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Dependence on the parameters of the solution of a mixed problem for a nonlinear integro-differential equation of the fifth order with a degenerate kernel

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Abstract. In this paper, it is considered a fifth order nonlinear partial integro-differential equations with mixed conditions and two real parameters. The Fourier spectral method of separation of variables is applied. A countable system of nonlinear functional-integral equations is derived. Theorem on a uniqueness and existence of the solution of mixed problem is proved for regular values of parameters. The method of compressing mapping in Banach space is applied. The solution of the mixed problem is obtained in the form of Fourier series. Theorem on absolute and uniform convergence of Fourier series is proved. Continuous dependence on parameters of the classical solution of mixed problem is studied.

Key Words and Phrases: Mixed problem, fifth order differential equation, unique solvability, real parameters, regular values, dependence on the parameters.

2010 Mathematics Subject Classifications: Primary 34B08, 34B10; Secondary 35G05.

1. Formulation of the problem statement

Differential equations of parabolic and hyperbolic types are the base of the equations of mathematical physics. Along with equations of parabolic and hyperbolic types, so-called pseudoparabolic and pseudohyperbolic differential equations are often studied.

Let us consider differential equations of the form

$$\left[\frac{\partial^k}{\partial t^k} + (-1)^m \frac{\partial^{k+2m}}{\partial t^k \partial x^{2m}} + (-1)^m \omega \frac{\partial^{2m}}{\partial x^{2m}}\right] U(t,x) = f(t,x), \ \ k = 1,2,3, \ \ m = 1,2,3,...,n.$$

This equation is sometimes called a Barenblatt–Zheltev–Kochina equation at $k = 1$. And when $k = 2$, it is often called a Boussinesq type differential equation. Many works have been devoted to the study of this equation for $k = 1, 2$ (see, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51]). However, we have not yet encountered a single work on the study of an equation in the case of $k = 3$.

Integro-differential equations are studied in the works of many mathematicians (see, for examples [52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78]). To study integro-differential equations with degenerate kernel are devoted only the few works [79, 80, 81, 82, 83, 84, 85, 86, 87, 88].

In this paper we consider the case $k = 3$. For simplicity, we take $m = 1$. So, we study the solvability of the mixed problem for a fifth order integro-differential equation with two real parameter and degenerate kernel. So, in the rectangular domain, $\Omega = \{0 < t < T,\ 0 < x < 1\}$ we consider the following partial integro-differential equation

$$\left[\frac{\partial^3}{\partial t^3} - \frac{\partial^5}{\partial t^3 \partial x^2} - \omega \frac{\partial^2}{\partial x^2}\right] V(t,x) =$$

$$= \nu \int_0^T K(t,s)\, V(s,x) ds + F\left(t, x, \int_0^T \int_0^1 G(s,y)\, V(s,y) dy ds\right), \tag{1}$$

where $K(t,s) = \sum_{i=1}^{p} \alpha_i(t)\ \beta_i(s),\ 0 < \alpha_i(t),\ \beta_i(s) \in C[0,T],\ \alpha_i(t)$ and $\beta_i(s)$ are linear independent, $F(t,x,y) \in C^{0,2,0}_{t,x,u}(\bar{\Omega} \times R),\ 0 < G(t,x) \in C(\bar{\Omega}),\ T$ is given positive number, ω is positive finite parameter, ν is nonzero real parameter, $\bar{\Omega} = \{0 \le t \le T,\ 0 \le x \le 1\}$.

In solving partial integro-differential equation (1), we use the following spectral and initial value conditions

$$V(t,0) = V(t,1) = 0, \quad 0 \le t \le T, \tag{2}$$

$$V(0,x) = \varphi_1(x), \quad V_t(0,x) = \varphi_2(x), \quad V_{tt}(0,x) = \varphi_3(x), \quad 0 \le x \le 1, \tag{3}$$

where $\varphi_k(x)$ are enough smooth functions on the segment $[0,1]$. For these functions the following conditions are fulfilled $\varphi_k(0) = \varphi_k(1) = 0,\ \ k = 1,2,3$.

Problem statement. To find a function

$$V(t,x) \in C(\overline{\Omega}) \cap C^{3,2}_{t,x}(\Omega), \tag{4}$$

which satisfies integro-differential equation (1) and conditions (2) and (3).

2. Countable system of nonlinear equations

The solution of the problem (1)–(3) we search in the form of Fourier series

$$V(t,x) = \sum_{n=1}^{\infty} a_n(t)\, b_n(x), \tag{5}$$

where $b_n(x) = \sqrt{2} \sin \sqrt{\lambda_n} x$ are eigenfunctions of the spectral problem $b''(x) + \lambda b(x) = 0,\ b(0) = b(1) = 0$, corresponding to the eigenvalues $\lambda_n = (n\,\pi)^2$,

$$a_n(t) = \int_0^1 V(t,x)\, b_n(x)\, dx.$$

So, we assume that

$$F(t,x,y)=\sum_{n=1}^{\infty}F_n(t,\cdot)\,b_n(x), \tag{6}$$

where

$$F_n(t,\cdot)=\int_0^1 F(t,y,\cdot)\,b_n(y)\,dy.$$

Substituting the Fourier series (5) and (6) into the given integro-differential equation (1), and taking into account that the functions $\left\{b_n(x)\right\}_{n=1}^{\infty}$ form a complete system of orthonormal systems, we obtain a countable system of third order ordinary differential equations

$$a_n'''(t)+\mu_n(\omega)a_n(t)=f_n(t),\quad \mu_n(\omega)=\frac{\lambda_n}{1+\lambda_n}\omega,\quad \lambda_n=(n\,\pi)^2, \tag{7}$$

where

$$f_n(t)=\frac{1}{1+\lambda_n}\left[\nu\sum_{i=1}^{p}\alpha_i(t)\tau_{n,i}+F_n(t,\cdot)\right], \tag{8}$$

$$\tau_{n,i}=\int_0^T\beta_i(s)a_n(s)ds. \tag{9}$$

The characteristic equation $\sigma^3+\mu_n(\omega)=0$ for the homogeneous equation $a_n'''(t)+\mu_n(\omega)a_n(t)=0$ has the roots

$$\sigma_1=-\sqrt[3]{\mu_n(\omega)},\quad \sigma_{2/3}=\left(\frac{1}{2}\pm\frac{\sqrt{3}}{2}i\right)\sqrt[3]{\mu_n(\omega)}.$$

So, the general solution of the homogeneous equation can be presented as

$$a_n(t)=A_{1,n}a_{1,n}(t)+A_{2,n}a_{2,n}(t)+A_{3,n}a_{3,n}(t), \tag{10}$$

where $A_{k,n}$ $(k=1,2,3)$ are yet arbitrary coefficients, which will be determined later,

$$\left[\begin{array}{l} a_{1,n}(t)=e^{-\sqrt[3]{\mu_n(\omega)}t},\\ a_{2,n}(t)=e^{\frac{\sqrt[3]{\mu_n(\omega)}t}{2}}\cos\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega)}t,\\ a_{3,n}(t)=e^{\frac{\sqrt[3]{\mu_n(\omega)}t}{2}}\sin\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega)}t. \end{array}\right. \tag{11}$$

