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Existence and Uniqueness of a Solution to the Optimal Control Problem with Lions Type Functional for the Linear Nonstationary Guasi-Optics Equation

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Abstract. In this paper we consider the optimal control problem with an integral performance criterion of the Lions functional type for the linear nonstationary quasi-optics equation with a special gradient term. Such quality criteria are usually formed on the basis of the Dirichlet-Neumann mapping on the boundary. The theorems on the existence and uniqueness of the solution to the considered optimal control problem are proved.

Key Words and Phrases: Nonstationary quasi-optics equations, special gradient term, optimal control problem, Lions functional.

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1. Introduction

Optimal control problems for the nonstationary quasi-optics equation with a special gradient term often arise in nonlinear optics when studying the propagation of a light beam in the inhomogeneous medium, in which the role of control is usually played by the refractive and absorption indices of the medium, as well as the initial phase of the emitted wave [1]. Note that identification problems for a non-stationary quasi-optics equation without a special gradient term earlier were studied in [2-6] and others, when the quality criteria were built mainly on the basis of final observations.

In this paper, we consider the optimal control problem for the nonstationary quasi-optics equation with a special gradient term, in which the role of control is usually played by the refractive and absorption indices of the medium and the performance criterion is a Lions-type functional. Such quality criteria are usually formed on the basis of the Dirichlet-Neumann mapping on the boundary. This approach for determining the initial function in the inverse problem for the parabolic equation was used by J.-L. Lyons [7]. And then it was widely used in the works of A.D. Iskenderov to determine the coefficients of the basic types of equations of mathematical physics and the nonstationary Schrödinger equation [8-11].

It should be noted that a similar problem for the nonstationary quasi-optics equation without a special gradient term, when the coefficients of the equation are boundedly measurable functions, was previously studied in [10]. Taking into account the above, we can assert that the problem considered in this work differs from the works [2-6, 10] and others in the formulation. Therefore, this work is relevant and is of interest both from a theoretical and practical point of view.

2. Problem Formulation

Let $l > 0, T > 0, L > 0$ be given numbers; $0 \leq x \leq l, 0 \leq t \leq T, 0 \leq z \leq L, \Omega_t = (0,l) \times (0,t)$, $\Omega_z = (0,l) \times (0,z)$, $\Omega_{tz} = (0,l) \times (0,t) \times (0,z)$, $\Omega = \Omega_{TL}$, $Q = (0,T) \times (0,L)$; $C^k([0,T], B)$ be a Banach space of the defined and $k \geq 0$ time continuousely differentable on the interval $[0,T]$ function with values from Banach space; $L_p(0,l)$ be a Lebesque space of the functions summing on the interval $(0,l)$ with order $p \geq 1$; $W_p^k(0,l), W_p^{k,m}(\Omega_L), W_p^{k,m}(\Omega_T), p \geq 1, k \geq 0, m \geq 0$ be a Sobolev spaces defined as for example in [12]; $W_2^{0,1,1}(\Omega)$ be a Hilbert space of the elements $u = u(x,t,z)$ from having generalized derivatives $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial z}$ from the space $L_2(\Omega)$ with scalar production and norm defined as follows

$$(u_1, u_2)_{W_2^{0,1,1}(\Omega)} = \int_\Omega \left(u_1 \bar{u}_2 + \frac{\partial u_1}{\partial t} \frac{\partial \bar{u}_2}{\partial t} + \frac{\partial u_1}{\partial z} \frac{\partial \bar{u}_2}{\partial z} \right) dxdtdz,$$

$$\|u\|_{W_2^{0,1,1}(\Omega)} = \sqrt{(u,u)_{W_2^{0,1,1}(\Omega)}};$$

$W_2^{2,0,0}(\Omega)$ be a Hilbert space of the elements $u = u(x,t,z)$ from $L_2(\Omega)$ having generalized derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ from $L_2(\Omega)$ with scalar product and norm defined as follows

$$(u_1, u_2)_{W_2^{2,0,0}(\Omega)} = \int_\Omega \left(u_1 \bar{u}_2 + \frac{\partial u_1}{\partial x} \frac{\partial \bar{u}_2}{\partial x} + \frac{\partial^2 u_1}{\partial x^2} \frac{\partial^2 \bar{u}_2}{\partial x^2} \right) dxdtdz,$$

$$\|u\|_{W_2^{2,0,0,}(\Omega)} = \sqrt{(u,u)_{W_2^{2,0,0}(\Omega)}};$$

$$W_2^{2,1,1}(\Omega) \equiv W_2^{2,0,0}(\Omega) \bigcap W_2^{0,1,1}(\Omega);$$

$\overset{0}{W}{}_2^{2,1,1}(\Omega)$ be subspace of the space $W_2^{2,1,1}(\Omega)$ the elements of which turn to zero on $S = \{(x,t,z) : x = 0, l, t \in (0,T), z \in (0,L)\}$.

Consider the optimal control problem on minimizing the functional:

$$J_\alpha(v) = \|\psi_1 - \psi_2\|^2_{L_2(\Omega)} + \alpha \|v - \omega\|^2_H \tag{1}$$

on the set

$$V \equiv \left\{ v = (v_0, v_1) : v_m \in L_2(0,l), |v_m(x)| \leq b_m, \overset{0}{\forall} x \in (0,l), m = 0, 1 \right\}$$

under the conditions

$$i\frac{\partial\psi_p}{\partial t}+ia_0\frac{\partial\psi_p}{\partial z}-a_1\frac{\partial^2\psi_p}{\partial x^2}+ia_2(x)\frac{\partial\psi_p}{\partial x}+a(x)\psi_p+v_0\left(x\right)\psi_p+iv_1\left(x\right)\psi_p=$$

$$=f_p(x,t,z),p=1,2,\ \ (x,t,z)\in\Omega, \tag{2}$$

$$\psi_p\left(x,0,z\right)=\varphi_{0p}\left(x,z\right),p=1,2,(x,z)\in\Omega_L, \tag{3}$$

$$\psi_p\left(x,t,0\right)=\varphi_{1p}\left(x,t\right),p=1,2,(x,t)\in\Omega_T, \tag{4}$$

$$\psi_1\left(0,t,z\right)=\psi_1\left(l,t,z\right)=0,(t,z)\in Q_{TL}, \tag{5}$$

$$\frac{\partial\psi_2\left(0,t,z\right)}{\partial x}=\frac{\partial\psi_2\left(l,t,z\right)}{\partial x}=0,(t,z)\in Q_{TL}, \tag{6}$$

