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On the smoothness of a generalized solution of a semi-nonlocal boundary value problem for a fourth-order mixed type equation of the second kind

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Abstract. In this paper, we investigate the unique solvability of a regular solution and the smoothness of a generalized solution of a semi-nonlocal boundary value problem for a fourth-order mixed-type equation of the second kind in Sobolev spaces. Theorems of uniqueness are proved by the energy integral method, and the existence and smoothness of the solution are proved by the methods of "ε-regularization", a priori estimates, modified Galerkin methods, and the Fourier method.

Key Words and Phrases: Mixed type equations of the second kind of the fourth order, semi-nonlocal boundary value problem, existence and uniqueness theorem, smoothness of the generalized solution, "ε-regularization" method, a priori estimates method, Galerkin method.

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1. Introduction

The first nonlocal boundary value problems for second-order mixed-type equations were initially studied using classical methods in the works [1-3]. Subsequently, the methods presented in their studies were further developed using functional approaches in the works [4-10]. Local and nonlocal boundary value problems for fourth-order partial differential equations were studied in the work [11-16]. Boundary value problems with local conditions for high-order mixed-type equations in various spaces were studied in the works [17-22]. However, there has been insufficient study of forward problems with nonlocal boundary conditions for high-order mixed-type equations [24].

In this paper, using the results of [17-19] and applying the modified Galerkin method, the "ε-regularization" method, and the method of a priori estimates, we study the unique solvability and smoothness of a generalized solution to a semi-nonlocal boundary value problem for a fourth-order mixed-type equation of second kind in Sobolev spaces.

2. Problem statement

Consider fourth-order mixed-type equations of second kind in domain $Q = (0,1) \times (0,T) = \{(x,t); 0 < x < 1;\ \ 0 < t < T < +\infty\}$:

$$Lu = Pu + Mu = f(x,t), \tag{1}$$

here $Pu = \sum_{i=0}^{4} K_i(x,t) D_t^i u$; $\quad Mu = au_{xxxx} - bu_{xxtt} - cu_{xx}$;
$K_4(x,t) = K_4(t)$, $K_4(0) = K_4(0) = 0$, $D_t^i u = \frac{\partial^i u}{\partial t^i}$ $(i = 0,1,2,3,4)$, $D_t^0 u = u$.

Let the following conditions be satisfied for the coefficients of equation (1):

$$K_4(t) \in C^3(0,T) \cap C[0,T];\ K_i(x,t) \in C^2(Q) \cap C(\overline{Q}); a, b, c - consts > 0$$

for all $x \in [0,1]$.

Equation (1) is a mixed-type equation of the second kind since no restrictions are imposed on the sign of function $K_4(t)$ with respect to variable t inside segment $[0,T]$ [17, 18, 25].

Semi-nonlocal boundary value problem: find solution $u(x,t)$ to equation (1) from the Sobolev space $W_2^4(Q)$, satisfying the following boundary conditions:

$$\gamma D_t^p\, u|_{t=0} = D_t^p\, u|_{t=T};\ \ p = 0,1,2 \tag{2}$$

$$u|_{x=0} = u|_{x=1} = 0; \tag{3}$$

$$u_{xx}|_{x=0} = u_{xx}|_{x=1} = 0, \tag{4}$$

where γ is the value different from zero, which will be specified below.

In what follows, we need the following definitions and auxiliary propositions. Let $\overrightarrow{e}(e_t, e_x); (e_t = \cos(\overrightarrow{e}, t), e_x = \cos(\overrightarrow{e}, x))$ be the unit vector of the inner normal to boundary ∂Q. When obtaining various a priori estimates, we often use the Cauchy inequality with σ [26], that is,

$$\forall u,\ \vartheta \geq 0,\ \ \forall \sigma > 0,\ \ 2u\,\vartheta \leq \sigma u^2 + \sigma^{-1}\vartheta^2.$$

Let us denote the class of smooth functions from space $W_2^4(Q)$, satisfying conditions (2)-(4) by C_L.