Taking (10) into account, we search a particular solution of the equation (7) as

$$\tilde{a}_n(t)=A_{1,n}(t)a_{1,n}(t)+A_{2,n}(t)a_{2,n}(t)+A_{3,n}(t)\,a_{3,n}(t). \tag{12}$$

In (12) we supposed that $A_{k,n}(t)$ are unknown functions. To find these functions we consider the following system of algebraic-differential equations

$$\begin{cases} A'_{1,n}(t)a_{1,n}(t) + A'_{2,n}(t)a_{2,n}(t) + A'_{3,n}(t)a_{3,n}(t) = 0, \\ A'_{1,n}(t)a'_{1,n}(t) + A'_{2,n}(t)a'_{2,n}(t) + A'_{3,n}(t)a'_{3,n}(t) = 0, \\ A'_{1,n}(t)a''_{1,n}(t) + A'_{2,n}(t)a''_{2,n}(t) + A'_{3,n}(t)a''_{3,n}(t) = f_n(t). \end{cases}$$

We solve the system as functional-algebraic equations by the Cramer rule and found:

$$A_{1,n}(t) = \frac{1}{12c_n^2} \int\limits_0^t \frac{1}{a_{1,n}(s)} f_n(s)ds, \tag{13}$$

$$A_{2,n}(t) = -\frac{1}{12c_n^2} \int\limits_0^t \left[a_{1,n}(s)a_{2,n} + \sqrt{3}a_{1,n}(s)a_{3,n}(s)\right] f_n(s)ds, \tag{14}$$

$$A_{3,n}(t) = \frac{1}{12c_n^2} \int\limits_0^t \left[\sqrt{3}a_{1,n}(s)a_{2,n}(s) - a_{1,n}(s)a_{3,n}(s)\right] f_n\left(s\right) ds, \tag{15}$$

where $2c_n = \sqrt[3]{\mu_n(\omega)}$. Substituting (13)–(15) into (12) and taking into account (11), we obtain a particular solution of the equation (7) as

$$\tilde{a}_n(t,\omega) = \frac{1}{\sqrt[3]{\mu_n^2(\omega)}} \int\limits_0^t Q(t,s,\omega) f_n(s)ds, \tag{16}$$

where

$$Q(t,s,\omega) = \frac{1}{3}\left\{e^{-\sqrt[3]{\mu_n(\omega)}(t-s)} - 2e^{\frac{\sqrt[3]{\mu_n(\omega)}}{2}(t-s)} \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega)}(s-t) + \frac{\pi}{6}\right)\right\}.$$

From (16) we derive a particular solution of the equation (1) in the form of Fourier series

$$\tilde{V}(t,x,\omega) = \sum_{n=1}^{\infty} \frac{b_n(x)}{\sqrt[3]{\mu_n^2(\omega)}} \int\limits_0^t Q(t,s,\omega) f_n(s)ds. \tag{17}$$

Taking into account (16) the general solution of the equation (7) can be presented as:

$$a_n(t) = B_{1,n}a_{1,n}(t) + B_{2,n}a_{2,n}(t) + B_{3,n}a_{3,n}(t) +$$

$$+ A_{1,n}(t)a_{1,n}(t) + A_{2,n}(t)a_{2,n}(t) + A_{3,n}(t)a_{3,n}(t) \tag{18}$$

and we find the constants $B_{k,n}$ $(k = 1,2,3)$.

We suppose that the functions $\varphi_k(x)$ are expanding into a Fourier series and using the Fourier coefficients, from initial value conditions (3) we obtain

$$a_n(0) = \int_0^1 V(0,y)\, b_n(y)\, dy = \int_0^1 \varphi_1(y)\, b_n(y)\, dy = \varphi_{1,n}, \tag{19}$$

$$a'_n(0) = \int_0^1 V_t(0,y)\, b_n(y)\, dy = \int_0^1 \varphi_2(y)\, b_n(y)\, dy = \varphi_{2,n}, \tag{20}$$

$$a''_n(0) = \int_0^1 V_{tt}(0,y)\, b_n(y)\, dy = \int_0^1 \varphi_3(y)\, b_n(y)\, dy = \varphi_{3,n}. \tag{21}$$

To find the unknown (arbitrary) coefficients $A_{k,n}$ $(k = 1,2,3)$, we use the boundary conditions (19)–(21) in the presentation (18) and obtain

$$\begin{cases} B_{1,n}a_{1,n} = \frac{1}{3}a_{1,n}\varphi_{1,n} - \frac{1}{6b_n}a_{1,n}\varphi_{2,n} + \frac{1}{12b_n^2}a_{1,n}\varphi_{3,n}, \\ B_{2,n}a_{2,n} = \frac{2}{3}a_{2,n}\varphi_{1,n} + \frac{1}{6b_n}a_{2,n}\varphi_{2,n} - \frac{1}{12b_n^2}a_{2,n}\varphi_{3,n}, \\ B_{3,n}a_{3,n} = \frac{1}{2\sqrt{3}c_n}a_{3,n}\varphi_{2,n} + \frac{1}{4\sqrt{3}c_n^2}a_{3,n}\varphi_{3,n}. \end{cases}$$

Hence, we obtain that

$$B_{1,n}a_{1,n} + B_{2,n}a_{2,n} + B_{3,n}a_{3,n} =$$

$$= \frac{a_{1,n} + 2a_{2,n}}{3}\varphi_{1,n} + \frac{-a_{1,n} + a_{2,n} + \sqrt{3}a_{3,n}}{6c_n}\varphi_{2,n} + \frac{a_{1,n} - c_{2,n} + \sqrt{3}c_{3,n}}{12c_n^2}\varphi_{3,n}. \tag{22}$$