where i is an imaginary unit, $a_0>0$, $a_1>0$, $\alpha\geq 0$, $b_0>0$, $b_1>0$- are given numbers; $0\leq x\leq l, 0\leq t\leq T, 0\leq z\leq L$, $\Omega_t=(0,l)\times(0,t)$, $\Omega_z=(0,l)\times(0,z)$, $\Omega_{tz}=(0,l)\times(0,t)\times(0,z)\,,\Omega=\Omega_{TL}$, $Q_{tz}=(0,t)\times(0,z)$; $a\left(x\right),a_2\left(x\right)$ are given measurable bounded functions satisfying the following conditions:

$$\tilde{\mu}_0\leq a\left(x\right)\leq\mu_0,\overset{0}{\forall}x\in\left(0,l\right),\mu_0,\tilde{\mu}_0=const>0; \tag{7}$$

$$\left|a_2\left(x\right)\right|\leq\mu_1,\left|\frac{da_2\left(x\right)}{dx}\right|\leq\mu_2,\overset{0}{\forall}x\in\left(0,l\right),\mu_1,\mu_2=const>0,a_2\left(0\right)=a_2\left(l\right)=0; \tag{8}$$

and the measurable functions $\varphi_{0p}\left(x,z\right),\varphi_{1p}\left(x,t\right)$,$f_p\left(x,t,z\right),p=1,2$ satisfy the conditions

$$\varphi_{01}\in\overset{0}{W}{}_2^{2,1}\left(\Omega_L\right),\varphi_{11}\in\overset{0}{W}{}_2^{2,1}\left(\Omega_T\right); \tag{9}$$

$$\varphi_{02}\in W_2^{2,1}\left(\Omega_L\right),\frac{\partial\varphi_{02}\left(0,z\right)}{\partial x}=\frac{\partial\varphi_{02}\left(l,z\right)}{\partial x}=0,z\in\left(0,L\right); \tag{10}$$

$$\varphi_{12}\in W_2^{2,1}\left(\Omega_T\right),\frac{\partial\varphi_{02}\left(0,t\right)}{\partial x}=\frac{\partial\varphi_{02}\left(l,t\right)}{\partial x}=0,t\in\left(0,T\right); \tag{11}$$

$$f_p\in W_2^{0,1,1}\left(\Omega\right),p=1,2; \tag{12}$$

$\omega=(\omega_0,\omega_1)\in H\equiv L_2\left(0,l\right)\times L_2\left(0,l\right)$ is a given element; the symbol $\overset{0}{\forall}$ stands "for almost all".

The problem of determining functions from conditions (2)-(6) for each will be called a reduced problem. It is clear that the reduced problem consists of two initial-boundary value problems for a linear nonstationary quasi-optics equation with a special gradient term. For each $v\in V$ the first initial boundary value problem is the problem of determining the function $\psi_1=\psi_1\left(x,t\right)\equiv\psi_1\left(x,t;v\right)$ from conditions (2)-(5) for $p=1$ and the second initial boundary value problem is the problem of determining $\psi_2=\psi_2\left(x,t\right)\equiv\psi_2\left(x,t;v\right)$ from conditions (2)-(4), (6) for $p=2$.

Definition 2.1. *By the solution of the reduced problem* (2)-(6) *for each* $v \in V$ *we mean the functions* $\psi_1 = \psi_1(x,t,z) \equiv \psi_1(x,t,z;v)$ *from the functional space* $\overset{0}{W}{}_2^{2,1,1}(\Omega)$ *and* $\psi_2 = \psi_2(x,t,z) \equiv \psi_2(x,t,z;v)$ *from the functional space* $W_2^{2,1,1}(\Omega)$ *satisfying* (2) *for almost all* $(x,t,z) \in \Omega$ *initial conditions* (3), (4) *for almost all* $(x,z) \in \Omega_L$ *and boundary conditions (5), (6) for almost all* $(t,z) \in Q$, *respectively.*

Reduced problem (2)-(6), as noted above, consists of two initial-boundary value problems, i.e. of the first initial-boundary value problem (2) - (5) and the second initial-boundary value problem (2)-(4), (6), which were studied earlier in the works [13, 14]. It should be noted that the initial-boundary value problems for a linear nonstationary quasi-optics equation without a special gradient term, when the coefficients of the equation are square-summable functions, were previously studied in [15]. Based on the results of the works [13, 14], we can establish the validity of the statement:

Theorem 2.2. *Let conditions* (7)-(12) *be satisfied. Then reduced problem* (2)-(6) *for each* $v \in V$ *has a unique solution* $\psi_1 \in \overset{0}{W}{}_2^{2,1,1}(\Omega), \psi_2 \in W_2^{2,1,1}(\Omega)$ *and for this solution the following estimates are valid:*

$$\|\psi_1\|^2_{\overset{0}{W}{}_2^{2,1,1}(\Omega)} \leq c_1 \left(\|\varphi_{01}\|^2_{\overset{0}{W}{}_2^{2,1}(\Omega_L)} + \|\varphi_{11}\|^2_{\overset{0}{W}{}_2^{2,1}(\Omega_T)} + \|f_1\|^2_{W_2^{0,1,1}(\Omega)} \right), \tag{13}$$

$$\|\psi_2\|^2_{W_2^{2,1,1}(\Omega)} \leq c_2 \left(\|\varphi_{02}\|^2_{W_2^{2,1}(\Omega_L)} + \|\varphi_{12}\|^2_{W_2^{2,1}(\Omega_T)} + \|f_2\|^2_{W_2^{0,1,1}(\Omega)} \right), \tag{14}$$

where the constants $c_1 > 0, c_2 > 0$ *do not depend on* $\varphi_{0p}, \varphi_{1p}, f_p, p = 1,2$.

This theorem implies that functional (1) makes sense in the considered class of solutions to the reduced problem (2)-(6) for each given $v \in V$.

3. Existence and uniqueness of the solution of the optimal control problems

First, let us establish that optimal control problem (1)-(6) has a unique solution for $\alpha > 0$. To this end, we first formulate the following theorem, which is known from [16].