Definition 1. *We call function $u(x,t)$ a regular solution to problem* (1), (2)-(4), *if $u \in C_L$ satisfies equation* (1) *almost everywhere in domain Q.*

Theorem 1. *Let the above conditions for the coefficients of equation* (1) *be satisfied, coefficient $K_1(x,t) > 0$ is sufficiently large, and let the following inequalities be satisfied for the coefficients of equation* (1)*; $-(2K_3 - 3K_{4t} + 3\lambda K_4) \geq \delta_3 > 0$, $2K_1 - K_{2t} + \lambda K_2 \geq \delta_2 > 0$, $\lambda K_0 - K_{0t} \geq \delta_1 > 0$ for any $(x,t) \in \overline{Q}$, where $\lambda = \frac{2}{T}\ln|\gamma| > 0$, $|\gamma| > 1$, for all $x \in [0,1]$.*

Then for any $f(x,t) \in L_2(Q)$. If there exists a regular solution $u(x,t)$ to problem (1), (2)-(4) *from the Sobolev space $W_2^4(Q)$, then it is unique and the following estimate is true for it:*

$$\|u\|^2_{W_2^2(Q)} \leq c\,\|f\|_0^2.$$

Proof. We will prove the uniqueness of the solution to problem (1), (2)-(4) using the method of energy integrals. Let there exist a regular generalized solution to problem (1), (2)-(4) $u(x,t)$ from the Sobolev space $W_2^4(Q)$. Consider the following identity:

$$2\int\limits_Q Lu\, e^{-\lambda t}\, u_t\, dxdt = 2\int\limits_Q f\, e^{-\lambda t}\, u_t\, dxdt. \tag{5}$$

By virtue of the conditions of Theorem 1 and boundary conditions (2)-(4), by integrating identity (5) by parts, and applying the Cauchy inequalities with σ [24], from identity (5), it is easy to obtain the following inequality:

$$2\int\limits_Q e^{-\lambda t} Lu\, u_t\, dxdt \geq \int\limits_Q e^{-\lambda t}\Big\{-(2K_3 - 3K_{4t} + 3\lambda K_4)\, u_{tt}^2 + \lambda a u_{xx}^2 + \lambda b u_{xt}^2 + \lambda c u_x^2 +$$

$$+ (2K_1 - K_{2t} + \lambda K_2)\, u_t^2 + (\lambda K_0 - K_{0t})\, u^2\Big\} dxdt - 2\sigma\|u_{tt}\|_0^2 - 4\lambda^4 K\sigma^{-1}\|u_t\|_0^2 -$$

$$-\int\limits_{\partial Q} e^{-\lambda t}\Big\{2K_4 u_{ttt}u_t - 2(K_{4t} - \lambda K_4)\, u_{tt}u_t - K_4 u_{tt}^2 + 2K_3 u_{tt}u_t + 2K_2 u_t^2 - K_0 u^2 - a u_{xx}^2 -$$

$$+ b u_{xt}^2 + c u_x^2\Big\} e_t\, ds - \int\limits_{\partial Q} e^{-\lambda t}\Big\{2a u_{xxx}u_t - 2a u_{xx}u_{tx} - 2b u_{xxt}u_t - 2c u_x u_t\Big\} e_x\, ds, \tag{6}$$

where $K = max\left\{\|K_4\|^2_{C^2(Q)}, \|K_3\|^2_{C^1(Q)}\right\}$.

The conditions of Theorem 1 ensure the nonnegativity of the integral over domain Q and zero boundary integrals. From inequality (6), we obtain:

$$2\left|\int\limits_Q Lue^{-\lambda t}u_t dxdt\right| \geq \int\limits_Q e^{-\lambda t}\left\{\delta_3 u_{tt}^2 + \lambda a u_{xx}^2 + \lambda b u_{xt}^2 + \delta_2 u_t^2 + \lambda c u_x^2 + \delta_1 u^2\right\}dxdt-$$

$$-2\sigma \left\|u_{tt}\right\|_0^2 - 4\lambda^4\sigma^{-1}K \left\|u_t\right\|_0^2. \tag{7}$$