Substituting (22) into (18) and taking into account (8), (11), (16), we obtain

$$a_n(t,\omega) = P_n(t,\omega) +$$

$$+ \frac{1}{\sqrt[3]{\left(\lambda_n^2 + \lambda_n^3\right)\omega}} \left[\nu \sum_{i=1}^{p} \tau_{n,i} \int_0^t Q_n(t,s,\omega)\, \alpha_i(s)\, ds + \int_0^t Q_n(t,s,\omega)\, F_n(s,\cdot)\, ds \right], \tag{23}$$

where

$$P_n(t,\omega) = \varphi_{1,n}\psi_{1,n}(t,\omega) + \frac{1}{\sqrt[3]{\mu_n(\omega)}}\varphi_{2,n}\psi_{2,n}(t,\omega) + \frac{1}{\sqrt[3]{\mu_n^2(\omega)}}\varphi_{3,n}\psi_{3,n}(t,\omega), \tag{24}$$

$$Q_n(t,s,\omega) = \frac{1}{3}\left[e^{-\sqrt[3]{\mu_n(\omega)}(t-s)} + 2e^{\frac{\sqrt[3]{\mu_n(\omega)}}{2}(t-s)} \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega)}(t-s) + \frac{\pi}{6} \right) \right], \tag{25}$$

$$\psi_{1,n}(t,\omega) = \frac{1}{3}\left[e^{-\sqrt[3]{\mu_n(\omega)}t} + 2e^{\frac{\sqrt[3]{\mu_n(\omega)}t}{2}} \cos\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega)}t \right], \tag{26}$$

$$\psi_{2,n}(t,\omega)=\frac{1}{3}\left[e^{-\sqrt[3]{\mu_n(\omega)}t}-2e^{\frac{\sqrt[3]{\mu_n(\omega)}}{2}t}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega)}t+\frac{\pi}{6}\right)\right], \tag{27}$$

$$\psi_{3,n}(t,\omega)=\frac{1}{3}\left[e^{-\sqrt[3]{\mu_n(\omega)}t}-2e^{\frac{\sqrt[3]{\mu_n(\omega)}}{2}t}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega)}t-\frac{\pi}{6}\right)\right], \tag{28}$$

$$F_n(t,\cdot)=\int\limits_0^1 F\left(t,y,\int\limits_0^T\int\limits_0^1 G(s,z)V(s,z)dzds\right)b_n(y)\,dy.$$

There is another unknown quantity in (23). To find it we substitute (23) into (9), and obtain a countable system of algebraic system of equations (CSASE)

$$\tau_{n,i}=\nu\sum_{j=1}^{p}\tau_{n,j}\Phi_{n,i,j}(\omega)+\Psi_{n,i}(a_r,\omega),\quad i=1,2,\cdots,p, \tag{29}$$

$$\Phi_{n,i,j}(\omega)=\frac{1}{\sqrt[3]{(\lambda_n^2+\lambda_n^3)\omega}}\int\limits_0^T\beta_i(s)\int\limits_0^s Q_n(s,\theta,\omega)\alpha_j(\theta)\,d\theta ds, \tag{30}$$

$$\Psi_{n,i}(a_r,\omega)=\int\limits_0^T\beta_i(s)P_n(s,\omega)ds+\frac{1}{\sqrt[3]{(\lambda_n^2+\lambda_n^3)\omega}}\int\limits_0^T\beta_i(s)\int\limits_0^s Q_n(s,\theta,\omega)\times$$

$$\times\int\limits_0^1 F\left(\theta,y,\int\limits_0^T\int\limits_0^1 G(\xi,z)\sum_{r=1}^{\infty}a_r(\xi)b_r(z)dzd\xi\right)b_n(y)\,dyd\theta ds. \tag{31}$$

To solve the CSASE we consider the following determinants

$$Z_n(\nu,\omega)=\begin{vmatrix} 1-\nu\,\Phi_{1\,1} & \nu\,\Phi_{1\,2} & \ldots & \nu\,\Phi_{1p}\\ \nu\,\Phi_{2\,1} & 1-\nu\,\Phi_{2\,2} & \ldots & \nu\,\Phi_{2p}\\ \ldots & \ldots & \ldots & \ldots\\ \nu\,\Phi_{p1} & \nu\,\Phi_{p2} & \ldots & 1-\nu\,\Phi_{pp}\end{vmatrix}, \tag{32}$$

$$Z_{in}(a_r,\nu,\omega)=\begin{vmatrix} 1-\nu\,\Phi_{1\,1} & \ldots & \nu\,\Phi_{1\,(i-1)} & \Psi_1 & \nu\,\Phi_{1\,(i+1)} & \ldots & \nu\,\Phi_{1p}\\ \nu\,\Phi_{2\,1} & \ldots & \nu\,\Phi_{2\,(i-1)} & \Psi_2 & \nu\,\Phi_{2\,(i+1)} & \ldots & \nu\,\Phi_{2p}\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ \nu\,\Phi_{p1} & \ldots & \nu\,\Phi_{p(i-1)} & \Psi_p & \nu\,\Phi_{p\,(i+1)} & \ldots & 1-\nu\,\Phi_{pp}\end{vmatrix}, \tag{33}$$

where $\Phi_{ij}=\Phi_{nij}(\omega)$, $\Psi_\kappa=\Psi_{n\kappa}(a_r,\omega)$, $\kappa=\overline{1,p}$.

CSASE (29) is uniquely solvable for any finite right-hand sides, if the following non-degeneracy condition for the Fredholm determinant is satisfied $Z_n(\nu,\omega)\neq 0$. The determinant (32) $Z_n(\nu,\omega)$ is a polynomial with respect to ν of degree not higher p. The equation $Z_n(\nu,\omega)=0$ has at most p different real roots. We denote them by $\theta_\ell\,(\ell=\overline{1\,,p_\ell},\ 1\le p_\ell\le p)$. Then $\nu=\nu_{n+\ell}=\theta_\ell$ called the irregular values of the parameter

ν. Other values of the parameter $\nu \neq \theta_\ell$, for which $|\, Z_n(\nu,\omega)\,| > 0$ and $\sum\limits_{n=1}^{\infty} \frac{1}{[Z_n(\nu,\omega)]^2} < \infty$, are called regular.

For regular values of the parameter ν the solution of the CSASE (29) has the form

$$\tau_{n\kappa}(\nu,\omega) = \frac{Z_{\kappa n}(a_r,\nu,\omega)}{Z_n(\nu,\omega)}, \quad \kappa = \overline{1,\, p}, \tag{34}$$

where $Z_{\kappa n}(\nu,\omega)$ is defined from (33). Substituting for regular values of the parameter ν the presentation of solution (34) of the CSASE (29) into representation (23) of the Fourier coefficients $a_n(t)$ of unknown function $V(t,x,\nu,\omega)$, we derive a nonlinear countable system of functional integral equations (NCSFIE)

$$a_n(t,\nu,\omega) = J(t;a_n) \equiv P_n(t,\nu,\omega)+$$

$$+\frac{\nu}{\sqrt[3]{(\lambda_n^2+\lambda_n^3)\omega}} \sum_{\kappa=1}^{p} \frac{Z_{\kappa n}(a_r,\nu,\omega)}{Z_n(\nu,\omega)} \int\limits_0^t Q_n(t,s,\omega)\alpha_j(s)\,ds + \frac{1}{\sqrt[3]{(\lambda_n^2+\lambda_n^3)\omega}}\times$$

$$\times \int\limits_0^t Q_n(t,s,\omega) \int\limits_0^1 F\left(s,y,\int\limits_0^T\int\limits_0^1 G(\theta,z)\sum_{r=1}^{\infty} a_r(\theta)b_r(z)dzd\theta\right) b_n(y)\,dyds. \tag{35}$$