Theorem 3.1. *Let* $\tilde{X}$ *be a uniformly convex space,* U *be a closed unbounded set from* $\tilde{X}$, *the functional* $I(v)$ *is lower bounded and lower semicontinuous on* U, $\alpha > 0$, $\beta \geq 1$ *are given numbers. Then there is a dense subset* G *of the space* $\tilde{X}$ *such that for any* $\omega \in G$ *the functional*

$$J_\alpha(v) = I(v) + \alpha \|v - \omega\|^\beta_{\tilde{X}}$$

reaches its lowest value on U. *If* $\beta > 1$, *then the minimum value of the functional* $J_\alpha(v)$ *is attained at the single element of* U.

Theorem 3.2. *Let the conditions of Theorem 2.2 be satisfied. Let, in addition,* $\omega \in H$ *be a given element. Then there exists a dense subset* G *of the space* H *such that, for any* $\omega \in G$ *at* $\alpha > 0$ *optimal control problem* (1)-(6) *has a unique solution.*

Proof. First, we prove the continuity of the functional

$$J_0(v) = \|\psi_1 - \psi_2\|^2_{L_2(\Omega)} \tag{15}$$

on the set V. Let $\Delta v \in B = L_\infty(0,l) \times L_\infty(0,l)$ be an increment of any element $v \in V$ such that $v + \Delta v \in V$ and $\Delta\psi_p = \Delta\psi_p(x,t,z) \equiv \psi_p(x,t,z;v+\Delta v) - \psi_p(x,t,z;v)$, $p = 1,2$, where $\psi_p(x,t,z;v), p = 1,2$ is a solution of the reduced problem (2) -(6) at $v \in V$. From conditions (2)-(6) follows that $\Delta\psi_p = \Delta\psi_p(x,t,z), p = 1,2$ is a solution to the following system of initial boundary value problems

$$i\frac{\partial\Delta\psi_p}{\partial t} + ia_0\frac{\partial\Delta\psi_p}{\partial z} - a_1\frac{\partial^2\Delta\psi_p}{\partial x^2} + ia_2(x)\frac{\partial\Delta\psi_p}{\partial x} + a(x)\Delta\psi_p + (v_0(x) + \Delta v_0(x))\Delta\psi_p +$$

$$+ i(v_1(x) + \Delta v_1(x))\Delta\psi_p = -\Delta v_0(x)\psi_p - i\Delta v_1(x)\psi_p, p = 1,2, (x,t,z) \in \Omega, \tag{16}$$

$$\Delta\psi_p(x,0,z) = 0, p = 1,2, (x,z) \in \Omega_L, \tag{17}$$

$$\Delta\psi_p(x,t,0) = 0, p = 1,2, (x,t) \in \Omega_T, \tag{18}$$

$$\Delta\psi_1(0,t,z) = \Delta\psi_1(l,t,z) = 0, (t,z) \in Q, \tag{19}$$

$$\frac{\partial\Delta\psi_2(0,t,z)}{\partial x} = \frac{\partial\Delta\psi_2(l,t,z)}{\partial x} = 0, (t,z) \in Q. \tag{20}$$

Let's estimate the solution of this system. For this purpose, we multiply both sides of equations (16) by functions $\Delta\bar{\psi}_p = \Delta\bar{\psi}_p(x,t,z), p = 1,2$ and integrate the obtained equalities over the domain Ω_{tz}. Then we get the validity of the equalities

$$\int_{\Omega_{tz}} \left(i\frac{\partial\Delta\psi_p}{\partial t} + ia_0\frac{\partial\Delta\psi_p}{\partial z} - a_1\frac{\partial^2\Delta\psi_p}{\partial x^2} + ia_2(x)\frac{\partial\Delta\psi_p}{\partial x} + a(x)\Delta\psi_p\right)\Delta\bar{\psi}_p dx d\tau d\theta +$$

$$+ \int_{\Omega_{tz}} \left((v_0(x) + \Delta v_0(x))|\Delta\psi_p|^2 + i(v_1(x) + \Delta v_1(x))|\Delta\psi_p|^2\right) dx d\tau d\theta =$$

$$= -\int_{\Omega_{tz}} (\Delta v_0(x)\psi_p + i\Delta v_1(x)\psi_p)\Delta\bar{\psi}_p dx d\tau d\theta, p = 1,2, \forall t \in [0,T], \forall z \in [0,L].$$

Applying the formula of integration by parts on the left-hand side of these equalities and using the boundary conditions (19) and (20), it is easy to obtain the validity of the equalities

$$\int_{\Omega_{tz}} \left(i\frac{\partial\Delta\psi_p}{\partial t}\Delta\bar{\psi}_p + ia_0\frac{\partial\Delta\psi_p}{\partial z}\Delta\bar{\psi}_p + a_1\left|\frac{\partial\Delta\psi_p}{\partial x}\right|^2 +\right.$$

$$\left. + ia_2(x)\frac{\partial\Delta\psi_p}{\partial x}\Delta\bar{\psi}_p + a(x)|\Delta\psi_p|^2\right) dx d\tau d\theta +$$

$$+ \int_{\Omega_{tz}} \left((v_0(x) + \Delta v_0(x))|\Delta\psi_p|^2 + i(v_1(x) + \Delta v_1(x))|\Delta\psi_p|^2\right) dx d\tau d\theta =$$

$$= -\int_{\Omega_{tz}} \left(\Delta v_0\left(x\right)\psi_p + i\Delta v_1\left(x\right)\psi_p\right)\Delta\bar{\psi}_p dxd\tau d\theta, p=1,2, \forall t\in[0,T], \forall z\in[0,L]. \quad (21)$$

It is clear that the complex conjugation of these equalities has the form:

$$\int_{\Omega_{tz}} \left(-i\frac{\partial\Delta\bar{\psi}_p}{\partial t}\Delta\psi_p - ia_0\frac{\partial\Delta\bar{\psi}_p}{\partial z}\Delta\psi_p + a_1\left|\frac{\partial\Delta\psi_p}{\partial x}\right|^2 - \right.$$

$$\left. -ia_2(x)\frac{\partial\Delta\bar{\psi}_p}{\partial x}\Delta\psi_p + a(x)\left|\Delta\psi_p\right|^2\right) dxd\tau d\theta +$$

$$+\int_{\Omega_{tz}} \left(\left(v_0\left(x\right)+\Delta v_0\left(x\right)\right)\left|\Delta\psi_p\right|^2 - i\left(v_1\left(x\right)+\Delta v_1\left(x\right)\right)\left|\Delta\psi_p\right|^2\right) dxd\tau d\theta =$$

$$= -\int_{\Omega_{tz}} \left(\Delta v_0\left(x\right)\bar{\psi}_p - i\Delta v_1\left(x\right)\bar{\psi}_p\right)\Delta\psi_p dxd\tau d\theta, p=1,2, \forall t\in[0,T], \forall z\in[0,L].$$

