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On a $(\omega, c)-$periodic solution for an impulsive system of differential equations with product of two nonlinear functions and mixed maxima

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Abstract. In this article the questions of existence and uniqueness of $(\omega, c)-$periodic solution of boundary value problem for an impulsive system of ordinary differential equations with product of two nonlinear functions and mixed maxima are studied. This problem is reduced to the investigation of solvability of the system of nonlinear functional-integral equations. The method of contracted mapping is used in the proof of unique solvability of nonlinear functional-integral equations in the space $BD\left([0, \omega], \mathbb{R}^n\right)$. Obtained an estimate for the $(\omega, c)-$periodic solution of the studying problem.

Key Words and Phrases: Impulsive system of differential equations, product of two nonlinear functions, $(\omega, c)-$periodic solution, mixed maxima, contracted mapping, existence and uniqueness.

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1. Introduction. Problem statement

The dynamics of evolving processes sometimes undergoes abrupt changes. Often we have to consider differential equations, the solutions of which are functions with first kind discontinuities at times. Impulsive differential and integro-differential equations have applications in biological, chemical and physical sciences, ecology, biotechnology, industrial robotic, pharmacokinetics, optimal control, etc. [1]–[7]. In particular, some problems with impulsive effects appear in biophysics at micro- and nano-scales [8]–[11]. A lot of publications are devoted to study differential equations with impulsive effects [12]–[37]. $(\omega, c)-$periodic solutions of the differential equations are studied in the works [38]–[43].

A continuous function $f(t):\mathbb{R}\to\mathbb{X}$ is $(\omega,c)-$periodic, if there exists a real number $c\in(0,1)\cup(1,\infty)$, that $f(t+\omega)=c\cdot f(t)$ for all $t\in\mathbb{R}$, where $0<\omega<\infty$, $\mathbb{X}\in\mathbb{R}^n$ is closed set.

On the interval $\Omega\equiv[0,\omega]\setminus\{t_i\}$ for $i=1,2,...,p$ we consider the questions of existence of the $(\omega,c)-$periodic solutions of the nonlinear impulsive system of differential equations with mixed maxima

$$x'(t)=$$

$$=g\Big(t,x(t),\max\big\{x(\tau)\big|\tau\in\big[\delta_1(t)\,\hat{|}\,\delta_2(t)\big]\big\}\Big)\,f\Big(t,x(t),\max\big\{x(\tau)\big|\tau\in\big[\delta_1(t)\,\hat{|}\delta_2(t)\big]\big\}\Big),\tag{1}$$

where $\big[\delta_1(t)\,\hat{|}\,\delta_2(t)\big]=\big[\min\{\delta_1(t);\,\delta_2(t)\},\,\max\{\delta_1(t);\,\delta_2(t)\}\big]$, $\delta_\kappa(t)\in C\big[\delta_0,h\big]$, $0<\delta_0=\min\limits_{0\le t\le\omega}\delta_\kappa(t)$, $\delta_0<h=\max\limits_{0\le t\le\omega}\delta_\kappa(t)<\omega$, $\kappa=1,2$, $g(t,x,y)\in C\big([0,\omega]\times\mathbb{X}\times\mathbb{X},\mathbb{R}^n\big)$, $f(t,x,y)\in C\big([0,\omega]\times\mathbb{X}\times\mathbb{X},\mathbb{R}^n\big)$.

The equation (1) we study with $(\omega,c)-$periodic condition

$$x(\omega)=c\cdot x(0),\quad c\in(0,1)\cup(1,\infty).\tag{2}$$

We assume that on the interval

$$I_1=[0,t_1]\cup[t_2,t_3]\cup\ldots\cup[t_{p-3},t_{p-2}]\cup[t_{p-1},t_p]$$

there holds $\delta_1(t)<\delta_2(t)$ and the equation (1) has the form

$$x'(t)=$$

$$=g\left(t,x(t),\max\big\{x(\tau)|\tau\in\big[\delta_1(t),\delta_2(t)\big]\big\}\right)f\left(t,x(t),\max\big\{x(\tau)|\tau\in\big[\delta_1(t),\delta_2(t)\big]\big\}\right).$$

