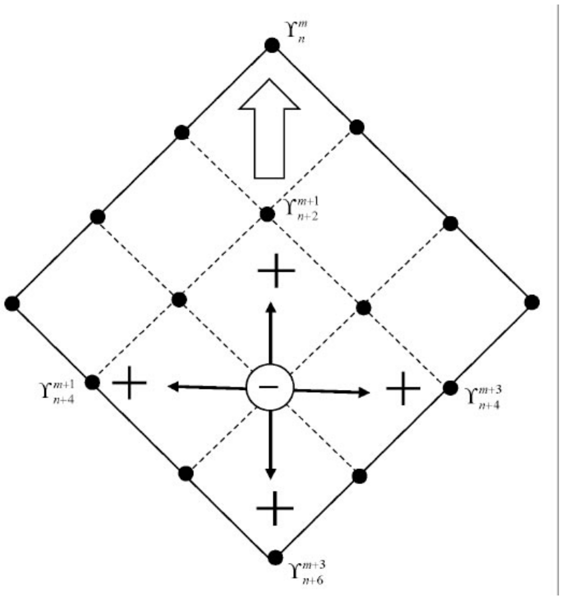

Figure 3:

we obtain

$$\begin{aligned}
\Upsilon_n^m &= \Upsilon_{n+6}^{m+3} + \Upsilon_{n+2}^{m+1} - \left(\Upsilon_{n+4}^{m+1} + \Upsilon_{n+4}^{m+3}\right) \\
&= P_{n-m+3}P_{m+3} + P_{n-m+1}P_{m+1} - (P_{n-m+3}P_{m+1} + P_{n-m+1}P_{m+3}) \\
&= (P_{n-m+1} - P_{n-m+3})\, P_{m+1} + (P_{n-m+3} - P_{n-m+1})\, P_{m+3} \\
&= -P_{n-m}P_{m+1} + P_{n-m}P_{m+3} \\
&= P_{n-m}\,(P_{m+3} - P_{m+3}) \\
&= P_{n-m}P_m.
\end{aligned}$$

2. Using (7) and Figure 2 we can illustrated the Figure 4:

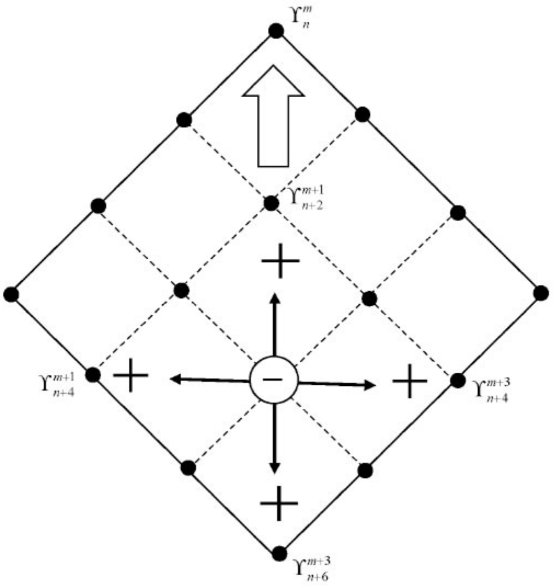

Figure 4:

By Figure 2 and (7) we obtain

$$\begin{aligned}\Upsilon_{n-2}^{m-1}\Upsilon_{n+2}^{m+1} &= \Upsilon_{n}^{m-1}\Upsilon_{n}^{m+1},\\ P_{m-n-1}P_{m-1}P_{n-m+1}P_{m+1} &= P_{m-n+1}P_{m-1}P_{n-m-1}P_{m+1},\\ 1 &= 1.\end{aligned}$$

3. Using (7) and Figure 2 we can illustrated the Figure 5:

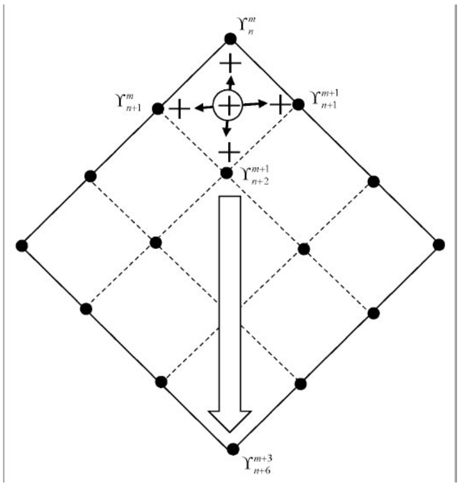

Figure 5:

By Figure 2 and (7) we obtain

$$\begin{aligned}\Upsilon_{n+6}^{m+3} &= \Upsilon_{n+2}^{m+1} + \Upsilon_{n+1}^{m+1} + \Upsilon_{n+1}^{m} + \Upsilon_{n}^{m} \\ &= P_{n-m+1}P_{m+1} + P_{n-m}P_{m+1} + P_{n-m+1}P_m + P_{n-m}P_m \\ &= P_{n-m+3}P_{m+1} + P_{n-m+3}P_m \\ &= P_{n-m+3}P_{m+3}.\end{aligned}$$

4. Using (7) and Figure 2 we can illustrated the Figure 6:

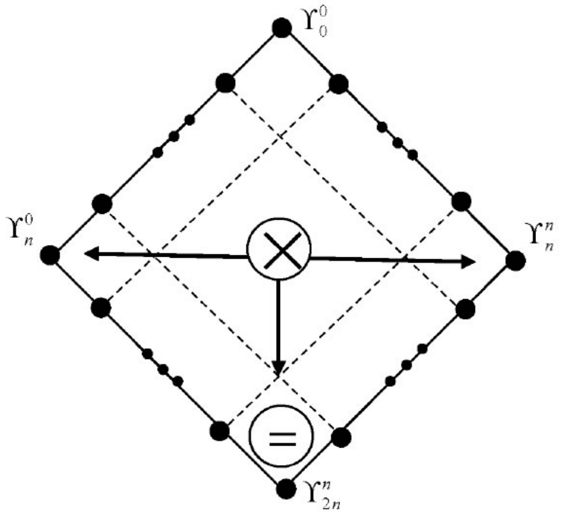

Figure 6:

By Figure 2 and (7) we obtain

$$\begin{aligned}\Upsilon_{2n}^{n} &= \Upsilon_{n}^{0}\Upsilon_{n}^{n} \\ &= P_nP_0P_0P_n \\ &= P_nP_n.\end{aligned}$$

□

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Orhan Dişkaya
Graduate School of Natural and Applied Sciences, Mersin University, Mersin, Turkey
E-mail: orhandiskayal@mersin.edu.tr

Hamza Menken
Department of Mathematics, Science and Letters Faculty, Mersin University, Mersin, Turkey
E-mail: hmenken@mersin.edu.tr

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