Subtracting these equalities from equalities (21), we get

$$i\int_{\Omega_{tz}} \left(\frac{\partial\Delta\psi_p}{\partial t}\Delta\bar{\psi}_p + \frac{\partial\Delta\bar{\psi}_p}{\partial t}\Delta\psi_p\right) dxd\tau d\theta + ia_0\int_{\Omega_{tz}} \left(\frac{\partial\Delta\psi_p}{\partial z}\Delta\bar{\psi}_p + \frac{\partial\Delta\bar{\psi}_p}{\partial z}\Delta\psi_p\right) dxd\tau d\theta +$$

$$+i\int_{\Omega_{tz}} a_2\left(x\right)\left(\frac{\partial\Delta\psi_p}{\partial x}\Delta\bar{\psi}_p + \frac{\partial\Delta\bar{\psi}_p}{\partial x}\Delta\psi_p\right) dxd\tau d\theta +$$

$$+2i\int_{\Omega_{tz}} \left(\left(v_1\left(x\right)+\Delta v_1\left(x\right)\right)\left|\Delta\psi_p\right|^2\right) dxd\tau d\theta = -\int_{\Omega_{tz}} \Delta v_0\left(x\right)\left(\psi_p\Delta\bar{\psi}_p - \bar{\psi}_p\Delta\psi_p\right) dxd\tau d\theta -$$

$$-2i\int_{\Omega_{tz}} \Delta v_1\left(x\right)\left(\psi_p\Delta\bar{\psi}_p + \bar{\psi}_p\Delta\psi_p\right) dxd\tau d\theta, p=1,2, \forall t\in[0,T], \forall z\in[0,L].$$

Using these equalities it is easy to establish the validity of the following equalities

$$\int_{\Omega_{tz}} \frac{\partial}{\partial t}\left|\Delta\psi_p\right|^2 dxd\tau d\theta + a_0\int_{\Omega_{tz}} \frac{\partial}{\partial z}\left|\Delta\psi_p\right|^2 dxd\tau d\theta + \int_{\Omega_{tz}} \frac{\partial}{\partial x}\left(a_2\left(x\right)\left|\Delta\psi_p\right|^2\right) dxd\tau d\theta =$$

$$= -2\int_{\Omega_{tz}} \left(v_1\left(x\right)+\Delta v_1\left(x\right)\right)\left|\Delta\psi_p\right|^2 dxd\tau d\theta + \int_{\Omega_{tz}} \frac{da_2\left(x\right)}{dx}\left|\Delta\psi_p\right|^2 dxd\tau d\theta +$$

$$= -2\int_{\Omega_{tz}} \Delta v_0\left(x\right) Im\left(\psi_p\Delta\bar{\psi}_p\right) dxd\tau d\theta - 2\int_{\Omega_{tz}} \Delta v_1\left(x\right) Re\left(\psi_p\Delta\bar{\psi}_p\right) dxd\tau d\theta, p=1,2$$

for $\forall t\in[0,T]$, $\forall z\in[0,L]$. On the left-hand side of these equalities, the third term is equal to zero by virtue of the conditions $a_2\left(0\right)=a_2\left(l\right)=0$. Therefore, from this, using conditions$\left|v_1\left(x\right)+\Delta v_1\left(x\right)\right| \le b_1, \overset{0}{\forall} x\in(0,l)$, (8) and initial conditions (17), (18), as well as the Cauchy-Bunyakovsky inequality, we can establish the validity of the inequalities

$$\left\|\Delta\psi_p\left(\cdot,t,\cdot\right)\right\|^2_{L_2(\Omega_z)} + a_0\left\|\Delta\psi_p\left(\cdot,\cdot,z\right)\right\|^2_{L_2(\Omega_t)} \le \left(2+\mu_2\right)\int_{\Omega_{tz}} \left|\Delta\psi_p\right|^2 dxd\tau d\theta +$$

$$+\left(\|\Delta v_0\|^2_{L_\infty(0,l)}+\|\Delta v_1\|^2_{L_\infty(0,l)}\right)\|\psi_p\|^2_{L_2(\Omega)}\,, p=1,2,\forall t\in[0,T]\,,\,\forall z\in[0,L]\,.$$

Hence, by virtue of estimates (13), (14), we obtain the validity of the following inequalities:

$$\|\Delta\psi_p(\cdot,t,\cdot)\|^2_{L_2(\Omega_z)}+a_0\|\Delta\psi_p(\cdot,\cdot,z)\|^2_{L_2(\Omega_t)}\le c_3\left(\|\Delta v_0\|^2_{L_\infty(0,l)}+\|\Delta v_1\|^2_{L_\infty(0,l)}\right)+$$

$$+(2+\mu_2)\int_{\Omega_{tz}}|\Delta\psi_p|^2\,dxd\tau d\theta\,, p=1,2,\forall t\in[0,T]\,,\,\forall z\in[0,L]\,.$$

From these inequalities, we can derive the following two types of inequalities:

$$\|\Delta\psi_p(\cdot,t,\cdot)\|^2_{L_2(\Omega_L)}\le c_3\left(\|\Delta v_0\|^2_{L_\infty(0,l)}+\|\Delta v_1\|^2_{L_\infty(0,l)}\right)+$$

$$+(2+\mu_2)\int_0^t\|\Delta\psi_p(.,\tau,.)\|^2_{L_2(\Omega_L)}\,d\tau, p=1,2,\forall t\in[0,T]\,, \tag{22}$$

$$\|\Delta\psi_p(\cdot,\cdot,z)\|^2_{L_2(\Omega_T)}\le c_4\left(\|\Delta v_0\|^2_{L_\infty(0,l)}+\|\Delta v_1\|^2_{L_\infty(0,l)}\right)+$$

$$+c_5\int_0^z\|\Delta\psi_p(.,.,\theta)\|^2_{L_2(\Omega_L)}\,d\theta, p=1,2,\forall z\in[0,L]\,. \tag{23}$$