And on the interval

$$I_2=[t_1,t_2]\cup[t_3,t_4]\cup\ldots\cup[t_{p-2},t_{p-1}]\cup[t_p,t_{p+1}]$$

there holds $\delta_1(t)>\delta_2(t)$ and the equation (1) has the form

$$x'(t)=$$

$$=g\left(t,x(t),\max\big\{x(\tau)|\tau\in\big[\delta_2(t),\delta_1(t)\big]\big\}\right)f\left(t,x(t),\max\big\{x(\tau)|\tau\in\big[\delta_2(t),\delta_1(t)\big]\big\}\right).$$

If we put $\delta_1\big(t_i\big)=\delta_2\big(t_i\big)$, $i=1,2,\ldots,p$, then there is no impulsive effect in the equation (1). So, we suppose that $\delta_1\left(t_i^+\right)\neq\delta_2\left(t_i^-\right)$, $\delta_2\left(t_i^+\right)\neq\delta_1\left(t_i^-\right)$, $i=1,2,\ldots,p$. Consequently, the problem (1), (2) we study with nonlinear impulsive effects

$$x\left(t_i^+\right)-x\left(t_i^-\right)=F_i\big(x\big(t_i\big)\big),\quad i=1,2,...,p,\tag{3}$$

where $0 = t_0 < t_1 < \ \dots \ < t_p < t_{p+1} = \omega,\ x\left(t_i^+\right)$ and $x\left(t_i^-\right)$ right and left-hand side limits, $F_i(x) \in C\big(\mathbb{X}, \mathbb{R}^n\big),\ F_i = F_{i+p},\ t_{i+p} = t_i + \omega.$

In general, the solutions of differential equations with maxima have properties different from the properties of the solutions of differential equations without maxima. If we consider the equation

$$x'(t) = f\left(t, x(t), \max\left\{x(\tau)|\tau \in \big[h_1(t), h_2(t)\big]\right\}\right),$$

then the increasing solutions coincide with the increasing solutions of the equation

$$x'(t) = f\left(t, x(t), x\big[h_2(t)\big]\right).$$

The decreasing solutions coincide with the decreasing solutions of the equation

$$x'(t) = f\left(t, x(t), x\big[h_1(t)\big]\right).$$

Periodic solutions of differential equations behave differently from periodic solutions of differential equations without maxima. If we consider the function $\max\left\{\sin\tau|\tau \in \big[t - h(t), t\big]\right\}$, the properties of this function differ from the properties of the functions $\sin t$ and $\sin(t - h(t))$. For example, the function $\max\left\{\sin\tau|\tau \in \big[t - \pi, t\big]\right\}$ is not negative on the axis $(-\infty, \infty)$. The function $\max\left\{\sin\tau\big|\tau \in \big[t - 2\pi, t\big]\right\}$ is constant on the axis $(-\infty, \infty)$ and equal to 1.

If we consider $(2\pi, c)-$periodic function $e^t \sin t$, this function has resonance on the interval $(0, \infty)$, where $c = e^{2\pi}$. The function $\max\left\{e^\tau \sin\tau\big|\tau \in \big[t - \pi, t\big]\right\} = e^t \max\left\{\sin\tau\big|\tau \in \big[t - \pi, t\big]\right\}$ has positive resonance on the interval $(0, \infty)$.

Therefore, the study of differential equations with maxima is relevant. In the case of differential equations with mixed maxima, the presence of impulsive effects is not fictitious. Differential equations with product of two nonlinear functions appear in solving nonlinear partial differential equations of parabolic and hyperbolic types (see, [44]). We note that the present paper is further development of the works [45]–[47].