Let σ be a sufficiently small positive number. We choose constant values of δ_3 and δ_2 in inequality (7) such that $\delta_3 - 2\sigma \geq \delta_{03} > 0,\ \delta_2 - 4\lambda^4\sigma^{-1}K \geq \delta_{02} > 0$, now denoting by $\delta = \min\{\delta_{03}, \lambda a, \lambda b, \lambda c, \delta_{02}, \delta_1\}$, we obtain the first a priori estimate from (7) for solving problem (1)-(4):

$$\|u\|_{W_2^2(Q)}^2 \leq c_1 \|f\|_{L_2(Q)}^2.$$

In what follows, we denote different positive constants by c_i.

Now we will prove the uniqueness of the regular solution to problem (1)-(4). We will prove the theorem by contradiction. Let problem (1)-(4) have two different solutions $u_1(x,t),\ u_2(x,t)$. Then the new function $\vartheta(x,t) = u_1(x,t) - u_2(x,t)$ satisfies the homogeneous equation (1) with conditions (2)-(4) and the first estimate $\|\vartheta\|_2^2 \leq 0$ is valid for it. From this follows the uniqueness of the regular solution to problem (1)-(4).

Now we will prove the solvability of the regular solution to problem (1)-(4).

3. Fifth-order equation with a small parameter (auxiliary problem)

The solvability of problem (1)-(4) will be proven by the "ε-regularization" method, in combination with the modified Galerkin method and the method of a priori estimates; in domain $Q = (0,1) \times (0,T)$ we will consider a family of fifth-order equations with a small parameter:

$$L_\varepsilon u_\varepsilon = -\varepsilon\frac{\partial \Delta^2 u_\varepsilon}{\partial t} + Lu_\varepsilon = f(x,t) \tag{8}$$

with semi-nonlocal boundary conditions:

$$\gamma D_t^q\, u_\varepsilon|_{t=0} = D_t^q\, u_\varepsilon|_{t=T};\ \ q = 0,1,2,3,4, \tag{9}$$

$$u_\varepsilon|_{x=0} = u_\varepsilon|_{x=1} = 0, \tag{10}$$

$$u_{\varepsilon xx}|_{x=0} = u_{\varepsilon xx}|_{x=1} = 0, \tag{11}$$

where ε is a small positive number, $D_z^q w = \frac{\partial^q w}{\partial z^q}$, $q = 1,2,3,4,5$; $D_z^0 w = w$, $\Delta^2 u = (\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2})^2 u = (\frac{\partial^4 u}{\partial t^4} + 2\frac{\partial^4 u}{\partial x^2 \partial t^2} + \frac{\partial^4 u}{\partial x^4})$ is a biharmonic operator.

Below we will use fifth-order equations with a small parameter (8) as the "ε-regularizing" equation for the mixed-type fourth-order equation of the second kind (1) [4, 5, 6, 17, 25].

By $V(Q)$, we will denote the class of functions such that $u_\varepsilon(x,t) \in W_2^4(Q)$, $\frac{\partial \Delta^2 u_\varepsilon}{\partial t} \in L_2(Q)$, satisfy the corresponding boundary conditions (9)-(11).

Definition 2. *We call function $u_\varepsilon(x,t)$ a regular solution to problem* (8), (9)-(11), *if $u_\varepsilon \in V(Q)$ satisfies equation* (8) *almost everywhere in domain Q.*

Theorem 2. *Let all the conditions of Theorem 1 be satisfied, and let the following conditions be satisfied for the coefficients of equation* (8)*; coefficient $K_3(x,t) > 0$ is sufficiently large and $-(2K_3 + (2j-3)K_{4t} + 3\lambda K_4) \geq \delta_3 > 0,\ j = 0,1,2.$*

Then, for any function $f(x,t) \in W_2^1(Q)$, such that $\gamma f(x,0) = f(x,T)$, there exists a unique regular solution $u_\varepsilon(x,t)$ to problem (8), (9)-(11) *from space $V(Q)$ and the following estimates are valid for it:*