Applying Gronwall's lemma, it is easy to establish the following estimates in these inequalities

$$\|\Delta\psi_p(\cdot,t,\cdot)\|^2_{L_2(\Omega_L)}\le c_6\left(\|\Delta v_0\|^2_{L_\infty(0,l)}+\|\Delta v_1\|^2_{L_\infty(0,l)}\right)\,, p=1,2,\forall t\in[0,T]\,, \tag{24}$$

$$\|\Delta\psi_p(\cdot,\cdot,z)\|^2_{L_2(\Omega_T)}\le c_7\left(\|\Delta v_0\|^2_{L_\infty(0,l)}+\|\Delta v_1\|^2_{L_\infty(0,l)}\right)\,, p=1,2,\forall z\in[0,L]\,, \tag{25}$$

where the constants $c_6>0, c_7>0$ do not depend on $\Delta v=(\Delta v_0,\Delta v_1)$. Summing up these estimates, we have:

$$\begin{aligned}&\|\Delta\psi_p(\cdot,t,\cdot)\|^2_{L_2(\Omega_L)}+\|\Delta\psi_p(\cdot,\cdot,z)\|^2_{L_2(\Omega_T)}\le\\&\le c_8\left(\|\Delta v_0\|^2_{L_\infty(0,l)}+\|\Delta v_1\|^2_{L_\infty(0,l)}\right), p=1,2\end{aligned} \tag{26}$$

for $\forall t\in[0,T]\,,\,\forall z\in[0,L]$, where the constant $c_8>0$ does not depend on $\Delta v=(\Delta v_0,\Delta v_1)$.

Directly from these estimates we have:

$$\|\Delta\psi_p\|^2_{L_2(\Omega)}\le c_9\|\Delta v\|^2_B\,, p=1,2 \tag{27}$$

where the constant $c_9>0$ does not depend on Δv.

Now let's consider the increment of the functional $J_0(v)$ on any element $v\in V$. Using formula (15), the increment of the functional $J_0(v)$ can be represented as

$$\Delta J_0(v)=J_0(v+\Delta v)-J_0(v)=$$

$$=2\int_\Omega Re\left[(\psi_1(x,t,z)-\psi_2(x,t,z))\left(\Delta\bar\psi_1(x,t,z)-\Delta\bar\psi_2(x,t,z)\right)\right]dxdtdz+$$

$$+\|\Delta\psi_1\|^2_{L_2(\Omega)}+\|\Delta\psi_2\|^2_{L_2(\Omega)}-2\int_\Omega Re\left(\Delta\psi_1(x,t,z)\,\Delta\bar{\psi}_2(x,t,z)\right)dxdtdz. \qquad (28)$$

Hence, by virtue of the Cauchy-Bunyakovsky inequality and estimates (13), (14), (27), it is easy to establish the validity of the inequality

$$|\Delta J_0(v)|\le c_{10}\left(\|\Delta v\|_B+\|\Delta v\|^2_B\right).$$

From this inequality we obtain the following limiting relation

$$|\Delta J_0(v)|\to 0 \ at \ \|\Delta v\|_B\to 0 \ for \ \forall v\in V.$$

This inequality means the continuity of the functional $J_0(v)$ on the set V. The set V is a closed bounded and convex set of space $B=L_\infty(0,l)\times L_\infty(0,l)$. It is easy to prove that it is a closed bounded and convex set in a uniform convex space $H=L_2(0,l)\times L_2(0,l)$ [17]. Then, by virtue of Theorem 3.1 formulated above, there exists a dense subset G of the space H such that for any $\omega\in G$ and for any $\alpha>0$ optimal control problem (1) - (6) has a unique solution. *Theorem 3.2 is proved.*

This theorem shows that optimal control problem (1)-(6) for $\alpha>0$ has a solution not for every $\omega\in H$. The next statement shows that the problem is that the optimal control problem (1)-(6) has at least one solution at $\alpha\ge 0$ for any $\omega\in H$.

Theorem 3.3. *Let the conditions of Theorem 3.2 be satisfied. Then there exists at least one solution to the optimal control problem* (1)-(6) *at* $\alpha\ge 0$ *for any* $\omega\in H$.

Proof. Take any minimizing sequence $\{v^k\}\subset V$

$$\lim_{k\to\infty}J_\alpha\left(v^k\right)=J_{\alpha*}=\inf_{v\in V}J_\alpha(v)$$

and set $\psi_{pk}=\psi_{pk}(x,t,z)\equiv\psi_p\left(x,t,z;v^k\right), p=1,2,\ k=1,2,....$ By virtue of Theorem 2.2, for each $v^k\subset V$ reduced problem (2) - (6) has a unique solution from the space $\psi_{1k}\in \overset{0}{W}{}_2^{2,1,1}(\Omega), \psi_{2k}\in W_2^{2,1,1}(\Omega)$ and the following estimates hold

$$\|\psi_{1k}\|^2_{\overset{0}{W}{}_2^{2,1,1}(\Omega)}\le c_1\left(\|\varphi_{01}\|^2_{\overset{0}{W}{}_2^{2,1}(\Omega_L)}+\|\varphi_{11}\|^2_{\overset{0}{W}{}_2^{2,1}(\Omega_T)}+\|f_1\|^2_{W_2^{0,1,1}(\Omega)}\right), k=1,2,..., \qquad (29)$$

$$\|\psi_{2k}\|^2_{W_2^{2,1,1}(\Omega)}\le c_2\left(\|\varphi_{02}\|^2_{W_2^{2,1}(\Omega_L)}+\|\varphi_{12}\|^2_{W_2^{2,1}(\Omega_T)}+\|f_2\|^2_{W_2^{0,1,1}(\Omega)}\right), k=1,2,..., \qquad (30)$$

where the right hand side does not depend on k.

Since there V is a closed bounded and convex set from the Banach space $B=L_\infty(0,l)\times L_\infty(0,l)$, then from the sequence $\{v^k\}\subset V$ one can choose such a subsequence $\{v^{k_l}\}$ (which we again denote by $\{v^k\}$) such that

$$v^k_m\to v_m,\ m=0,1\ (*)-weekly\ in\ L_\infty(0,l)\ at\ k\to\infty. \qquad (31)$$

In addition, the set V is (*)-weakly closed set from space B. Therefore $v \in V$. By virtue of the limit relation (31), we can write the following limit relation

$$\int_0^l v_m^k(x)\, q(x)\, dx \to \int_0^l v_m(x)\, q(x)\, dx, m=0,1, \tag{32}$$

at $k \to \infty$ for any function $q \in L_1(0,l)$.