We recall that by $C\left([0, \omega], \mathbb{R}^n\right)$ is denoted the Banach space with continuous vector functions $x(t)$ on the segment $[0, \omega]$ and this space is equipped with the norm

$$\parallel x(t) \parallel_{C[0,\omega]} = \sqrt{\sum_{j=1}^{n} \max_{0 \le t \le \omega} \big| x_j(t) \big|}.$$

By $PC\big([0, \omega], \mathbb{R}^n\big)$ is denoted the following linear vector space

$$PC\left([0, \omega], \mathbb{R}^n\right) = \left\{x : [0, \omega] \to \mathbb{R}^n;\ x(t) \in C\left(\left(t_i, t_{i+1}\right], \mathbb{R}^n\right),\ i = 1, ..., p\right\},$$

where limits $x\left(t_i^+\right)$ and $x\left(t_i^-\right)$ $(i = 0, 1, ..., p)$ exist and bounded; $x\left(t_i^-\right) = x\big(t_i\big)$. Note, that the linear vector space $PC\big([0, \omega], \mathbb{R}^n\big)$ is Banach space, if we equip it with the norm

$$\| \, x(t) \, \|_{PC[0,\omega]} = \max \Big\{ \| \, x(t) \, \|_{C(t_i, t_{i+1}]} \, , \ i = 1, 2, \, \ldots \, , \, p \Big\} \, .$$

We use also the vector space $BD\left([0, \omega], \mathbb{R}^n\right)$, which is Banach space with the following norm

$$\| \, x(t) \, \|_{BD[0,\omega]} = \| \, x(t) \, \|_{PC[0,\omega]} + h \cdot \| \, x'(t) \, \|_{PC[0,\omega]} \, ,$$

where $0 < h = \text{const}$.

Formulation of problem. To find the $(\omega, c)-$periodic function $x(t) \in BD\left([0, \omega], \mathbb{R}^n\right)$, which for all $t \in \Omega$ satisfies the system of differential equations (1), $(\omega, c)-$periodic condition (2) and for $t = t_i$, $i = 1, 2, \, \ldots \, , \, p$ satisfies the nonlinear limit condition (3).

2. Reduction to functional-integral equations

Let the function $x(t) \in BD\left([0, \omega], \mathbb{R}^n\right)$ is a solution of the $(\omega, c)-$periodic boundary value problem (1)–(3). Then, after integration on the intervals $(0, t_1]$, $(t_1, t_2]$, $\ldots$, $(t_p, t_{p+1}]$, we have:

$$\int\limits_0^{t_1} g(s, x, y) f(s, x, y) ds = \int\limits_0^{t_1} x'(s) ds = x\big(t_1^-\big) - x(0^+), \tag{4}$$

$$\int\limits_{t_1}^{t_2} g(s, x, y) f(s, x, y) ds = \int\limits_{t_1}^{t_2} x'(s) ds = x\big(t_2^-\big) - x\big(t_1^+\big), \tag{5}$$

$$\int\limits_{t_2}^{t_3} g(s, x, y) f(s, x, y) ds = \int\limits_{t_2}^{t_3} x'(s) ds = x\big(t_3^-\big) - x\big(t_2^+\big), \tag{6}$$

$$\vdots$$

$$\int\limits_{t_p}^{\omega} g(s, x, y) f(s, x, y) ds = \int\limits_{t_p}^{t_{p+1}} x'(s) ds = x\big(t_{p+1}^-\big) - x\big(t_p^+\big). \tag{7}$$

From the formulas (4)–(7) and $x(0^+) = x(0)$, $x\left(t_{p+1}^-\right) = x(t)$, on the interval $(0, \omega]$ we have

$$\int_0^t g(s,x,y)f(s,x,y)ds =$$

$$= -x(0) - \left[x\left(t_1^+\right) - x\left(t_1\right)\right] - \left[x\left(t_2^+\right) - x\left(t_2\right)\right] - \ \ldots \ - \left[x\left(t_p^+\right) - x\left(t_p\right)\right] + x(t).$$

Hence, taking into account the impulsive condition (3) in the last equality, we obtain

$$x(t) = x(0) + \sum_{0<t_i<t} F_i\left(x\left(t_i\right)\right) + \int_0^t g(s,x,y)f(s,x,y)ds. \tag{8}$$

From (8) we have

$$x(\omega) = x(0) + \sum_{0<t_i<\omega} F_i\left(x\left(t_i\right)\right) + \int_0^\omega g(s,x,y)f(s,x,y)ds. \tag{9}$$