I). $\varepsilon\Big(\|u_{\varepsilon ttt}\|_0^2 + \|u_{\varepsilon ttx}\|_0^2 + \|u_{\varepsilon txx}\|_0^2 \Big) + \|u_\varepsilon\|_2^2 \leq c_1 \| f \|_0^2,$

II). $\varepsilon \left\| \frac{\partial}{\partial t} \Delta^2 u_\varepsilon \right\| + \|u_\varepsilon\|_4^2 \leq c_2 \| f \|_1^2.$

Proof. The inequality I) is proven in the same way as the first estimate of Theorem 1, from which the uniqueness of a regular solution to problem (8), (9)-(11) follows [17-19].

Now we present the *proof* of the first a priori estimate. Let $\phi_j(x,t) \in W_2^4(Q)$ be the eigenfunctions of the following problem:

$$-\Delta^2 \phi_j = -\Big(\frac{\partial^4 \phi_j}{\partial^4 t} + \frac{\partial^4 \phi_j}{\partial^4 x} \Big) = \mu_j^4 \phi_j, \tag{12}$$

$$D_t^p \phi_j \big|_{t=0} = D_t^p \phi_j \big|_{t=T}, \quad p = 0,1,2,3, \tag{13}$$

$$\phi_j \big|_{x=0} = \phi_j \big|_{x=1} = 0, \tag{14}$$

$$\phi_{jxx} \big|_{x=0} = \phi_{jxx} \big|_{x=1} = 0. \tag{15}$$

From the general theory [17-19] of linear self-adjoint elliptic operators, it is known that all eigenfunctions of problem (12)-(15) belong to $W_2^4(Q)$ and form a complete orthonormal system in $L_2(Q)$. Now, using these sequences of functions, we construct a solution to the auxiliary problem:

$$P\omega_j \equiv \exp\Big(-\frac{\lambda t}{2} \Big) \frac{\partial \omega_j}{\partial t} = \phi_j, \tag{16}$$

$$\gamma \cdot \omega_j(x,0) = \omega_j(x,T), \tag{17}$$

where $\gamma - const \neq 0$, and $|\gamma| > 1$. Obviously, problem (16), (17) is uniquely solvable and its solution has the following form:

$$P^{-1}\phi_j = \omega_j = \int\limits_0^t \exp\Big(\frac{\lambda\tau}{2}\Big)\phi_j d\tau + \frac{1}{\gamma - 1}\int\limits_0^T \exp\Big(\frac{\lambda t}{2}\Big)\phi_j dt.$$

Functions $\omega_j(x,t) \in W_2^5(Q)$ are linearly independent. Indeed, if $\sum_{j=1}^N c_j\omega_j = 0$ for some set of sequences of functions $\omega_1, \omega_2, ..., \omega_N$, then, acting on this sum by operator P, we have $\sum_{j=1}^N c_j P\omega_j = \sum_{j=1}^N c_j\phi_j = 0$, and from this, it follows that for all $j = \overline{1,N}$, coefficients are $c_j = 0$. Note that the following conditions of functions $\omega_j(x,t) \in W_2^5(Q)$, follow from the construction of function $\phi_j(x,t)$:

$$\gamma\, D_t^q\, \omega_j|_{t=0} = D_t^q\, \omega_j|_{t=T}, \quad q = 0,1,2,3,4, \tag{18}$$

$$\omega_j|_{x=0} = \omega_j|_{x=1} = 0, \tag{19}$$

$$\omega_{jxx}|_{x=0} = \omega_{jxx}|_{x=1} = 0. \tag{20}$$

Now we seek an approximate solution to problem (8)-(11) in the form $w(x,t) = u_\varepsilon^N(x,t) = \sum_{j=1}^N c_j\omega_j(x,t)$, where coefficients $c_j$for any$j$ from 1 to N are determined as a solution to the linear algebraic system:

$$2\int\limits_Q L_\varepsilon u_\varepsilon^N \exp\Big(-\frac{\lambda\, t}{2}\Big)\,\phi_j dxdt = 2\int\limits_Q f\, \exp\Big(-\frac{\lambda\, t}{2}\Big)\,\phi_j dxdt. \tag{21}$$