As, in the case of Fourier series (17), from (35) we obtain a formal solution of the mixed problem (1)–(3)

$$V(t,x,\nu,\omega) = \sum_{n=1}^{\infty} b_n(x)\Big[P_n(t,\omega)+$$

$$+\frac{\nu}{\sqrt[3]{(\lambda_n^2+\lambda_n^3)\omega}} \sum_{\kappa=1}^{p} \frac{Z_{\kappa n}(a_r,\nu,\omega)}{Z_n(\nu,\omega)} \int\limits_0^t Q_n(t,s,\omega)\alpha_j(s)\,ds + \frac{1}{\sqrt[3]{(\lambda_n^2+\lambda_n^3)\omega}}\times$$

$$\times \int\limits_0^t Q_n(t,s,\omega) \int\limits_0^1 F\left(s,y,\int\limits_0^T\int\limits_0^1 G(\theta,z)\sum_{r=1}^{\infty} a_r(\theta)b_r(z)dzd\theta\right) b_n(y)dyds\Big]. \tag{36}$$

We note that the functions in (25)–(28) become zero at some values of parameter ω. We obtain the following transcendental equation

$$\sin\left(\frac{\sqrt{3}}{2}y+\frac{\pi}{6}\right) = -\frac{1}{2}e^{\frac{-3}{2}y}, \quad y = \sqrt[3]{\mu_n(\omega)}\,(t-s) > 0,$$

for the case of function (25) and

$$\cos\frac{\sqrt{3}}{2}y = -\frac{1}{2}e^{\frac{-3}{2}y}, \quad y = \sqrt[3]{\mu_n(\omega)}\,t > 0,$$

for the case of function (26), respectively, $\mu_n(\omega) = \frac{\lambda_n}{1+\lambda_n}\omega$. Functions in the formulas (27) and (28) become zero at some values of parameter ω. We replace these equations by the following transcendental equations

$$\sin\left(\frac{\sqrt{3}}{2}y + \frac{\pi}{6}\right) = \frac{1}{2}e^{\frac{-3y}{2}}, \quad y = \sqrt[3]{\mu_n(\omega)} \;\; t > 0,$$

$$\sin\left(\frac{\sqrt{3}}{2}y - \frac{\pi}{6}\right) = \frac{1}{2}e^{\frac{-3y}{2}}, \quad y = \sqrt[3]{\mu_n(\omega)} \;\; t > 0,$$

respectively.

The values of parameter ω, for which the functions (25)–(28) become zero, we denote by Λ_j, $j = 1, 2, 3, 4$, respectively. However, from the fact $\Lambda_1 \cap \Lambda_2 \cap \Lambda_3 \cap \Lambda_4 = \emptyset$ we deduce that the problem (1)–(3) is correct.

3. Solvability of the countable system

To prove the classical solvability of the mixed problem (1)–(3) we require in some properties of the given functions, which we call as smoothness conditions.

Smoothness conditions. Let the functions $\varphi_k(x) \in C^4[0,1]$ $(k = 1, 2, 3)$, $F(t, x, \cdot) \in C^{0,2}_{t,x}(\Omega \times R)$ have continuous derivatives with respect to x up to the fourth and second order, respectively. We integrate by parts

$$\varphi_{k,n} = \int_0^1 \varphi_k(y)\, b_n(y) dy, \quad F_n(t, \cdot) = \int_0^1 F\left(t, y, \int_0^T \int_0^1 G(s,z)V(s,z)dzds\right) b_n(y)\, dy$$

four and second times, respectively. Then we obtain the estimates

$$\varphi_{k,n} \le \left(\frac{1}{\pi}\right)^4 \frac{\left|\varphi^{(IV)}_{k,n}\right|}{n^4}, \quad F_n \le \left(\frac{1}{\pi}\right)^2 \frac{|\, F''_n(t,\cdot)\,|}{n^2},$$

where

$$\varphi^{(IV)}_{k,n} = \int_0^1 \frac{\partial^4 \varphi_k(y)}{\partial y^4} b_n(y)dy, \;\; F''_n(t, \cdot) = \int_0^1 \frac{\partial^2}{\partial y^2}F\left(t, y, \int_0^T \int_0^1 G(s,z)V(s,z)dzds\right) b_n(y)dy.$$

In estimating approximations, we use also Bessel inequalities in the form:

$$\left\|\, \vec{\varphi}^{(IV)}_k \,\right\|_{\ell_2} \le 4 \left\| \frac{\partial^4 \varphi_k(x)}{\partial\, x^4} \right\|_{L_2[0,1]}, \quad \left\|\vec{F''}(t, \cdot)\right\|_{B_2[0,T]} \le 2 \max_{0\le t\le T} \left\| \frac{\partial^4 F(t,x,\cdot)}{\partial\, x^4} \right\|_{L_2[0,1]}.$$

Theorem 3.1. *Let the smoothness conditions be fulfilled and*

1). $\alpha_0 \sum_{\kappa=1}^{p} \left| Z_{\kappa n}(a^0_r, \nu, \omega) \right| \le \delta_0$, $\alpha_0 = \max\limits_{0\le t\le T} \int\limits_0^t \alpha_j(s)ds$, $0 < \delta_0$, $\alpha_0 = \text{const} < \infty$;

2). $\max\limits_{0\le t\le T} \| F(t,x,\cdot) \|_{L_2[0,1]} \le \delta_1, \ 0<\delta_1 = \text{const} < \infty$*;*

3). $| F(t,x,u_1) - F(t,x,u_2) | \le l(x) \, | u_1 - u_2 |, \ 0 < l(x) \in L_2[0,1]$*;*

4). $\rho = M_3 \int\limits_0^T \| G(t,x) \|_{L_2[0,1]} \, dt < 1$*, where* M_3 *determines from* (41) *below.*

Then for regular values of the parameter ν *NCSFIE* (35) *has a unique solution in the space* $B_2[0,T]$ *with norm*

$$\| \vec{a}(t) \|_{B_2[0,T]} = \sqrt{\sum_{n=1}^{\infty} \left(\max_{t\in[0,T]} | a_n(t) | \right)^2} < \infty.$$

Proof. We define the successive approximations for NCSFIE (35) as:

$$\begin{cases} a_n^0(t,\nu,\omega) = P_n(t,\nu,\omega), \\ a_n^{m+1}(t,\nu,\omega) = J(t; a_n^m), \ \ m = 1, 2, 3, \dots \end{cases} \tag{37}$$

We estimate the zero approximation. By virtue of formulas (24), (26)–(28), and

$$\mu_n = \frac{\lambda_n}{1+\lambda_n} < 1, \lim_{\lambda_n\to\infty} \frac{1}{\sqrt[3]{\mu_n^2(\omega)}} = \lim_{\lambda_n\to\infty} \sqrt[3]{\left(\frac{1+\lambda_n}{\lambda_n}\right)^2 \frac{1}{\omega^2}} = e^{\frac{2}{3}} \omega^{-\frac{2}{3}}, \ \ \omega>0,$$

we can put

$$\max\left\{ \max_t | Q_n(t,s,\omega) | ; \max_{k=1,2,3} \max_t | \psi_{k,n}(t,\omega) | \right\} \le M_0 < \infty, \quad 0<M_0 = \text{const} < \infty.$$