It follows from estimates (29), (30) that the sequences $\{\psi_{pk}(x,t,z)\}, p=1,2$ are uniformly bounded in the norm of the spaces $\overset{0}{W}{}_2^{2,1,1}(\Omega)$ and $W_2^{2,1,1}(\Omega)$, respectively. Then from these sequences one can choose such subsequences $\{\psi_{pk_l}(x,t,z)\}, p=1,2$, (for simplicity of presentation we again denote them by $\{\psi_{pk}(x,t,z)\}, p=1,2$) such that

$$\psi_{pk} \to \psi_p,\ p=1,2 \textit{ weekly in } W_2^{2,1,1} \textit{ at } k \to \infty. \tag{33}$$

From these limit relations we can derive the following limit relations:

$$\psi_{pk} \to \psi_p,\ p=1,2 \textit{ weekly in } L_2(\Omega), \tag{34}$$

$$\frac{\partial \psi_{pk}}{\partial x} \to \frac{\partial \psi_p}{\partial x},\ p=1,2 \textit{ weekly in } L_2(\Omega), \tag{35}$$

$$\frac{\partial^2 \psi_{pk}}{\partial x^2} \to \frac{\partial^2 \psi_p}{\partial x^2},\ p=1,2 \textit{ weekly in } L_2(\Omega), \tag{36}$$

$$\frac{\partial \psi_{pk}}{\partial t} \to \frac{\partial \psi_p}{\partial t},\ p=1,2 \textit{ weekly in } L_2(\Omega), \tag{37}$$

$$\frac{\partial \psi_{pk}}{\partial z} \to \frac{\partial \psi_p}{\partial z},\ p=1,2 \textit{ weekly in } L_2(\Omega) \tag{38}$$

at $k \to \infty$.

Since the elements of subsequences $\{\psi_{pk}(x,t,z)\}, p=1,2$ are almost everywhere a solution to the reduced problem (2)-(6) from the space $W_2^{2,1,1}(\Omega)$, i.e.

$$\psi_{1k} \in \overset{0}{W}{}_2^{2,1,1}(\Omega), \psi_{2k} \in W_2^{2,1,1}(\Omega), k=1,2,....$$

Therefore, the elements of subsequences $\{\psi_{pk}(x,t,z)\}, p=1,2$ will satisfy the integral identities

$$\int_\Omega \Big(i\frac{\partial \psi_{pk}}{\partial t} + ia_0\frac{\partial \psi_{pk}}{\partial z} - a_1\frac{\partial^2 \psi_{pk}}{\partial x^2} + ia_2(x)\frac{\partial \psi_{pk}}{\partial x} + a(x)\psi_{pk} +$$

$$+ v_0^k(x)\psi_{pk} + iv_1^k(x)\psi_{pk} - f(x,t,z)\,\bar{\eta}_p(x,t,z)\Big)\, dxdtdz = 0, p=1,2, k=1,2,... \tag{39}$$

for any functions $\eta_p \in L_2(\Omega), p=1,2$ and the following initial

$$\psi_{pk}(x,0,z) = \varphi_{0p}(x,z),\ p=1,2, \overset{0}{\forall}(x,z) \in \Omega_L, k=1,2,..., \tag{40}$$

$$\psi_{pk}(x,t,0) = \varphi_{1p}(x,t),\ p=1,2, \overset{0}{\forall}(x,t) \in \Omega_T, k=1,2,..., \tag{41}$$

and boundary conditions

$$\psi_{1k}(0,t,z)=\psi_{1k}(l,t,z)=0,\overset{0}{\forall}(t,z)\in Q,k=1,2,..., \tag{42}$$

$$\frac{\partial\psi_{2k}(0,t,z)}{\partial x}=\frac{\partial\psi_{2k}(l,t,z)}{\partial x}=0,\overset{0}{\forall}(t,z)\in Q,k=1,2,.... \tag{43}$$

Let us now show that the limit functions $\psi_p=\psi_p(x,t,z),p=1,2$ are a solution to the reduced problem (2)-(6). For this purpose, we first show that the limit functions $\psi_p(x,t,z),p=1,2$ from $W_2^{2,1,1}(\Omega)$ satisfy (2) for $\overset{0}{\forall}(x,t,z)\in\Omega$. By virtue of the limit relations (34)-(38), we can write the following limit relations

$$\int_{\Omega}\left(i\frac{\partial\psi_{pk}}{\partial t}+ia_0\frac{\partial\psi_{pk}}{\partial z}-a_1\frac{\partial^2\psi_{pk}}{\partial x^2}+ia_2\frac{\partial\psi_{pk}}{\partial x}+a(x)\psi_{pk}\right)\bar{\eta}_p dxdtdz\to$$

$$\to\int_{\Omega}\left(i\frac{\partial\psi_{p}}{\partial t}+ia_0\frac{\partial\psi_{p}}{\partial z}-a_1\frac{\partial^2\psi_{p}}{\partial x^2}+ia_2\frac{\partial\psi_{p}}{\partial x}+a(x)\psi_{p}\right)\bar{\eta}_p dxdtdz,p=1,2 \tag{44}$$

for any functions $\eta_p\in L_2(\Omega),p=1,2$ at $k\to\infty$. Now let us show that the following limit relations hold

$$\int_{\Omega}v_m^k(x)\psi_{pk}(x,t,z)\bar{\eta}_p(x,t,z)dxdtdz\to$$

$$\to\int_{\Omega}v_m(x)\psi_{p}(x,t,z)\bar{\eta}_p(x,t,z)dxdtdz,m=0,1,p=1,2 \tag{45}$$

for any functions $\eta_p\in L_2(\Omega),p=1,2$ at $k\to\infty$. It is clear that the following equalities hold

$$\int_{\Omega}v_m^k(x)\psi_{pk}(x,t,z)\bar{\eta}_p(x,t,z)dxdtdz=$$

$$=\int_{\Omega}\left(v_m^k(x)-v_m(x)\right)\psi_{p}(x,t,z)\bar{\eta}_p(x,t,z)dxdtdz+$$

$$+\int_{\Omega}v_m^k(x)\left(\psi_{pk}(x,t,z)-\psi_p(x,t,z)\right)\bar{\eta}_p(x,t,z)dxdtdz+$$