Let us prove the unique solvability of algebraic system (21). Multiplying each equation from (21) by c_j and summing over j from 1 to N, considering problem (16), (17) and boundary conditions (18)-(20), and algebraic system (21), we obtain the following identity:

$$2\int\limits_Q L_\varepsilon w\, \exp(-\lambda\, t)\, w_t dxdt = 2\int\limits_Q f\, \exp(-\lambda\, t)\, w_t dxdt, \tag{22}$$

from which, by virtue of the conditions of Theorem 2 and by integrating identity (22) by parts, we obtain estimate I) for an approximate solution to problem (8)-(11), i.e.

$$\varepsilon\,\Big(\,\|u_{\varepsilon ttt}^N\|_0^2 + \|u_{\varepsilon ttx}^N\|_0^2 + \|u_{\varepsilon txx}^N\|_0^2\Big) + \|u_\varepsilon^N\|_2^2 \le c_1\,\|\, f\,\|_0^2. \tag{23}$$

From this follows the solvability of algebraic system (21) [26]. Estimate (23) allows (by virtue of Theorem 1 on weak compactness [26, 28]) to pass to the limit as $N \to \infty$ and conclude that some subsequence $\left\{u_\varepsilon^{N_k}(x,t)\right\}$ converges weakly, by virtue of the uniqueness of the solution to the problem (Theorem 1), to the sought-for solution $u_\varepsilon(x,t)$ of problem (8)-(11) in space $V(Q)$, possessing the properties specified in Theorem 2. For $u_\varepsilon(x,t)$, by virtue of (23), the following inequality is valid:

$$\varepsilon\left(\|u_{\varepsilon ttt}\|_0^2 + \|u_{\varepsilon ttx}\|_0^2 + \|u_{\varepsilon txx}\|_0^2 \right) + \|u_\varepsilon\|_2^2 \le c_1 \, \| f \|_0^2 . \tag{24}$$

Now passing to the limit as $N \to \infty$ in (21), we obtain the unique regular generalized solution to problem (8)-(11) from space $V(Q)$.

Let us prove the second a priori estimate II). Using problem (12)-(17), from identity (21), we obtain:

$$-\frac{2}{\mu_j^2}\int\limits_D L_\varepsilon w \, \exp\Big(-\frac{\lambda\, t}{2}\Big)\, \Delta^2 P\omega_j \, dxdt = -\frac{2}{\mu_j^2}\int\limits_D f \, \exp\Big(-\frac{\lambda\, t}{2}\Big)\, \Delta^2 P\omega_j \, dxdt. \tag{25}$$

Multiplying equation (25) by $-2\,\mu_j^2 c_j$, summing over j from 1 to N and considering conditions (18)-(20), from (25) we obtain the following identity:

$$2\Big(L_\varepsilon w, e^{-\lambda\, t}\, Pw\Big)_0 = 2\Big(f, e^{-\lambda\, t}\, Pw\Big)_0, \tag{26}$$

where $Pw \equiv \frac{\partial \Delta^2 w}{\partial\, t} - 2\lambda \frac{\partial^2}{\partial\, t^2}\Delta w + 3\lambda^2 \frac{\partial}{\partial t}\Delta w - \frac{\lambda}{2} w_{tt} + \frac{\lambda^2}{16} w_t$,

$$\frac{\partial \Delta^2 w}{\partial\, t} = \frac{\partial}{\partial\, t}\Big(w_{tttt} + 2w_{ttxx} + w_{xxxx}\Big).$$