By virtue of smoothness conditions, applying the Cauchy–Shwartz inequality and Bessel inequality, from approximations (37) we have

$$\| \vec{a}^0(t,\nu,\omega) \|_{B_2[0,T]} \le \sqrt{\sum_{n=1}^{\infty} \left[\max_{0\le t\le T} | a_n^0(t,\nu,\omega) | \right]^2} \le$$

$$\le \sum_{n=1}^{\infty} \max_{0\le t\le T} | P_n(t,\omega) | \le M_0 \left[\sum_{n=1}^{\infty} | \varphi_{1,n} | + \sqrt[3]{\frac{e}{\omega}} \sum_{n=1}^{\infty} | \varphi_{2,n} | + \sqrt[3]{\left(\frac{e}{\omega}\right)^2} \sum_{n=1}^{\infty} | \varphi_{3,n} | \right] \le$$

$$\le C_0 M_0 \left(\frac{1}{\pi}\right)^4 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^8}} \left[\left\| \vec{\varphi}_1^{(IV)} \right\|_{\ell_2} + \left\| \vec{\varphi}_2^{(IV)} \right\|_{\ell_2} + \left\| \vec{\varphi}_3^{(IV)} \right\|_{\ell_2} \right] \le$$

$$\le M_1 \left[\left\| \frac{\partial^4 \varphi_1(x)}{\partial x^4} \right\|_{L_2[0,1]} + \left\| \frac{\partial^4 \varphi_2(x)}{\partial x^4} \right\|_{L_2[0,1]} + \left\| \frac{\partial^4 \varphi_3(x)}{\partial x^4} \right\|_{L_2[0,1]} \right] < \infty, \tag{38}$$

where

$$M_1 = C_0 M_0 \left(\frac{\sqrt{2}}{\pi}\right)^4 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^8}}, \quad C_0 = \max\left\{ 1; \sqrt[3]{\frac{e}{\omega}}; \sqrt[3]{\left(\frac{e}{\omega}\right)^2} \right\}.$$

Due to the conditions of the Theorem 3.1, formulas (25), (30), (31), estimate (38) and applying the Cauchy–Shwartz inequality and Bessel inequality, for the first difference $a_n^1(t) - a_n^0(t)$ we obtain

$$\| \vec{a}^1(t,\nu,\omega) - \vec{a}^0(t,\nu,\omega) \|_{B_2[0,T]} \leq$$

$$\leq \frac{|\nu|}{\sqrt[3]{\omega}} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \sum_{\kappa=1}^{p} \left| \frac{Z_{\kappa n}(a_r^0,\nu,\omega)}{Z_n(\nu,\omega)} \right| \max_{0\leq t\leq T} \int_0^t |Q_n(t,s,\omega)| \, \alpha_j(s)\, ds +$$

$$+ \frac{1}{\sqrt[3]{\omega}} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \max_{0\leq t\leq T} \int_0^t |Q_n(t,s,\omega)| \left| \int_0^1 F\left(s, y, \int_0^T \int_0^1 G(\theta,z) \sum_{r=1}^{\infty} a_r^0(\theta,\omega) b_r(z) dz d\theta \right) b_n(y) dy \right| ds \leq$$

$$\leq \frac{M_0}{\sqrt[3]{\omega}} \left(\frac{1}{\pi}\right)^2 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^4}} \left[|\nu| \, \delta_0 \sqrt{\sum_{n=1}^{\infty} \frac{1}{|Z_n(\nu,\omega)|^2}} + T \left\| \int_0^1 F(t,y,\cdot)\, b_n(y)\, dy \right\|_{B_2[0,T]} \right] \leq$$

$$\leq M_2 \left[|\nu| \, \delta_0 \sqrt{\sum_{n=1}^{\infty} \frac{1}{|Z_n(\nu,\omega)|^2}} + T \max_{0\leq t\leq T} \|F(t,x,\cdot)\|_{L_2[0,1]} \right] < \infty, \tag{39}$$

where $M_2 = \frac{M_0}{\sqrt[3]{\omega}} \left(\frac{1}{\pi}\right)^2 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^4}}$.

Now we consider the arbitrary consecutive difference $a_n^{m+1}(t) - a_n^m(t)$. We take into account that the quantities

$$\int_0^1 l(y) b_n(y) dy, \quad \int_0^1 |G(t,z)| \, b_r(z) dz$$

are Fourier coefficients. By the same way as the estimate (39) above, we obtain

$$\left\| \vec{a}^{m+1}(t,\nu,\omega) - \vec{a}^m(t,\nu,\omega) \right\|_{B_2[0,T]} \leq$$

$$\leq \frac{|\nu|}{\sqrt[3]{\omega}} \sum_{n=1}^{\infty} \frac{M_0}{\lambda_n} \sum_{i=1}^{p} \left| \frac{Z_{in}(a_r^m,\nu,\omega) - Z_{in}(a_r^{m-1},\nu,\omega)}{Z_n(\nu,\omega)} \right| \max_{0\leq t\leq T} \int_0^t \alpha_j(s) ds +$$

$$+ M_0 \frac{1}{\sqrt[3]{\omega}} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \max_{0\leq t\leq T} \int_0^t \left| \int_0^1 l(y) \int_0^T \int_0^1 |G(t,z)| \sum_{r=1}^{\infty} \left| a_r^m(t,\omega) - a_r^{m-1}(t,\omega) \right| b_r(z) dz dt \, b_n(y) dy \right| ds \leq$$

$$\leq \frac{|\nu|}{\sqrt[3]{\omega^2}} \sum_{n=1}^{\infty} \frac{\alpha_0 M_0^2}{\lambda_n^2 Z_n(\nu,\omega)} \sum_{\kappa=1}^{p} \bar{\delta}_{0,\kappa} \int_0^T \beta_\kappa(s) ds$$

$$\left| \int_0^1 l(y) b_n(y) dy \int_0^T \int_0^1 |\, G(t,z) \,| \sum_{r=1}^{\infty} \left| \, a_r^m(t,\omega) - a_r^{m-1}(t,\omega) \, \right| b_r(z) dz dt \right| +$$

$$+ M_0 \frac{T}{\sqrt[3]{\omega}} \left| \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \int_0^1 l(y)\, b_n(y)\, dy \int_0^T \int_0^1 |\, G(t,z) \,| \sum_{r=1}^{\infty} \left| \, a_r^m(t,\omega) - a_r^{m-1}(t,\omega) \, \right| b_r(z) dz dt \right| \leq$$