Let the function $x(t) \in BD\left([0,\omega], \mathbb{R}^n\right)$ in (8), satisfies the boundary value condition (2). Then from (9) we have

$$x(0) = \frac{1}{c-1} \sum_{0<t_i<\omega} F_i\left(x\left(t_i\right)\right) + \frac{1}{c-1} \int_0^\omega g(s,x,y)f(s,x,y)ds. \tag{10}$$

Substituting (10) into (8), we obtain the functional differential equation

$$x(t) = J(t;x) \equiv \frac{1}{\nu} \sum_{0<t_i<\omega} F_i\left(x\left(t_i\right)\right) + \sum_{0<t_i<t} F_i\left(x\left(t_i\right)\right) +$$

$$+\frac{1}{\nu} \int_0^\omega g\left(s, x(s), \max\left\{x(\tau) \middle| \tau \in \left[\delta_1 \hat{|} \delta_2\right]\right\}\right) f\left(s, x(s), \max\left\{x(\tau) \middle| \tau \in \left[\delta_1 \hat{|} \delta_2\right]\right\}\right) ds +$$

$$+\int_0^t g\left(s, x(s), \max\left\{x(\tau) \middle| \tau \in \left[\delta_1 \hat{|} \delta_2\right]\right\}\right) f\left(s, x(s), \max\left\{x(\tau) \middle| \tau \in \left[\delta_1 \hat{|} \delta_2\right]\right\}\right) ds, \tag{11}$$

where $\delta_\kappa = \delta_1(s)$, $\kappa = 1, 2$, $\nu = c - 1 \neq 0$.

3. Main results

Lemma 1. *Assume that there exist positive quantities M_g, M_f, M_{F_i} such that for all $t \in [0,\omega]$ are fulfilled the following conditions:*
1). $\| g(t,x,y) \|_{C[0,\omega]} \le M_g < \infty, \quad \| f(t,x,y) \|_{C[0,\omega]} \le M_f < \infty$;
2). $\max\limits_{1 \le i \le p} \| F_i\,(x(t_i)) \|_{C[0,\omega]} \le M_{F_i} < \infty$.

Then for the equation (11) *is true the following estimate*

$$\| J(t;x) \|_{PC[0,\omega]} \le p\frac{1+|\,\nu\,|}{|\,\nu\,|} M_{F_i} + \frac{\omega + T\,|\,\nu\,|}{|\,\nu\,|} M_g M_f. \tag{12}$$

Proof. From the equation (11) we obtain

$$\| J(t;x) \|_{PC[0,\omega]} \le \frac{1}{|\,\nu\,|} \sum_{0<t_i<\omega} \|F_i\|_{C[0,\omega]} + \sum_{0<t_i<t} \|F_i\|_{C[0,\omega]} +$$

$$+\frac{1}{|\,\nu\,|}\int\limits_0^\omega \| g \|_{C[0,\omega]} \| f \|_{C[0,\omega]}\, ds + \int\limits_0^t \| g \|_{C[0,\omega]} \| f \|_{C[0,\omega]}\, ds \le$$

$$\le p\frac{1+|\,\nu\,|}{|\,\nu\,|} \max_{1 \le i \le p} \| F_i \|_{C[0,\omega]} + \frac{\omega + T\,|\,\nu\,|}{|\,\nu\,|} \| g \|_{C[0,\omega]} \| f \|_{C[0,\omega]}\,. \tag{13}$$

From the estimate (13) follows (12). Lemma 1 is proved.

Lemma 2. *For the difference of two functions with maxima there holds the following estimate*

$$\Big\| \max\left\{x(\tau) \middle| \tau \in \left[\lambda_1(s)\hat{|}\lambda_2(s)\right]\right\} - \max\left\{y(\tau) \middle| \tau \in \left[\lambda_1(s)\hat{|}\lambda_2(s)\right]\right\} \Big\|_{PC[0,\omega]} \le$$

$$\le \| x(t) - y(t) \|_{PC[0,\omega]} + h \left\| x'(t) - y'(t) \right\|_{PC[0,\omega]} = \| x(t) - y(t) \|_{BD[0,\omega]}\,, \tag{14}$$

where $h = \max\limits_{0 \le t \le \omega} \left|\, \lambda_2(t) - \lambda_1(t)\, \right|$.