Integrating (26) by parts, considering the conditions of Theorem 2 and boundary conditions (18), (20), we obtain the following inequality:

$$c_2 \,\| \, f \, \|_1^2 \; \ge \varepsilon \, \left\| \frac{\partial\, \Delta^2 w}{\partial t} \right\|_0^2 + \int\limits_Q e^{-\lambda t}\Big\{ -(2K_3 + K_{4t} + 3\lambda\, K_4) w_{tttt}^2 - (2K_3 - K_{4t} +$$

$$+3\lambda K_4) w_{ttxx}^2 - (\,2K_3 + K_{4t} + 3\lambda\, K_4) w_{tttx}^2 + \lambda\, a w_{xxxx}^2 + \lambda\, b w_{xxtt}^2 + \lambda\, a w_{xxxt}^2 \Big\} dxdt +$$

$$+\rho\, \|w\|_3^2 - N_1 \sigma(\|w_{tttt}\|_0^2 + \|w_{ttxx}\|_0^2 + \|w_{tttx}\|_0^2) - N_2 \sigma\,(\|w_{xxxx}\|_0^2 + \|w_{ttxx}\|_0^2 + \|w_{txxx}\|_0^2) -$$

$$-c\,(\sigma^{-1}, \lambda, K)\, \|w\|_3^2 + \int\limits_{\partial Q} e^{-\lambda \cdot t} B(u(s), K_i(s))\, ds, \quad i = \overline{0,4}, \tag{27}$$

where $\rho,\ N_i(i=1,2)$ are positive numbers depending on the norm of function $K_i(x,t);\ i=\overline{0,3,}$ in space $C^3(\overline{Q}),\ K=\max\Big\{\|K_4(t)\|_{C^3[0,T]},\ \|K_i(x,t)\|_{C^3(\overline{Q})}\Big\}$, $\sigma,\ c(\sigma^{-1})$ are the coefficients of the Cauchy inequality [26], $B(u(s),K_i(s))$ are functions depending on the traces of functions $u(x,t),\ K_i(x,t)$ on the boundary of domain Q. Let us denote $\delta_0=\min\Big\{\delta_3,\lambda a,\lambda b,\lambda c,\delta_2,\delta_1\Big\}$, by $N=\max\Big\{N_1,N_2\Big\}$. Considering the condition of Theorem 2, boundary conditions (18)-(20), and $\gamma^2=e^{\lambda T}$, we obtain that in (27) the boundary integrals will vanish. Now choosing σ so that $\delta_0-N\sigma\geq\sigma_0>0,\ \ \rho-c(\sigma^{-1},\lambda,K)\geq\rho_0>0$, from inequality (27), we obtain the second estimate:

$$\varepsilon\left\|\frac{\partial\Delta^2u_\varepsilon^N}{\partial t}\right\|_0^2+\left\|u_\varepsilon^N\right\|_4^2\leq c_2\left(\|\,f\,\|_0^2+\|\,f_t\,\|_0^2\right)\leq c_2\,\|f\|_1^2\,. \qquad (28)$$

The constant on the right-hand side of inequality (28) does not depend on N, therefore, the second estimate for the approximate solution to problem (8), (9)-(11) follows from (28). Estimate (24) together with estimate (28) allow us to pass to the limit as $N\to\infty$ and conclude that some subsequence $\left\{u_\varepsilon^{N_k}(x,t)\right\}$ converges weakly (due to the uniqueness of the solution to problem (8)-(11)) in $V(Q)$ with the derivatives of the fourth and fifth orders to the sought for solution of problem (8), (9)-(11), which has the properties specified in Theorem 2 [26, 28]. Therefore, for $u_\varepsilon(x,t)$ by virtue of (28), the following inequality is true:

$$\varepsilon\left\|\frac{\partial}{\partial t}\Delta^2u_\varepsilon\right\|_0^2+\|u_\varepsilon\|_4^2\leq c_2\left(\|\,f\,\|_0^2+\|\,f_t\,\|_0^2\right)\leq c_2\,\|\,f\,\|_1^2\,.$$

This implies the existence of a regular generalized solution $u_\varepsilon(x,t)$ to problem (8)-(11) from space $V(Q)$. Thus, Theorem 2 is proven.

4. Existence of a regular solution to problem (1), (2)-(4)

Let us proceed to the proof of the solvability of problem (1), (2)-(4).