$$+\int_{\Omega}v_m(x)\psi_{p}(x,t,z)\bar{\eta}_p(x,t,z)dxdtdz,m=0,1,p=1,2,k=1,2,... \tag{46}$$

for any functions $\eta_p\in L_2(\Omega),p=1,2$. Since $\psi_p\in L_2(\Omega),p=1,2$ and $\eta_p\in L_2(\Omega),p=1,2$ we can state that the functions $q_p(x)=\int_Q\psi_p(x,t,z)\bar{\eta}_p(x,t,z)dtdz$ belong to the space $L_1(0,l)$. Therefore, using limit relations of the form (32), we establish the validity of the fact that the first term on the right-hand side of equalities (46) tends to zero at $k\to\infty$ i.e. the following limit relations hold

$$\int_{\Omega}\left(v_m^k(x)-v_m(x)\right)\psi_{p}(x,t,z)\bar{\eta}_p(x,t,z)dxdtdz\to 0,m=0,1,p=1,2 \tag{47}$$

for any functions $\eta_p \in L_2(\Omega), p = 1,2$ at $k \to \infty$.

Now let's evaluate the second term on the right-hand side of equalities (46). Applying the Cauchy-Bunyakovsky inequality, we get

$$\left| \int_\Omega v_m^k(x)\left(\psi_{pk}(x,t,z) - \psi_p(x,t,z)\right)\bar{\eta}_p(x,t,z)\,dxdtdz \right| \leq$$

$$\leq \int_\Omega \left|v_m^k(x)\right| \left|\psi_{pk}(x,t,z) - \psi_p(x,t,z)\right| \left|\eta_p(x,t,z)\right| dxdtdz \leq$$

$$\leq b_m \|\eta_p\|_{L_2(\Omega)} \|\psi_{pk} - \psi_p\|_{L_2(\Omega)}, m = 0,1, p = 1,2, k = 1,2, ... \quad (48)$$

Due to the compact embedding of space $W_2^{2,1,1}(\Omega)$ into the space $L_2(\Omega)$, we can write the following limit relations

$$\psi_{pk} \to \psi_p,\ p = 1,2 \textit{ strongly in } L_2(\Omega) \textit{ at } k \to \infty. \quad (49)$$

By virtue of these limit relations, passing to the limit in both sides of inequalities (48) for, we obtain the following limit relations

$$\int_\Omega v_m^k(x)\left(\psi_{pk}(x,t,z) - \psi_p(x,t,z)\right)\bar{\eta}_p(x,t,z)\,dxdtdz \to 0, m = 0,1, p = 1,2 \quad (50)$$

for any functions $\eta_p \in L_2(\Omega), p = 1,2$ at $k \to \infty$. Thus, using the limit relations (47), (49), if we pass to the limit in both parts of (46), then for $k \to \infty$ we obtain the limit relations (45).

Now, using the limit relations (44), (45), if we pass to the limit in both parts of the integral identity (39), then for $k \to \infty$ we obtain the following integral identities

$$\int_\Omega \left(i\frac{\partial \psi_p}{\partial t} + ia_0\frac{\partial \psi_p}{\partial z} - a_1\frac{\partial^2 \psi_p}{\partial x^2} + ia_2(x)\frac{\partial \psi_p}{\partial x} + a(x)\psi_p + \right.$$

$$\left. + v_0(x)\psi_p + iv_1(x)\psi_p - f(x,t,z)\right)\bar{\eta}_p(x,t,z)\,dxdtdz = 0, p = 1,2 \quad (51)$$

for any functions $\eta_p \in L_2(\Omega), p = 1,2$. From this we obtain that the limit functions $\psi_p(x,t,z), p = 1,2$ from $W_2^{2,1,1}(\Omega)$ satisfy equations (2) for $\overset{0}{\forall}(x,t,z) \in \Omega$.

Now we show that the limit function $\psi_p(x,t,z), p = 1,2$ satisfies conditions (3) and (4). It is evident that the elements $\{\psi_{1k}\} \in \overset{0}{W}{}_2^{2,1,1}(\Omega)$ and $\{\psi_{2k}\} \in W_2^{2,1,1}(\Omega)$ satisfy the following relations

$$\psi_{1k} \in L_2\left(0,T;\ \overset{0}{W}{}_2^{2,1}(\Omega_L)\right), \frac{\partial \psi_{1k}}{\partial t} \in L_2\left(0,T;\ L_2(\Omega_L)\right), \quad (52)$$

$$\psi_{2k} \in L_2\left(0,T;\ W_2^{2,1}(\Omega_L)\right), \frac{\partial \psi_{2k}}{\partial t} \in L_2\left(0,T;\ L_2(\Omega_L)\right) \quad (53)$$

and there are limiting relations

$$\psi_{pk} \to \psi_p,\ p = 1,2 \textit{ weekly in } L_2\left(0,T;\ W_2^{2,1}(\Omega_L)\right), \quad (54)$$

$$\frac{\partial \psi_{pk}}{\partial t} \to \frac{\partial \psi_p}{\partial t},\ p = 1,2 \ weekly\ in\ L_2\left(0, T;\ L_2\left(\Omega_L\right)\right) \tag{55}$$

at $k \to \infty$. From these relations and from the embedding theorem (see [18, p. 33]) we establish that

$$\left\|\psi_{pk}\left(\cdot, t, \cdot\right) - \psi_p\left(\cdot, t, \cdot\right)\right\|_{L_2(\Omega_L)} \to 0,\ p = 1,2\ at\ k \to \infty,\ \forall t \in [0, T]\,. \tag{56}$$

Similarly, we establish that

$$\left\|\psi_{pk}\left(\cdot, \cdot, z\right) - \psi_p\left(\cdot, \cdot, z\right)\right\|_{L_2(\Omega_T)} \to 0,\ p = 1,2\ at\ k \to \infty,\ \forall z \in [0, L]\,. \tag{57}$$