Proof. For definiteness, we assume that $\lambda_1(t) < \lambda_2(t)$. It is obvious that there the following relation

$$\max\left\{x(\tau)\middle|\tau \in \left[\lambda_1(t), \lambda_2(t)\right]\right\} = \max\left\{[x(\tau) - y(\tau) + y(\tau)]\middle|\tau \in \left[\lambda_1(t), \lambda_2(t)\right]\right\} \le$$

$$\le \max\left\{[x(\tau) - y(\tau)]\middle|\tau \in \left[\lambda_1(t), \lambda_2(t)\right]\right\} + \max\left\{y(\tau)\middle|\tau \in \left[\lambda_1(t), \lambda_2(t)\right]\right\}$$

is true. Hence, we obtain that

$$\max\left\{x(\tau)\middle|\tau \in \left[\lambda_1(t), \lambda_2(t)\right]\right\} - \max\left\{y(\tau)\middle|\tau \in \left[\lambda_1(t), \lambda_2(t)\right]\right\} \le$$

$$\leq \max\left\{[x(\tau)-y(\tau)]\big|\tau\in\big[\lambda_1(t),\lambda_2(t)\big]\right\}. \tag{15}$$

We fix the variable t and denote by s_1 and s_2 the points of the segment $\big[\lambda_1(t),\lambda_2(t)\big]$, on which the maximums of the functions $x(t)$ and $y(t)$ are reached:

$$\max\left\{x(\tau)\big|\tau\in\big[\lambda_1(t),\lambda_2(t)\big]\right\}=x(s_1),\quad \max\left\{y(\tau)\big|\tau\in\big[\lambda_1(t),\lambda_2(t)\big]\right\}=y(s_1),$$

$$\max\left\{[x(\tau)-y(\tau)]\big|\tau\in\big[\lambda_1(t),\lambda_2(t)\big]\right\}=x(s_2)-y(s_2).$$

The inequality (15) we rewrite as

$$\max\left\{x(\tau)\big|\tau\in\big[\lambda_1(t),\lambda_2(t)\big]\right\}-\max\left\{y(\tau)\big|\tau\in\big[\lambda_1(t),\lambda_2(t)\big]\right\}\leq$$
$$\leq x(s_2)-y(s_2). \tag{16}$$

Subtracting $x(s_1)-y(s_1)$ from both sides of (16), we obtain

$$\max\left\{x(\tau)\big|\tau\in\big[\lambda_1(t),\lambda_2(t)\big]\right\}-\max\left\{y(\tau)\big|\tau\in\big[\lambda_1(t),\lambda_2(t)\big]\right\}-x(s_1)+y(s_1)\leq$$
$$\leq x(s_2)-y(s_2)-x(s_1)+y(s_1),\quad s_1,s_2\in\big[\lambda_1(t),\lambda_2(t)\big]. \tag{17}$$

We consider the right-hand side of (17) and use the Lagrange's Mean-value theorem:

$$x(s_2)-y(s_2)-x(s_1)+y(s_1)=[x(s_2)-x(s_1)]-[y(s_2)-y(s_1)]=$$
$$=(s_2-s_1)x'(\bar{s})-(s_2-s_1)y'(\bar{\bar{s}})=(s_2-s_1)[x'(\bar{s})-y'(\bar{\bar{s}})]\leq$$
$$\leq h_0\cdot\big|\,x'(\bar{s})-y'(\bar{\bar{s}})\,\big|,\quad h_0=|\,s_2-s_1\,|,\quad \bar{s},\bar{\bar{s}}\in(s_1,s_2). \tag{18}$$