$$\leq \frac{|\,\nu\,|}{\sqrt[3]{\omega^2}} \sum_{n=1}^{\infty} \frac{\alpha_0 \beta_0 M_0^2}{\lambda_n^2 Z_n(\nu,\omega)} \left| \int_0^1 l(y) b_n(y)\, dy \right| \left| \int_0^T \int_0^1 |\, G(t,z) \,| \sum_{r=1}^{\infty} \left| \, a_r^m(t,\omega) - a_r^{m-1}(t,\omega) \, \right| b_r(z) dz dt \right| +$$

$$+ M_0 \frac{T}{\sqrt[3]{\omega}} \left| \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \int_0^1 l(y)\, b_n(y)\, dy \right| \left| \int_0^T \int_0^1 |\, G(t,z) \,| \sum_{r=1}^{\infty} \left| \, a_r^m(t,\omega) - a_r^{m-1}(t,\omega) \, \right| b_r(z) dz dt \right| \leq$$

$$\leq \alpha_0 \beta_0 \delta_2 M_0^2 \frac{|\,\nu\,|}{\sqrt[3]{\omega^2}} \left\| \frac{1}{\vec{\lambda}^4 \vec{Z}^2(\nu,\omega)} \right\|_{\ell_2} \left| \int_0^T \int_0^1 |\, G(t,z) \,| \sum_{r=1}^{\infty} \left| \, a_r^m(t,\omega) - a_r^{m-1}(t,\omega) \, \right| b_r(z) dz dt \right| +$$

$$+ M_0 \delta_2 \frac{T}{\sqrt[3]{\omega}} \left\| \frac{1}{\vec{\lambda}^2} \right\|_{\ell_2} \left| \int_0^T \int_0^1 |\, G(t,z) \,| \sum_{r=1}^{\infty} \left| \, a_r^m(t,\omega) - a_r^{m-1}(t,\omega) \, \right| b_r(z) dz dt \right| \leq$$

$$\leq M_3 \int_0^T \sum_{r=1}^{\infty} \left| \int_0^1 |\, G(t,z) \,| \, b_r(z) dz \right| \left| \, a_r^m(t,\nu,\omega) - a_r^{m-1}(t,\nu,\omega) \, \right| dt \leq$$

$$\leq \rho \cdot \left\| \, \vec{a}^m(t,\nu,\omega) - \vec{a}^{m-1}(t,\nu,\omega) \, \right\|_{B_2[0,T]}, \qquad (40)$$

where

$$\rho = M_3 \int_0^T \|\, G(t,x) \,\|_{L_2[0,1]} \, dt, \qquad (41)$$

$$M_3 = \alpha_0 \beta_0 \delta_2 M_0^2 \frac{|\,\nu\,|}{\sqrt[3]{\omega^2}} \left\| \frac{1}{\vec{\lambda}^4 \vec{Z}^2(\nu,\omega)} \right\|_{\ell_2} + M_0 \delta_2 \frac{T}{\sqrt[3]{\omega}} \left\| \frac{1}{\vec{\lambda}^2} \right\|_{\ell_2},$$

$$\beta_0 = \sum_{\kappa=1}^{p} \bar{\delta}_{0,\kappa} \int_0^T \beta_\kappa(s) ds, \quad \bar{\delta}_{0,\kappa} = \left| \, \bar{Z}_{\kappa n}(\nu,\omega) \, \right|,$$

$$\delta_2 = \|\, l(x) \,\|_{L_2[0,1]} \geq \sqrt{\sum_{n=1}^{\infty} \left| \int_0^1 l(y) b_n(y)\, dy \right|^2},$$

$$\bar{Z}_{in}(\nu,\omega)=\begin{vmatrix} 1-\nu\,\Phi_{11} & \dots & \nu\,\Phi_{1\,(i-1)} & 1 & \nu\,\Phi_{1\,(i+1)} & \dots & \nu\,\Phi_{1p} \\ \nu\,\Phi_{21} & \dots & \nu\,\Phi_{2\,(i-1)} & 1 & \nu\,\Phi_{2\,(i+1)} & \dots & \nu\,\Phi_{2p} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \nu\,\Phi_{p1} & \dots & \nu\,\Phi_{p(i-1)} & 1 & \nu\,\Phi_{p\,(i+1)} & \dots & 1-\nu\,\Phi_{pp} \end{vmatrix}.$$

From estimates (38)–(40) it follows that the operator $J(t;a_n)$ on the right-hand side of (35) is contracting and there is unique fixed point. So, the existence and uniqueness of the solution $\vec{a}(t)\in B_2[0,T]$ to NCSFIE (35) are proved. The theorem 3.1 is proved. □

4. Continuously dependence of the solution to NCSFIE from parameter ω

In this section we use the following obvious lemma.

Lemma 4.1. *For two values ω_1, ω_2 of positive parameter ω there true the following estimates*

$$\left| e^{-\sqrt[3]{\mu_n(\omega_1)}(t-s)} - e^{-\sqrt[3]{\mu_n(\omega_2)}(t-s)} \right| \le L_{01}\,|\,\omega_1-\omega_2\,|\,, \quad 0<L_{01}=\mathrm{const};$$

$$\left| \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega_1)}(t-s)+\frac{\pi}{6}\right) - \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega_2)}(t-s)+\frac{\pi}{6}\right) \right| \le L_{02}\,|\,\omega_1-\omega_2\,|\,;$$

$$\left| e^{\sqrt[3]{\mu_n(\omega_1)}\frac{t-s}{2}} - e^{\sqrt[3]{\mu_n(\omega_2)}\frac{t-s}{2}} \right| \le L_{03}\,|\,\omega_1-\omega_2\,|\,, \quad 0<L_{03}=\mathrm{const};$$

$$\left| \cos\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega_1)}(t-s) - \cos\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega_2)}(t-s) \right| \le L_{02}\,|\,\omega_1-\omega_2\,|\,, \quad 0<L_{02}=\mathrm{const};$$

$$\left| \frac{1}{\sqrt[3]{\omega_1}} - \frac{1}{\sqrt[3]{\omega_2}} \right| \le L_{04}\,|\,\omega_1-\omega_2\,|\,, \quad L_{04}=\mathrm{const};$$

$$\left| \frac{1}{\sqrt[3]{\omega_1^2}} - \frac{1}{\sqrt[3]{\omega_2^2}} \right| \le L_{05}\,|\,\omega_1-\omega_2\,|\,, \quad L_{05}=\mathrm{const}.$$

Theorem 4.2. *Let be fulfilled the conditions of the Theorem 3.1. Then the following estimate*

$$\|\,\vec{a}(t,\omega_1)-\vec{a}(t,\omega_2)\,\|_{B_2[0,T]} \le L_M\,|\,\omega_1-\omega_2\,|\,, \quad 0<L_M=\mathrm{const} \tag{42}$$

holds.