Using the limit relations (56), (57) and conditions (40), (41) with passing to the limit by $k \to \infty$ in the inequalities

$$\left\|\psi_p\left(\cdot, 0, \cdot\right) - \varphi_0\right\|_{L_2(\Omega_L)} \leq \left\|\psi_p\left(\cdot, 0, \cdot\right) - \psi_{pk}\left(\cdot, 0, \cdot\right)\right\|_{L_2(\Omega_L)} + \left\|\psi_{pk}\left(\cdot, 0, \cdot\right) - \varphi_0\right\|_{L_2(\Omega_L)}, p = 1,2,$$

$$\left\|\psi_p\left(\cdot, \cdot, 0\right) - \varphi_1\right\|_{L_2(\Omega_T)} \leq \left\|\psi_p\left(\cdot, \cdot, 0\right) - \psi_{pk}\left(\cdot, \cdot, 0\right)\right\|_{L_2(\Omega_T)} + \left\|\psi_{pk}\left(\cdot, \cdot, 0\right) - \varphi_1\right\|_{L_2(\Omega_T)}, p = 1,2$$

we obtain the validity of the relations

$$\left\|\psi_p\left(\cdot, 0, \cdot\right) - \varphi_0\right\|_{L_2(\Omega_L)} = 0, \left\|\psi_p\left(\cdot, \cdot, 0\right) - \varphi_1\right\|_{L_2(\Omega_T)} = 0, p = 1,2.$$

It follows that the limit the functions $\psi_p\left(x, t, z\right), p = 1,2$ satisfy conditions (3) and (4) for almost all $(x, z) \in \Omega_L$, $(x, t) \in \Omega_T$, respectively.

Finally, we show that the limit functions $\psi_p\left(x, t, z\right), p = 1,2$ satisfy boundary conditions (5) and (6), respectively, for almost all $(t, z) \in Q$. From the compactness of the embedding of the space $\overset{0}{W}{}_2^{2,1,1}(\Omega)$ into the space $L_2\left(Q; C\left[0, l\right]\right)$ we obtain

$$\left\|\psi_{1k}\left(s, ., .\right) - \psi_1\left(s, ., .\right)\right\|_{L_2(Q)} \to 0,\ s = 0, l\ at\ k \to \infty. \tag{58}$$

Using this and the boundary conditions (42) from the inequalities

$$\left\|\psi_1\left(s, ., .\right)\right\|_{L_2(Q)} \leq \left\|\psi_1\left(s, ., .\right) - \psi_{1k}\left(s, ., .\right)\right\|_{L_2(Q)} + \left\|\psi_{1k}\left(s, ., .\right)\right\|_{L_2(Q)}, s = 0, l$$

passing to the limit, we find that the limit function $\psi_1\left(x, t, z\right)$ satisfies the following relations

$$\left\|\psi_1\left(s, ., .\right)\right\|_{L_2(Q)}, s = 0, l.$$

It follows immediately from these relations that the limit function $\psi_1\left(x, t, z\right)$ satisfies the boundary conditions (5) for almost all $(t, z) \in Q$.

Now we show that the limit function $\psi_2\left(x, t, z\right)$ satisfies boundary conditions (6) for almost all $(t, z) \in Q$. Due to the fact that elements of a subsequence $\{\psi_{2k}\left(x, t, z\right)\}$ from space $W_2^{2,1,1}(\Omega)$ have a generalized derivative $\frac{\partial \psi_{2k}}{\partial x},\ k = 1, 2, ...$ from the space $L_2\left(Q; W_2^1\left(0, l\right)\right)$ i.e. from the space $W_2^{1,0,0}(\Omega)$.

It is not difficult to establish that this space is boundedly limitedly embedded into the space $C^0([0,l], L_2(Q))$. Then it is clear that for the subsequences $\left\{\frac{\partial\psi_{2k}(0,t,z)}{\partial x}\right\}, \left\{\frac{\partial\psi_{2k}(l,t,z)}{\partial x}\right\}$ the following limiting relations hold

$$\frac{\partial\psi_{2k}(s,.,.)}{\partial x} \to \frac{\partial\psi_2(s,.,.)}{\partial x},\ s=0,l \ weekly\ in\ L_2(Q) \tag{59}$$

at $k\to\infty$. On the other hand, one can write the following equalities

$$\int_Q \frac{\partial\psi_2(s,t,z)}{\partial x}\bar{\eta}(t,z)\,dtdz = \int_Q\left(\frac{\partial\psi_2(s,t,z)}{\partial x} - \frac{\partial\psi_{2k}(s,t,z)}{\partial x}\right)\bar{\eta}(t,z)\,dtdz+$$

$$+\int_Q \frac{\partial\psi_{2k}(s,t,z)}{\partial x}\bar{\eta}(t,z)\,dtdz,\ s=0,l,\ k=1,2,... \tag{60}$$

for any function $\eta\in L_2(Q)$. Using the boundary conditions (43), we can assert that the second terms on the right-hand sides of equalities (60) vanish. Taking this and the limit relations (59) into account, if we pass to the limit in equalities (60), then for $k\to\infty$ we obtain the following relations

$$\int_Q \frac{\partial\psi_2(s,t,z)}{\partial x}\bar{\eta}(t,z)\,dtdz = 0,\ s=0,l.$$

for any function $\eta\in L_2(Q)$. From these relations we obtain the validity of the following second boundary conditions

$$\frac{\partial\psi_2(0,t,z)}{\partial x} = \frac{\partial\psi_2(l,t,z)}{\partial x} = 0, \overset{0}{\forall}(t,z)\in Q.$$

Thus we proved that the limit functions $\psi_1 \in \overset{0}{W}{}_2^{2,1,1}(\Omega), \psi_2\in W_2^{2,1,1}(\Omega)$ are solutions for reduced problem (2)-(6) corresponding to the limit function $v=v(x)$ from V of the subsequence $\{v^k\}\subset V$ i.e. $\psi_p=\psi_p(x,t,z)\equiv\psi_p(x,t,z;v), p=1,2$. Moreover, this solution satisfies estimates (13), (14), which follow from estimates (29), (30) with passing to the lower limit along weakly converging to the functions $\psi_p(x,t,z), p=1,2$ subsequences $\{\psi_{pk}(x,t,z)\}, p=1,2$. By virtue of the limit relations (31) and (34) and the weak lower semicontinuity of the norms of the spaces $L_2(\Omega)$ and H, as well as the conditions $\alpha\geq 0$ for $\forall\omega\in H$ we obtain the relation

$$J_{\alpha *}\leq J_\alpha(v)\leq \underline{\lim}_{k\to\infty} J_\alpha(v_k) = \inf_{v\in V} J_\alpha(v) = J_{\alpha *}.$$

This means that the limit function $v=v(x)$ from of the subsequence from V gives minimum to the functional $J_\alpha(v)$ on the set V i.e. $v\in V$ is a solution to the optimal control problem (1)-(6). Theorem 3.3 is proved.

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