From the inequalities (17) and (18) we have

$$\max\left\{x(\tau)\big|\tau\in\big[\lambda_1(t),\lambda_2(t)\big]\right\}-\max\left\{y(\tau)\big|\tau\in\big[\lambda_1(t),\lambda_2(t)\big]\right\}-x(s_1)+y(s_1)\leq$$
$$\leq h_0\cdot\big|\,x'(\bar{s})-y'(\bar{\bar{s}})\,\big|$$

or

$$\max\left\{x(\tau)\big|\tau\in\big[\lambda_1(t),\lambda_2(t)\big]\right\}-\max\left\{y(\tau)\big|\tau\in\big[\lambda_1(t),\lambda_2(t)\big]\right\}\leq$$
$$\leq[x(s_1)-y(s_1)]+h_0\cdot\big|\,x'(\bar{s})-y'(\bar{\bar{s}})\,\big|.$$

Hence, we obtain

$$\Big|\max\left\{x(\tau)\big|\tau\in\big[\lambda_1(t),\lambda_2(t)\big]\right\}-\max\left\{y(\tau)\big|\tau\in\big[\lambda_1(t),\lambda_2(t)\big]\right\}\Big|\leq$$
$$\leq|\,x(s_1)-y(s_1)\,|+h_0\cdot\big|\,x'(\bar{s})-y'(\bar{\bar{s}})\,\big|. \tag{19}$$

Proceeding in (19) to the norm in the space of continuous functions $C[0,T]$, for $h=\max\limits_{0\leq t\leq T}|\,\lambda_2(t)-\lambda_1(t)\,|\geq h_0$ we arrive at (14). The case $\lambda_1(t)>\lambda_2(t)$ is proved similarly. The Lemma 2 is proved.

Theorem 1. *Assume that the conditions of the Lemma 1 are fulfilled and there exist positive quantities* L_1, L_2, L_{3i} *such that for all* $t \in \Omega$ *are fulfilled the following conditions:*

1). $\| g(t, x_1, y_1) - g(t, x_2, y_2) \| \le L_1 [\| x_1 - x_2 \| + \| y_1 - y_2 \|]$;

2). $\| f(t, x_1, y_1) - f(t, x_2, y_2) \| \le L_2 [\| x_1 - x_2 \| + \| y_1 - y_2 \|]$;

3). $\| F_i(t, x_1) - F_i(t, x_2) \| \le L_{3i} \| x_1 - x_2 \|$;

4). *The radius of the inscribed ball in* $\mathbb{X}$ *is greater than*

$$p\frac{1 + |\nu|}{|\nu|} M_{F_i} + \frac{\omega + T|\nu|}{|\nu|} M_g M_f;$$

5). $\rho < 1$, *where* $\rho = \beta_1 + h\beta_2$ *and* β_1, β_2 *are defined from* (23), (26) *below.*

Then the problem (1)–(3) *has a unique* $(\omega, c)-$*periodic solution for all* $t \in [0, \omega]$.

Proof. The theorem we proof by the fixed-point method. According to the theorem condition, we have

$$f(t + \omega, x(t + \omega), y(t + \omega)) = f(t + \omega, c\,x(t), c\,y(t)) = c\,f(t, x(t), y(t)).$$

We differentiate (11):

$$x'(t) = J(t; x') \equiv$$

$$\equiv g\left(t, x(t), \max\left\{x(\tau)\middle|\tau \in \left[\delta_1 \hat{\,} \delta_2\right]\right\}\right) f\left(t, x(t), \max\left\{x(\tau)\middle|\tau \in \left[\delta_1 \hat{\,} \delta_2\right]\right\}\right), \quad (20)$$

where $\delta_\kappa = \delta_\kappa(t)$, $\kappa = 1, 2$.

For the difference of two operators in (11), to obtain estimate, we use the following approach:

$$\| g(x)f(x) - g(y)f(y) \| \le \| g(x)f(x) - g(y)f(x) \| + \| g(y)f(x) - g(y)f(y) \| \le$$

$$\le \| f(x) \| \, \| g(x) - g(y) \| + \| g(y) \| \, \| f(x) - f(y) \|.$$

Consequently, from (11) we have

$$\|J(t; x) - J(t; y)\|_{PC[0,\omega]} \le$$

$$\le \frac{1}{|\nu|} \sum_{0<t_i<\omega} \|F_i(x(t_i)) - F_i(y(t_i))\|_{C[0,\omega]} + \sum_{0<t_i<t} \|F_i(x(t_i)) - F_i(y(t_i))\|_{C[0,\omega]} +$$