Proof. By virtue of the Lemma for the function (25) we obtain

$$|\,Q_n(t,s,\omega_1)-Q_n(t,s,\omega_2)\,| \le \frac{1}{3}\left| e^{-\sqrt[3]{\mu_n(\omega_1)}(t-s)} - e^{-\sqrt[3]{\mu_n(\omega_2)}(t-s)} \right| +$$

$$+\frac{2}{3}\left| e^{\frac{\sqrt[3]{\mu_n(\omega_1)}}{2}(t-s)} - e^{\frac{\sqrt[3]{\mu_n(\omega_2)}}{2}(t-s)} \right| \cdot \left| \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega_1)}(t-s)+\frac{\pi}{6}\right) \right| +$$

$$+\frac{2}{3}e^{\frac{\sqrt[3]{\mu_n(\omega_2)}}{2}(t-s)}\left|\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega_1)}(t-s)+\frac{\pi}{6}\right)-\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega_2)}(t-s)+\frac{\pi}{6}\right)\right|\leq$$

$$\leq\left[L_{01}+L_{03}+e^{\sqrt[3]{\omega_2}T}L_{02}\right]|\,\omega_1-\omega_2\,|=L_1\,|\,\omega_1-\omega_2\,|\,. \tag{43}$$

By virtue of the Lemma, for the function (26) we obtain

$$|\,\psi_{1,n}(t,\omega_1)-\psi_{1,n}(t,\omega_2)\,|\leq$$

$$\leq\frac{1}{3}\left|\,e^{-\sqrt[3]{\mu_n(\omega_1)}t}-e^{-\sqrt[3]{\mu_n(\omega_2)}t}\,\right|+\frac{2}{3}\left|\,e^{\frac{\sqrt[3]{\mu_n(\omega_1)}}{2}t}-e^{\frac{\sqrt[3]{\mu_n(\omega_2)}}{2}t}\,\right|\cdot\left|\,\cos\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega_1)}t\,\right|+$$

$$+\frac{2}{3}e^{\frac{\sqrt[3]{\mu_n(\omega_2)}}{2}t}\left|\,\cos\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega_1)}t-\cos\frac{\sqrt{3}}{2}\sqrt[3]{\mu_n(\omega_2)}t\,\right|\leq$$

$$\leq\left[L_{01}+L_{03}+e^{\sqrt[3]{\omega_2}T}L_{02}\right]|\,\omega_1-\omega_2\,|=L_1\,|\,\omega_1-\omega_2\,|\,. \tag{44}$$

By similarly way for the functions (27) and (28) we obtain

$$|\,\psi_{j,n}(t,\omega_1)-\psi_{j,n}(t,\omega_2)\,|\leq L_1\,|\,\omega_1-\omega_2\,|\,,\quad j=2,3. \tag{45}$$

By the aid of the estimates (43)–(45), taking properties of the functions (24), (31) and matrix (33), for the NCSFIE (35) we derive

$$\|\,\vec{a}(t,\nu,\omega_1)-\vec{a}(t,\nu,\omega_2)\,\|_{B_2[0,T]}\leq\sum_{n=1}^{\infty}\max_{0\leq t\leq T}|\,P_n(t,\omega_1)-P_n(t,\omega_2)\,|+$$

$$+|\,\nu\,|\left|\,\frac{1}{\sqrt[3]{\omega_1}}-\frac{1}{\sqrt[3]{\omega_2}}\,\right|\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\sum_{\kappa=1}^{p}\left|\,\frac{Z_{\kappa n}(a_r,\nu,\omega_1)}{Z_n(\nu,\omega_1)}\,\right|\max_{0\leq t\leq T}\int_0^t|\,Q_n(t,s,\omega_1)\,|\;\alpha_j(s)\,ds+$$

$$+\frac{|\,\nu\,|}{\sqrt[3]{\omega_2}}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\sum_{\kappa=1}^{p}\left|\,\frac{Z_{\kappa n}(a_r,\nu,\omega_1)}{Z_n(\nu,\omega_1)}\,\right|\max_{0\leq t\leq T}\int_0^t|\,Q_n(t,s,\omega_1)-Q_n(t,s,\omega_2)\,|\;\alpha_j(s)\,ds+$$

$$+\frac{|\,\nu\,|}{\sqrt[3]{\omega_2}}\sum_{n=1}^{\infty}\frac{\alpha_0M_0}{\lambda_n}\sum_{\kappa=1}^{p}\left[\,|Z_{\kappa n}(a_r,\nu,\omega_1)|\left|\frac{1}{Z_n(\nu,\omega_1)}-\frac{1}{Z_n(\nu,\omega_2)}\right|+\right.$$

$$\left.+\left|\frac{1}{Z_n(\nu,\omega_2)}\right|\,|\,Z_{\kappa n}(a_r,\nu,\omega_1)-Z_{\kappa n}(a_r,\nu,\omega_2)|\right]+$$

$$+\left|\,\frac{1}{\sqrt[3]{\omega_1}}-\frac{1}{\sqrt[3]{\omega_2}}\,\right|\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\max_{0\leq t\leq T}\int_0^t Q_n(t,s,\omega_1)\left|\int_0^1 F\,(s,y,\cdot)\;b_n(y)\,dy\,\right|ds+$$

$$+\frac{1}{\sqrt[3]{\omega_2}}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\max_{0\le t\le T}\int_0^t |\, Q_n(t,s,\omega_1)-Q_n(t,s,\omega_2)\,| \left|\int_0^1 F\,(s,y,\cdot)\; b_n(y)\,dy\right| ds+\frac{M_0}{\sqrt[3]{\omega_2}}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\times$$

$$\times\sum_{\kappa=1}^{p}\left|\int_0^1 l(y)b_n(y)\,dy\int_0^T\int_0^1 |\,G(t,z)\,|\sum_{r=1}^{\infty}|\,a_r(t,\omega_1)-a_r(t,\omega_2)\,|\,b_r(z)dzdt\right|. \qquad (46)$$