Theorem 3. *Let all the conditions of Theorem 2 be satisfied. Then a solution to problem* (1)-(4) *from* $W_2^4(Q)$ *exists and is unique.*

Proof. The uniqueness of a solution to problem (1)-(4) in space $W_2^4(Q)$ is proven in Theorem 1. Now we prove the existence of a solution to problem (1)-(4) in space $W_2^4(Q)$. To do this, we consider equation (8) and boundary conditions (9)-(11) for $\varepsilon>0$ in domain Q. Since all the conditions of Theorem 2 are satisfied, there exists a unique regular solution to problem (8), (9)-(11) for $\varepsilon>0$ from $V(Q)$ and the first and second estimates are valid for it. It follows that from the set of functions $\{u_\varepsilon(x,t)\}$, $\varepsilon > 0$, we can extract a weakly convergent subsequence of functions in $V(Q)$, such that $\{u_{\varepsilon_i}(x,t)\} \to u(x,t)$ as $\varepsilon_i \to 0$. We will show that the limit function $u(x,t)$ satisfies equation $Lu = f$ (equation (1)) almost everywhere in domain Q. Indeed, since subsequence $\{u_{\varepsilon_i}(x,t)\}$ weakly converges in $W_2^4(Q)$, subsequence $\left\{\sqrt{\varepsilon_i}\frac{\partial\,\Delta^2 u_{\varepsilon_i}(x,t)}{\partial\,t}\right\}$ is uniformly bounded in $L_2(Q)$, and operator L is linear, we obtain:

$$Lu - f = Lu - Lu_{\varepsilon_i} + \varepsilon_i \frac{\partial\,\Delta^2 u_{\varepsilon_i}}{\partial\,t} = L\,(u - u_{\,\varepsilon_i}) + \varepsilon_i\,\frac{\partial\,\Delta^2 u_{\,\varepsilon_i}}{\partial\,t}. \qquad (29)$$

From equality (29), passing to the limit as $\varepsilon_i \to 0$, we obtain a unique solution to problem (1)-(4) [25]. Thus, Theorem 3 is proven.

5. Smoothness of the generalized solution to problem (1), (2)-(4)

Now we proceed to the study of the smoothness of the generalized solution to problem (1), (2)-(4) in Sobolev spaces $W_2^{m+4}(Q)$, when $0 \le m$ is a finite integer. Below, for simplicity, we assume that the coefficients of equation (1) are sufficiently differentiable functions in the closed domain $\overline{Q}$.

Theorem 4. *Let all the conditions of Theorem 3 be satisfied, in addition, let* $p = 0,1,2,3,...,m$ *and* $q = 0,1,2,3,...,m$; $-2\,(K_3 + m\,K_{4t}) + (2j-3)K_{4t} + 3\lambda\,K_4 \ge \delta > 0$, $j = 0,1,2$ *for all* $(x,t) \in \overline{Q}$, $D_t^{p+1}\,K_4\,|_{t=0} = D_t^{p+1}\,K_4\,|_{t=T}$, $D_t^q\,K_i\,|_{t=0} = D_t^q\,K_i\,|_{t=T}$, $(i = 0,1,2,3)$.

Then for any function $f(x,t) \in W_2^m(Q)$, *such that* $\gamma D_t^q\,f|_{t=0} = D_t^q\,f|_{t=T}$ *for all* $x \in [0,1]$, *there exists a unique solution to problem* (1)-(4) *in Sobolev spaces* $W_2^{m+4}(Q)$, *where* $0 \le m$ *is a finite integer.*