$$+ \frac{1}{|\nu|} \int_0^\omega \left[\left\| f\left(s, x(s), \max\left\{x(\tau)\middle|\tau \in \left[\delta_1 \hat{\,} \delta_2\right]\right\}\right) \right\|_{C[0,\omega]} \times\right.$$

$$\times \left\| g\left(s, x(s), \max\left\{x(\tau)\middle|\tau \in \left[\delta_1 \hat{|} \delta_2\right]\right\}\right) - g\left(s, y(s), \max\left\{y(\tau)\middle|\tau \in \left[\delta_1 \hat{|} \delta_2\right]\right\}\right) \right\| +$$

$$+ \left\| g\left(s, y(s), \max\left\{y(\tau)\middle|\tau \in \left[\delta_1 \hat{|} \delta_2\right]\right\}\right) \right\|_{C[0,\omega]} \times$$

$$\times \left\| f\left(s, x(s), \max\left\{x(\tau)\middle|\tau \in \left[\delta_1 \hat{|} \delta_2\right]\right\}\right) - f\left(s, y(s), \max\left\{y(\tau)\middle|\tau \in \left[\delta_1 \hat{|} \delta_2\right]\right\}\right) \right\|\Big] ds +$$

$$+ \int_0^t \left[\left\| f\left(s, x(s), \max\left\{x(\tau)\middle|\tau \in \left[\delta_1 \hat{|} \delta_2\right]\right\}\right) \right\|_{C[0,\omega]} \times\right.$$

$$\times \left\| g\left(s, x(s), \max\left\{x(\tau)\middle|\tau \in \left[\delta_1 \hat{|} \delta_2\right]\right\}\right) - g\left(s, y(s), \max\left\{y(\tau)\middle|\tau \in \left[\delta_1 \hat{|} \delta_2\right]\right\}\right) \right\| +$$

$$+ \left\| g\left(s, y(s), \max\left\{y(\tau)\middle|\tau \in \left[\delta_1 \hat{|} \delta_2\right]\right\}\right) \right\|_{C[0,\omega]} \times$$

$$\times \left\| f\left(s, x(s), \max\left\{x(\tau)\middle|\tau \in \left[\delta_1 \hat{|} \delta_2\right]\right\}\right) - f\left(s, y(s), \max\left\{y(\tau)\middle|\tau \in \left[\delta_1 \hat{|} \delta_2\right]\right\}\right) \right\|\Big] ds,$$

where $\delta_\kappa = \delta_\kappa(s), \ \kappa = 1, 2$.

Then, by virtue of conditions of the theorem, we derive

$$\|J(t; x) - J(t; y)\|_{PC[0,\omega]} \leq$$

$$\leq p\frac{1 + |\,\nu\,|}{|\,\nu\,|} \max_{1 \leq i \leq p} L_{3i} \,\|\, x\,(t_i) - y\,(t_i)\,\|_{C[t_i, t_{i+1}]} +$$

$$+ \frac{M_f L_1 + M_g L_2}{|\,\nu\,|} \int_0^\omega \Big[\|\, x(s) - y(s)\,\|_{PC[0,\omega]} + \Big\| \max\left\{x(\tau)\middle|\tau \in \left[\delta_1(s) \hat{|} \delta_2(s)\right]\right\} -$$

$$- \max\left\{y(\tau)\middle|\tau \in \left[\delta_1(s) \hat{|} \delta_2(s)\right]\right\} \Big\|_{PC[0,\omega]}\Big] ds +$$

$$+ [M_f L_1 + M_g L_2] \int_0^t \Big[\|\, x(s) - y(s)\,\|_{PC[0,\omega]} + \Big\| \max\left\{x(\tau)\middle|\tau \in \left[\delta_1(s) \hat{|} \delta_2(s)\right]\right\} -$$

$$- \max\left\{y(\tau)\middle|\tau \in \left[\delta_1(s) \hat{|} \delta_2(s)\right]\right\} \Big\|_{PC[0,\omega]}\Big] ds. \qquad (21)$$