By the aid of Lemma, estimates (43)–(45) and conditions of the Theorem 3.1, from (46) we obtain

$$\|\,\vec{a}(t,\nu,\omega_1)-\vec{a}(t,\nu,\omega_2)\,\|_{B_2[0,T]}\le C_0M_0\left(\frac{\sqrt{2}}{\pi}\right)^4\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^8}}\left[L_1\left\|\frac{\partial^4\varphi_1(x)}{\partial x^4}\right\|_{L_2[0,1]}+\right.$$

$$\left.+(L_1+L_{04})\left\|\frac{\partial^4\varphi_2(x)}{\partial x^4}\right\|_{L_2[0,1]}+(L_1+L_{05})\left\|\frac{\partial^4\varphi_3(x)}{\partial x^4}\right\|_{L_2[0,1]}\right]|\,\omega_1-\omega_2\,|+$$

$$+M_0\delta_0\left(\frac{1}{\pi}\right)^2\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^4}}\left(|\,\nu\,|\,L_{04}\sqrt{\sum_{n=1}^{\infty}\frac{1}{|\,Z_n(\nu,\omega_1)\,|^2}}+\frac{|\,\nu\,|}{\sqrt[3]{\omega_2}}L_1\sqrt{\sum_{n=1}^{\infty}\frac{1}{|\,Z_n(\nu,\omega_1)\,|^2}}\right)|\,\omega_1-\omega_2\,|+$$

$$+\delta_0M_0^2\frac{|\,\nu\,|}{\sqrt[3]{\omega_2}}\left(\frac{1}{\pi}\right)^2\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^4}}\sqrt{\sum_{n=1}^{\infty}\frac{1}{\left|\,\bar{\bar{Z}}_n(\nu)\,\right|^2}}\left[L_{04}+L_1\frac{1}{\sqrt[3]{\omega_2}}\right]|\,\omega_1-\omega_2\,|+$$

$$+\alpha_0\beta_0\delta_2M_0^2\frac{|\,\nu\,|}{\sqrt[3]{\omega_2^2}}\left\|\frac{1}{\vec{\lambda}^2\vec{Z}^2(\nu,\omega_2)}\right\|_{\ell_2}\left|\int_0^T\int_0^1|\,G(t,z)\,|\sum_{r=1}^{\infty}\left|\,a_r^{\tau}(t,\omega_1)-a_r^{\tau-1}(t,\omega_2)\,\right|\,b_r(z)dzdt\right|+$$

$$+M_0T\left\|\frac{1}{\vec{\lambda}^2}\right\|_{\ell_2}\sqrt{\sum_{n=1}^{\infty}\left[\max_{0\le t\le T}\left|\int_0^1 F\,(t,y,\cdot)\,b_n(y)dy\right|\right]^2}\left(L_{04}+L_1\frac{1}{\sqrt[3]{\omega_2}}\right)|\omega_1-\omega_2|+$$

$$+\frac{M_0}{\sqrt[3]{\omega_2}}\left\|\frac{1}{\vec{\lambda}^2}\right\|_{\ell_2}\sqrt{\sum_{n=1}^{\infty}\left|\int_0^1 l(y)b_n(y)\,dy\right|^2}\times$$

$$\times\int_0^T\int_0^1|\,G(t,z)\,|\sum_{r=1}^{\infty}|\,a_r(t,\omega_1)-a_r(t,\omega_2)\,|\,b_r(z)dzdt, \qquad (47)$$

where

$$\bar{\bar{Z}}_n(\nu)=\frac{1}{\lambda_n}\begin{vmatrix}1-\nu\,\Delta_{11} & \nu\,\Delta_{12} & \ldots & \nu\,\Delta_{1p}\\ \nu\,\Delta_{21} & 1-\nu\,\Delta_{22} & \ldots & \nu\,\Delta_{2p}\\ \ldots & \ldots & \ldots & \ldots\\ \nu\,\Delta_{p1} & \nu\,\Delta_{p2} & \ldots & 1-\nu\,\Delta_{pp}\end{vmatrix},\quad \Delta_{ij}=\int_0^T\beta_i(s)\int_0^s\alpha_j(\theta)\,d\theta ds.$$

It is not difficult to check that from (47) we obtain

$$\| \vec{a}(t,\nu,\omega_1) - \vec{a}(t,\nu,\omega_2) \|_{B_2[0,T]} \leq M_4 \,|\, \omega_1 - \omega_2 \,| + \rho \cdot \| \vec{a}(t,\nu,\omega_1) - \vec{a}(t,\nu,\omega_2) \|_{B_2[0,T]} \,, \quad (48)$$

where

$$M_4 = C_0 M_0 \left(\frac{\sqrt{2}}{\pi} \right)^4 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^8}} \Bigg[L_1 \left\| \frac{\partial^4 \varphi_1(x)}{\partial x^4} \right\|_{L_2[0,1]} +$$

$$+ (L_1 + L_{04}) \left\| \frac{\partial^4 \varphi_2(x)}{\partial x^4} \right\|_{L_2[0,1]} + (L_1 + L_{05}) \left\| \frac{\partial^4 \varphi_3(x)}{\partial x^4} \right\|_{L_2[0,1]} \Bigg] +$$

$$+ |\, \nu \,| \, M_0 \delta_0 \left(\frac{1}{\pi} \right)^2 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^4}} \left[\sqrt{\sum_{n=1}^{\infty} \frac{1}{|\, Z_n(\nu,\omega_1) \,|^2}} + M_0 \frac{1}{\sqrt[3]{\omega_2}} \sqrt{\sum_{n=1}^{\infty} \frac{1}{\left|\, \bar{\bar{Z}}_n(\nu) \,\right|^2}} \right] \left[L_{04} + \frac{1}{\sqrt[3]{\omega_2}} L_1 \right] +$$

$$+ T \left[M_0 L_{04} + \frac{L_1}{\sqrt[3]{\omega_2}} \right] \left\| \frac{1}{\vec{\lambda}^2} \right\|_{\ell_2} \max_{0 \leq t \leq T} \| F(t,x,\,\cdot) \|_{L_2[0,1]} \,.$$

From the estimate (48) we obtain (42). Theorem 4.2 is proved. □

5. Convergence of the Fourier series

Theorem 5.1. *Let the conditions of the Theorem 3.1 be fulfilled. Then for regular values of the parameters ν the series* (36) *converges absolute and uniform in the domain Ω. Moreover, the solution of the mixed problem* (1)–(3) *belongs to the class of functions* (4).

The *proof* of the theorem 5.1 is based on obtaining the estimates for the Fourier series (36) and for its derivatives. The method for obtaining an estimate is the same as in the case of obtaining estimates (38)–(40), (47) and (48). The Theorem 5.1 is proved.

Corollary 5.2. *Let be fulfilled the conditions of the Theorem 3.1. Then the following estimate*

$$|\, V(t,\nu,\omega_1) - V(t,\nu,\omega_2) \,| \leq L_C \,|\, \omega_1 - \omega_2 \,|\,, \quad 0 < L_C = \text{const}. \quad (49)$$

holds.

6. Conclusion

It is considered a fifth order nonlinear partial integro-differential equations (1) with mixed conditions (2) and (3) and with two real parameters ν, ω. The Fourier spectral method of separation of variables (5) is applied. A countable system of nonlinear functional-integral equations (35) is derived. Theorem on a uniqueness and existence of the solution of mixed problem (1)–(3) is proved for regular values of parameters. The method of compressing mapping is applied for countable system (35) in Banach space $B_2[0,T]$. The solution of the mixed problem (1)–(3) is obtained in the form of Fourier series (36). Theorem on absolute and uniform convergence of Fourier series is proved. Continuous dependence on parameter ω of the classical solution of mixed problem is studied (see, estimates (42) and (49)).

We hope that this work can serve as a basis for further development of the theory of partial differential and integro-differential equations of the third and higher orders.

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