Proof. Considering the conditions of Theorem 2, Theorem 3 for $\varepsilon > 0$ and nonlocal boundary conditions at $t = 0,\ t = T$, from the equality

$$\left(e^{-\frac{\lambda\,t}{2}}\,L_\varepsilon u_\varepsilon\right)|_{t=0}^{t=T} = \left(-\,\varepsilon\,e^{\frac{-\lambda\,t}{2}}\,\frac{\partial}{\partial t}\Delta^2 u_\varepsilon + e^{\frac{-\lambda\,t}{2}}\,Lu_\varepsilon\right)|_{t=0}^{t=T} = \left(e^{\frac{-\lambda\,t}{2}}\,f(x,t)\right)|_{t=0}^{t=T}$$

we obtain

$$\gamma D_t^5 u_\varepsilon\big|_{t=0} = D_t^5 u_\varepsilon\big|_{t=T}\,.$$

It follows that function $\vartheta_\varepsilon(x,t) = u_{\varepsilon\,t}(x,t)$ belongs to class $V(D)$ and satisfies the following equation:

$$T_\varepsilon\vartheta_\varepsilon \equiv L_\varepsilon\vartheta_\varepsilon + K_{4t}\vartheta_{\varepsilon ttt} = f_t - \sum_{i=0}^{3} K_{it}D_t^i u_\varepsilon \equiv F_\varepsilon.$$

From Theorem 3, it follows that the family of functions $\{F_\varepsilon\}$ is uniformly bounded in space $L_2(D)$, i.e. $\|F_\varepsilon\|_0^2 \le c_1 \parallel f \parallel_1^2$.

Then, from the conditions of Theorem 3, it is easy to obtain that the coefficients of operator T_ε ($\varepsilon > 0$) satisfy the conditions of Theorem 4, hence, based on estimates I), II), and Theorem 3 for function $\{\vartheta_\varepsilon(x,t)\}$, we obtain similar estimates:

$$\varepsilon \left(\|\vartheta_{\varepsilon ttt}\|_0^2 + \|\vartheta_{\varepsilon ttx}\|_0^2 + \|\vartheta_{\varepsilon txx}\|_0^2 \right) + \|\vartheta_\varepsilon\|_2^2 \le c_1 \parallel f \parallel_0^2,$$

$$\varepsilon \left\| \frac{\partial}{\partial t} \Delta^2 \vartheta_\varepsilon \right\| + \|\vartheta_\varepsilon\|_4^2 \le c_2 \parallel f \parallel_1^2.$$

Functions $\{ u_\varepsilon\}$ satisfy the parabolic equation with conditions (2), (3):

$$\Pi\, u_\varepsilon \equiv u_{\varepsilon\, t} - u_{\varepsilon xx} = f + \varepsilon \frac{\partial}{\partial t} \Delta^2 u_\varepsilon - \sum_{i=0}^{4} K_i\, D_t^i u_\varepsilon - M\, u_\varepsilon + u_{\varepsilon\, t} - u_{\varepsilon xx} \equiv \Phi_\varepsilon$$

and $\Phi_\varepsilon \in W_2^1(Q)$; by virtue of what was proven above, the family of functions $\{\Phi_\varepsilon\}$ is uniformly bounded in space $W_2^1(Q)$, i.e.

$$\|\Phi_\varepsilon\|_1^2 \le c_2 \left(\parallel f \parallel_1^2 + \parallel f_{tt} \parallel_0^2 \right) \le c_2 \|f\|_2^2 < c_4 \|f\|_4^2. \tag{30}$$

Hence, based on a priori estimates for parabolic equations [25, 26] and inequality (30), we obtain

$$\parallel u_\varepsilon \parallel_5^2 \le c_4 \|f\|_4^2.$$

Repeating similar reasoning, the following inequality is also proven:

$$\|u_\varepsilon\|_{m+2}^2 \le c_{m+2} \|f\|_m^2, \ \ m = 1, 2, 3, ...$$

6. Conclusion

In this paper, we investigated the unique solvability and smoothness of solutions to a semi-nonlocal boundary value problem for a fourth-order mixed-type equation of the second kind in Sobolev spaces. The theorems of uniqueness were proved using the energy integral method, demonstrating the robustness of this approach for mixed-type equations. The existence and smoothness of solutions were established through the combined use of ε-regularization methods, a priori estimates, modified Galerkin methods, and the Fourier method. These results extend the theory of mixed-type equations and provide a deeper understanding of the regularity properties of solutions in Sobolev spaces. The proposed techniques and results can be further applied to other types of boundary value problems in mathematical physics.

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