Appling Lemma 2 (inequality (14)) to the inequality (21), we obtain that

$$\|J(t; x) - J(t; y)\|_{PC[0,\omega]} \leq p\frac{1 + |\,\nu\,|}{|\,\nu\,|} \max_{1 \leq i \leq p} L_{3i} \,\|\, x(t) - y(t)\,\|_{PC[0,\omega]} +$$

$$+ \frac{\omega + T\,|\,\nu\,|}{|\,\nu\,|} [M_f L_1 + M_g L_2] \Big[2 \,\|\, x(t) - y(t)\,\|_{PC[0,\omega]} + h \,\|\, x'(t) - y'(t)\,\|_{PC[0,\omega]}\Big] \leq$$

$$\leq \beta_1 \| x(t) - y(t) \|_{PC[0,\omega]} + \gamma_1 h \left\| x'(t) - y'(t) \right\|_{PC[0,\omega]}, \tag{22}$$

where

$$\beta_1 = p \frac{1 + |\nu|}{|\nu|} \max_{1 \leq i \leq p} L_{3i} + 2 \frac{\omega + T |\nu|}{|\nu|} (M_f L_1 + M_g L_2), \tag{23}$$

$$\gamma_1 = \frac{\omega + T |\nu|}{|\nu|} [M_f L_1 + M_g L_2].$$

Since $\beta_1 > \gamma_1$, then from (22) we have

$$\|J(t;x) - J(t;y)\|_{PC[0,\omega]} \leq \beta_1 \left[\| x(t) - y(t) \|_{PC[0,\omega]} + h \left\| x'(t) - y'(t) \right\|_{PC[0,\omega]} \right]. \tag{24}$$

Now for the difference of two operators in (21), similarly, we have estimate

$$\left\| J(t;x') - J(t;y') \right\| \leq$$

$$\leq [M_f L_1 + M_g L_2] \left[2 \| x(t) - y(t) \|_{PC[0,\omega]} + h \left\| x'(t) - y'(t) \right\|_{PC[0,\omega]} \right] \leq$$

$$\leq \beta_2 \| x(t) - y(t) \|_{PC[0,\omega]} + \gamma_2 h \left\| x'(t) - y'(t) \right\|_{PC[0,\omega]} \leq$$

$$\leq \beta_2 \left[\| x(t) - y(t) \|_{PC[0,\omega]} + h \left\| x'(t) - y'(t) \right\|_{PC[0,\omega]} \right], \tag{25}$$

where

$$\beta_2 = 2 [M_f L_1 + M_g L_2] > \gamma_2 = M_f L_1 + M_g L_2. \tag{26}$$

We multiply both sides of (25) to h term by term. Then, adding the estimates (24) and (25) term by term, we obtain that

$$\|J(t;x) - J(t;y)\|_{BD[0,\omega]} \leq \rho \cdot \| x(t) - y(t) \|_{BD[0,\omega]}, \tag{27}$$

where $\rho = \beta_1 + h\beta_2$.

According to the last condition of the theorem $\rho < 1$, so right-hand side of (11) as an operator is contraction mapping. From the estimates (12) and (27) implies that there exists a unique fixed point $x(t)$, satisfying equation (1) and $(\omega, c)-$periodic condition (2). The theorem is proved.

4. Conclusion

The theory of differential equations plays an important role in solving applied problems of sciences and technology. Especially, periodic and almost periodic boundary value problems for differential equations with impulsive actions have many applications in mathematical physics, mechanics and technology, in particular in nanotechnology.

In this paper, we investigated the questions of $(\omega, c)-$periodic solvability of the system of impulsive differential equations (1) with $(\omega, c)-$periodic (2) and impulsive (3) conditions for $t = t_i$, $i = 1, 2, \dots, p$. The nonlinear right-hand side of this equation consists the product of two nonlinear functions and construction of mixed maxima. The questions of existence and uniqueness of the $(\omega, c)-$periodic solution of the problem (1)–(3) are studied. The problem we reduce to the $(\omega, c)-$periodic solvability of the system of nonlinear functional integral equations (11). The estimate (12) are obtained for $(\omega, c)-$periodic solutions of the problem (1)–(3).

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